A basic inequality of submanifolds in quaternionic space forms

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Abstract

In this article, we establish a sharp inequality involving δ-invariant introduced by Chen for submanifolds in quaternionic space forms of constant quaternionic sectional curvature with arbitrary codimension.

Mathematics Subject Classification: 53B25, 53B35.
Key words: δ-invariant, scalar curvature, mean curvature, quaternionic space form.

1 Introduction

Let $\tilde{M}$ be a $4m$-dimensional Riemannian manifold with metric g. $\tilde{M}$ is called a quaternionic Kaehler manifold if there exists a 3-dimensional vector space V of tensors of type (1,1) with local basis of almost Hermitian structure ϕ_1, ϕ_2 and ϕ_3 such that for all $i \in \{1, 2, 3\}$:
(a) $\phi_i\phi_{i+1} = \phi_{i+2} = -\phi_{i+1}\phi_i$ and $\phi_i^2 = -1$ (i mod 3),
(b) for any local cross-section ξ of V, $\tilde{\nabla}_X\xi$ is also a cross-section of V, where X is an arbitrary vector field on $\tilde{M}$ and $\tilde{\nabla}$ the Riemannian connection on $\tilde{M}$.
In fact, condition (b) is equivalent to the following condition:
(b$'$) there exist local 1-forms q_1, q_2, q_3 such that

$$\tilde{\nabla}_X\phi_i = q_{i+2}(X)\phi_{i+1} - q_{i+1}(X)\phi_{i+2} \quad (i \text{ mod } 3).$$

Now, let X be a unit vector on $\tilde{M}$, then $X, \phi_1(X), \phi_2(X)$ and $\phi_3(X)$ form an orthonormal frame on $\tilde{M}$. We denote by $Q(X)$ the 4-plane spanned by them, and denote by $\pi(X, Y)$ the plane spanned by X, Y. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic

Balkan Journal of Geometry and Its Applications, Vol.9, No.2, 2004, pp. 92-102.

plane π is called the quaternionic sectional curvature of π. A quaternionic Kaehler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say $4c$. A quaternionic space form will be denoted by $\tilde{M}(4c)$. It is well-known that a quaternionic Kaehler manifold $\tilde{M}$ is a quaternionic space form if and only if its curvature tensor $\tilde{R}$ is of the following ([4]):

$$\begin{aligned}\tilde{R}(X,Y)Z &= c\{g(Y,Z)X - g(X,Z)Y \\ &\quad + \sum_{i=1}^{3}(g(\phi_i Y,Z)\phi_i X - g(\phi_i X,Z)\phi_i Y - 2g(\phi_i X,Y)\phi_i Z)\}\end{aligned}$$

for vectors X, Y, Z tangent to $\tilde{M}$.

An n-dimensional Riemannian manifold M isometrically immersed in $\tilde{M}(4c)$ is called invariant if ϕ_i $(i = 1, 2, 3)$ maps the tangent space T_pM into T_pM for each point $p \in M$. Also, M is called totally real or anti-invariant if ϕ_i maps the tangent space T_pM into $T_p^{\perp}M$ for each point $p \in M$, that is, $\phi_i(T_pM) \subset T_p^{\perp}M$, where $T_p^{\perp}M$ is the normal space of M in $\tilde{M}(4c)$. A submanifold M is said to admit a quasi anti-invariant structure of rank k in $\tilde{M}(4c)$ if the tangent bundle TM of M is decomposed as $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$ satisfying ([6]):

(i) $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are mutually orthogonal.

(ii) $\mathcal{D}^{\perp}$ is anti-invariant under the action of ϕ_i for every point p of M.

(iii) $\dim\mathcal{D}^{\perp} = k$.

Let ∇ be the induced Levi-Civita connection on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}_X V = -A_V X + D_X V$$

for vector fields X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, D the normal connection and A_V the shape operator in the direction of V. The second fundamental form and the shape operator are related by

$$g(h(X,Y),V) = g(A_V X, Y).$$

We also use g for the induced Riemannian metric on M as well as the quaternionic space form $\tilde{M}(4c)$. The mean curvature vector H on M in $\tilde{M}(4c)$ plays an important role in determining our basic inequality later that is defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) = \mathrm{tr}h$$

where $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of tangent bundle TM of M. A submanifold M in $\tilde{M}(4c)$ is called minimal if the mean curvature vector H vanishes identically over M.

2 Riemannian invariants

Riemannian invariants of a Riemannian manifold are the intrinsic characteristic of the Riemannian manifold. In this section, we recall a string of Riemannian invariants on a Riemannian manifold ([3]).

For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$. For any orthornormal basis $e_1, \cdots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined by to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.1}$$

Chen introduced an invariant δ_M on M by using

$$(\inf K)(p) = \inf\{K(\pi)|\ \pi \text{ is a plane section} \subset T_pM\}$$

in the following manner:

$$\delta_M = \tau - \inf K. \tag{2.2}$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \cdots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{2.3}$$

Given an orthonormal basis $\{e_1, \cdots, e_n\}$ of the tangent space T_pM, we simply denote by $\tau_{1\cdots r}$ the scalar curvature of the r-plane section spanned by $e_1, \cdots, e_r$. The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p, and if L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of L. Geometrically, $\tau(L)$ is nothing but the scalar curvature of the image $exp_p(L)$ of L at p under the exponential map at p. For an integer $k \geq 0$ denote by $\mathcal{S}(n, k)$ the finite set consisting of unordered k-tuples $(n_1, \cdots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of unordered k-tuples with $k \geq 0$ for a fixed n. For each k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$ the sequence of Riemannian invariant $\mathcal{S}(n_1, \cdots, n_k)(p)$ is defined by

$$\mathcal{S}(n_1, \ldots, n_k)(p) = \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, \cdots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, j = 1, \cdots, k$. The string of Riemannian curvature invariant $\delta(n_1, \cdots, n_k)$ is given by

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \mathcal{S}(n_1, \ldots, n_k)(p). \tag{2.4}$$

For each $(n_1, \cdots, n_k) \in \mathcal{S}(n)$, let $c(n_1, \cdots, n_k)$ and $b(n_1, \cdots, n_k)$ denote the positive constants given by

$$c(n_1, \cdots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)},$$

$$b(n_1, \cdots, n_k) = \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right).$$

3 An inequality for submanifolds in quaternionic space form

Let M be an n-dimensional Riemannian manifold isometrically immersed in a $4m$-dimensional quaternionic space form $\tilde{M}(4c)$ of constant quaternionic sectional curvature $4c$. Then, the Gauss equation on M is given by

(3.1)
$$\begin{aligned} g(R(X,Y)Z,W) = c\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + \\ &+ \textstyle\sum_{i=1}^{3}[g(\phi_i Y, Z)g(\phi_i X, W) - g(\phi_i X, Z)g(\phi_i Y, W) - 2g(\phi_i X, Y)g(\phi_i Z, W)]\} \end{aligned}$$

for vectors X, Y, Z tangent to $\tilde{M}$. For any $p \in M$ and for any $X \in T_pM$, we have $\phi_i X = P_i X + F_i X$, $P_i \in T_pM$, $F_i \in T_p^{\perp}M$, $i = 1, 2, 3$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM. We put

$$||P_k||^2 = \sum_{i,j=1}^{n} g^2(P_k e_i, e_j), \quad k = 1, 2, 3.$$

On the other hand, the scalar curvature τ satisfies

$$2\tau = n(n-1)c + 3c\sum_{i=1}^{3}||P_i||^2 + n^2||H||^2 - ||h||^2. \tag{3.2}$$

Let $L \subset T_pM$ be a subspace of T_pM, $\dim L = r$. We put

$$\alpha_k(L) = \sum_{1 \le i \le j \le r} g^2(P_k e_i, e_j), \quad k = 1, 2, 3,$$

where $\{e_1, \cdots, e_r\}$ is an orthonormal basis of L.

We now recall Chen's lemma:

Lemma 3.1.([2]) *Let $a_1, \cdots, a_n, c$ be $n+1$ ($n \geq 2$) real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + c\right).$$

Then, $2a_1a_2 \geq c$, with the equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Theorem 3.2. *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic space form $\tilde{M}(4c)$ of constant quaternionic sectional curvature $4c$. Then, for any point $p \in M$ and any plane section π in T_pM, we have*

$$\begin{aligned}\tau - K(\pi) \leq \frac{(n-2)(n+1)}{2}c + \frac{n^2(n-2)}{2(n-1)}||H||^2 \\ - 3c\sum_{i=1}^{3}\alpha_i(\pi) + \frac{3c}{2}\sum_{i=1}^{3}||P_i||^2.\end{aligned} \tag{3.3}$$

The equality holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_{n+1}, \cdots, e_{4m}\}$ for $T_p^{\perp}M$ such that (a) *$\pi = span\{e_1, e_2\}$* (b) *the shape operator $A_r = A_{e_r}, r = n+1, \cdots, 4m$, take the following forms:*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c \end{pmatrix}, \tag{3.4}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 & \dots & 0 \\ d_r & -c_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{3.5}$$

where $a + b = c$ and $c_r, d_r \in \mathbb{R}$.

Proof. Let p be a point of M and π be a plane section contained in the tangent space T_pM of M at p. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ for T_pM and $\{e_{n+1}, \cdots, e_{4m}\}$ for the normal space $T_p^{\perp}M$ at p such that e_1 and e_2 generator the plane section π and the normal vector e_{n+1} is in the driction of the mean curvature vector H. Then the Gauss equation (3.1) gives

$$K(\pi) = K(e_1 \wedge e_2) = c + 3c\sum_{i=1}^{3}\alpha_i(\pi) + h_{11}^{n+1}h_{22}^{n+1} + \sum_{r\geq n+2} h_{11}^r h_{22}^r - (h_{12}^{n+1})^2 - \sum_{r\geq n+2}(h_{12}^r)^2.$$

We put

$$\rho = 2\tau - \frac{n^2(n-2)}{n-1}||H||^2 - 3c\sum_{i=1}^{3}||P_i||^2 - n(n-1)c. \tag{3.6}$$

Substituting (3.2) into (3.6), we have

$$n^2||H||^2 = (n-1)(\rho + ||h||^2),$$

in other words,

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left(\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r\geq n+2}\sum_{i,j}(h_{ij}^r)^2 + \rho\right).$$

Applying Lemma 3.1, we get

$$h_{11}^{n+1}h_{22}^{n+1} \geq \frac{1}{2}\left(\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j}(h_{ij}^r)^2 + \rho\right).$$

Thus, we have

$$\begin{aligned} K(\pi) \geq{}& c + 3c\sum_{i=1}^{3}\alpha_i(\pi) + \frac{1}{2}\rho + \sum_{r=n+1}^{4m}\sum_{j>2}\{(h_{1j}^r)^2 + (h_{2j}^r)^2\} \\ &+ \frac{1}{2}\sum_{i\neq j>2}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{4m}\sum_{i,j>2}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{4m}(h_{11}^r + h_{22}^r)^2. \end{aligned} \tag{3.7}$$

Making use of (3.6), we get (3.3).

Suppose the equality of (3.3) holds. Then, the terms involving h_{ij}^r's in (3.7) vanish at the same time and thus

$$\begin{aligned} &h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad h_{ij}^{n+1} = 0, \quad i \neq j > 2, \\ &h_{1j}^r = h_{2j}^r = h_{ij}^r = 0, \quad r = n+2, \cdots, 4m; \quad i, j \geq 3, \\ &h_{11}^r + h_{22}^r = 0, \quad r = n+2, \cdots, 4m. \end{aligned}$$

Moreover, we may choose e_1 and e_2 such that $h_{12}^{n+1} = 0$. Also, Lemma 3.1 implies that

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.$$

Therefore, the shape operator $A_r (r = n+1, \cdots, 4m)$ take the form (3.4) and (3.5). The converse is obvious. □

Theorem 3.3. *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic projective space $\tilde{M}(4c)$ $(c > 0)$ of constant quaternionic sectional curvature $4c$. Then, for any point $p \in M$ and any plane section π in T_pM, we have*

$$\delta_M \leq \frac{1}{2}(n^2 + 8n - 2)c + \frac{n^2(n-2)}{2(n-1)}||H||^2 \tag{3.8}$$

with equality holding if and only if M is invariant.

Proof. We suppose that $c > 0$, we must maximize the term $\sum_{i=1}^{3} ||P_i||^2 - 2\sum_{i=1}^{3} \alpha_i(\pi)$ in (3.3). The maximum value is reached for $||P_i||^2 = n,\ \alpha_i(\pi) = 0$ $(i = 1, 2, 3)$, that is, M is invariant and we can also obtain (3.8). □

Theorem 3.4. *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic hyperbolic space $\tilde{M}(4c)$ $(c < 0)$ of constant quaternionic sectional curvature $4c$. Then, for any point $p \in M$ and any plane section π in T_pM, we have*

$$\delta_M \leq \frac{(n-2)(n+1)}{2}c + \frac{n^2(n-2)}{2(n-1)}||H||^2 \tag{3.9}$$

with equality holding if and only if M admits a quasi anti- invariant structure of rank $n-2$.

Proof. Assume that $c < 0$. We must minimize the last term $\sum_{i=1}^{3} ||P_i||^2 - 2\sum_{i=1}^{3} \alpha_i(\pi)$ in (3.3) in order to estimate δ_M. For an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM with $\pi = \mathrm{span}\{e_1, e_2\}$, we can write

$$\sum_{i=1}^{3} ||P_i||^2 - 2\sum_{i=1}^{3} \alpha_i(\pi) = \sum_{k=1}^{3} \left(\sum_{i,j=3}^{n} g^2(\phi_k e_i, e_j) + 2\sum_{j=3}^{n} \left(g^2(\phi_k e_1, e_j) + g^2(\phi_k e_2, e_j) \right) \right).$$

Thus, the minimum vale is zero. This occurs only when π =span$\{e_1, e_2\}$ is orthogonal to span$\{\phi_k e_i | i = 3, \cdots, n, \quad k = 1, 2, 3\}$. Furthermore, span$\{\phi_k e_i | i = 3, \cdots, n, \quad k = 1, 2, 3\}$ is orthogonal to the tangent space T_pM. Thus, we have (3.9) with equality holding if and only if M admits a quasi anti-invariant structure of rank $(n-2)$. □

Theorem 3.5. *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic projective space $\tilde{M}(4c)$ $(c > 0)$ of constant quaternionic sectional curvature $4c$. Then, we have*

$$\delta(n_1,\cdots,n_k) \le c(n_1,\cdots,n_k)||H||^2 + b(n_1,\cdots,n_k)c + \frac{9n}{2}c \tag{3.10}$$

for any k-tuple $(n_1,\cdots,n_k) \in \mathcal{S}(n)$.

The equality case of inequality (3.10) holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1,\cdots e_{4m}$ at p such that the shape operators of M in $\tilde{M}(4c)$ $(c>0)$ at p take the following forms:

$$A_r = \begin{pmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & \mathbf{0} \\ 0 & \dots & A_k^r & \\ & \mathbf{0} & & \mu_r I \end{pmatrix}, \quad r = n+1,\cdots,4m, \tag{3.11}$$

where I is an identity matrix and each A_j^r are symmetric $n_j \times n_j$ submatrices such that

$$trace(A_1^r) = \cdots = trace(A_k^r) = \mu_r.$$

Proof. Let M be a submanifold of a quaternionic projective space $\tilde{M}(4c)$ $(c>0)$ of constant quaternionic sectional curvature $4c$.

If $k=1$, this was done in Theorem 3.3. Hence, we assume $k>1$.

Let $(n_1,\cdots,n_k) \in \mathcal{S}(n)$. Put

$$\eta = 2\tau - n(n-1)c - \frac{n^2(n+k-1-\sum n_j)}{(n+k-\sum n_j)}||H||^2 - 3c\sum_{i=1}^{3}||P_i||^2. \tag{3.12}$$

Substituting (3.2) into (3.12), we have

$$n^2||H||^2 = \gamma(\eta + ||h||^2), \quad \gamma = n+k-\sum n_j. \tag{3.13}$$

Let $L_1,\cdots,L_k$ be mutually orthogonal subspaces of T_pM with dim $L_j = n_j, j=1,\cdots,k$. By choosing an orthonormal basis $e_1,\cdots,e_{4m}$ at p such that

$$L_j = \mathrm{Span}\{e_{n_1+\cdots+n_{j-1}+1},\ldots,e_{n_1+\cdots+n_j}\}, \quad j=1,\cdots,k$$

and e_{n+1} is in the direction of the mean curvature vector, we obtain from (3.13) that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \gamma\left(\eta + \sum_{i=1}^{n}(a_i)^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right), \tag{3.14}$$

where $a_i = h_{ii}^{n+1}, i=1,\cdots,n$, and $\gamma = n+k-\sum n_j$.

We set

$$\Delta_1 = \{1, \dots, n_1\}, \cdots, \Delta_k = \{n_1 + \cdots + n_{k-1} + 1, \cdots, n_1 + \cdots + n_k\}. \tag{3.15}$$

In other words, the equation (3.14) can be rewritten in the form

$$\begin{aligned}\left(\sum_{i=1}^{\gamma+1} \bar{a}_i\right)^2 &= \gamma(\eta + \sum_{i=1}^{\gamma+1}(\bar{a}_i)^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\ &- \sum_{2\le\alpha_1\neq\beta_1\le n_1} a_{\alpha_1}a_{\beta_1} - \sum_{\alpha_2\neq\beta_2} a_{\alpha_2}a_{\beta_2} - \cdots - \sum_{\alpha_k\neq\beta_k} a_{\alpha_k}a_{\beta_k}), \\ &\alpha_2, \beta_2 \in \Delta_2, \dots, \alpha_k, \beta_k \in \Delta_k\end{aligned} \tag{3.16}$$

where we put

$$\begin{aligned}&\bar{a}_1 = a_1, \bar{a}_2 = a_2 + \cdots + a_{n_1}, \\ &\bar{a}_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \dots, \bar{a}_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k}, \\ &\bar{a}_{k+2} = a_{n_1+\cdots+n_k+1}, \dots, \bar{a}_{\gamma+1} = a_n.\end{aligned}$$

Applying Lemma 3.1 to (3.15), we can obtain the following inequality

$$\begin{aligned}&\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \\ &\quad \ge \frac{\eta}{2} + \sum_{i<j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2, \\ &\quad \alpha_j, \beta_j \in \Delta_j, \qquad j = 1, \dots, k.\end{aligned} \tag{3.17}$$

Furthermore, from (2.3) and Gauss' equation we see that

$$\begin{aligned}\tau(L_j) &= \frac{n_j(n_j-1)}{2}c + 3c\sum_{l=1}^{3}\alpha_l(L_j) \\ &+ \sum_{r=n+1}^{4m}\sum_{\alpha_j<\beta_j}(h_{\alpha_j\alpha_j}^r h_{\beta_j\beta_j}^r - (h_{\alpha_j\beta_j}^r)^2), \quad \alpha_j, \beta_j \in \Delta_j, \quad j = 1, \cdots, k.\end{aligned} \tag{3.18}$$

Thus, combining (3.16) and (3.17) we get

$$\begin{aligned}\tau(L_1) + \cdots + \tau(L_k) &\ge \frac{\eta}{2} + \sum_{j=1}^{k}\left(\frac{n_j(n_j-1)}{2}c + 3c\sum_{l=1}^{3}\alpha_l(L_j)\right) \\ &+ \frac{1}{2}\sum_{r=n+1}^{4m}\sum_{(\alpha,\beta)\notin\Delta^2}(h_{\alpha\beta}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{4m}\sum_{j=1}^{k}\left(\sum_{\alpha_j\in\Delta_j} h_{\alpha_j\alpha_j}^r\right)^2 \\ &\ge \frac{\eta}{2} + \sum_{j=1}^{k}\left(\frac{n_j(n_j-1)}{2}c + 3c\sum_{l=1}^{3}\alpha_l(L_j)\right),\end{aligned} \tag{3.19}$$

where $\Delta = \Delta_1 \cup \cdots \cup \Delta_k, \quad \Delta^2 = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k)$.
Substituting (3.2) into (3.18), it follows that

$$\begin{aligned} \tau - \sum_{j=1}^{k} \tau(L_j) &\leq c(n_1, \cdots, n_k)||H||^2 + b(n_1, \cdots, n_k)c \\ &+ \frac{3}{2}c\left(\sum_{i=1}^{3}||P_i||^2 + 2\sum_{l=1}^{3}\sum_{j=1}^{k}\alpha_l(L_j)\right). \end{aligned} \tag{3.20}$$

Since $c > 0$, inequality (3.10) thus follows.

If the equality in (3.10) holds at a point p, then the inequalities in (3.16) and (3.18) are actually equalities at p. In this case, by applying Lemma 3.1 and (3.15)-(3.18), we also obtain (3.11). The converse can be verified by a straight-forward computation. □

Corollary 3.6. *Let M be an n-dimensional Riemannian manifold and $p \in M$. If there exists a k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$ and a point $p \in M$ such that*

$$\delta(n_1, \ldots, n_k) > \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1) + 9n\right)c, \tag{3.21}$$

then M admits no minimal submanifold into any $4m$-dimensional quaternionic projective space $\bar{M}(4c)$ ($c > 0$).

Theorem 3.7. *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic hyperbolic space $\tilde{M}(4c)$ ($c < 0$) of constant quaternionic sectional curvature $4c$. Then, we have*

$$\delta(n_1, \cdots, n_k) \leq c(n_1, \cdots, n_k)||H||^2 + b(n_1, \cdots, n_k)c \tag{3.22}$$

for any k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$.

The equality case of inequality (3.20) holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \cdots, e_{4m}$ at p such that the shape operators of M in $\tilde{M}(4c)$ ($c < 0$) at p take the forms (3.11).

Proof. By using (3.19) and $c < 0$, one gets (3.20). □

Corollary 3.8. *Let M be an n-dimensional Riemannian manifold and $p \in M$. If there exists a k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$ and a point $p \in M$ such that*

$$\delta(n_1, \ldots, n_k) > \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right)c, \tag{3.23}$$

then M admits no minimal submanifold into any m-dimensional quaternionic hyperbolic space $\bar{M}(4c)$ ($c < 0$).

Corollary 3.9. *Let M be an n-dimensional totally real submanifold of a $4m$-dimensional quaternionic space form $\tilde{M}(4c)$ of constant quaternionic sectional curvature $4c$. Then, we have*

$$\delta(n_1, \cdots, n_k) \leq c(n_1, \cdots, n_k)||H||^2 + b(n_1, \cdots, n_k)c \tag{3.24}$$

for any k-tuple $(n_1, \cdots, n_k) \in \mathcal{S}(n)$.

The equality case of inequality (3.21) holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, \cdots, e_{4m}$ at p such that the shape operators of M in $\tilde{M}(4c)$ at p take the forms (3.11).

Proof. Let M be an n-dimensional totally real submanifold of a $4m$-dimensional quaternionic space form $\tilde{M}(4c)$. Then we have $||P_i||^2 = 0,\ \alpha_i(L) = 0, i = 1, 2, 3$. Thus, from (3.19) we obtain (3.21). □

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