

# Some properties of strongly $S$ -decomposable operators

Ion Bacalu

## Abstract

In this paper we describe several basic properties of strongly  $S$ -decomposable operators, namely their behaviour regarding: the direct sums, restrictions, quotients, the Riesz-Dunfort functional calculus and the quasinilpotent equivalence.

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## Introduction

Let  $X$  be a Banach space,  $B(X)$  the algebra of all linear bounded operators on  $X$ , and  $\mathbf{C}$  the field of complex numbers. An operator  $T \in B(X)$  is said to have the *single-valued extension property*, if for any analytic function  $f : D_f \rightarrow X$ ,  $D_f \subset \mathbf{C}$  open, with  $(\lambda I - T)f(\lambda) \equiv 0$  implies  $f(\lambda) \equiv 0$ , ([3], [5]).

For an operator  $T \in B(X)$  having the single-valued extension property and for  $x \in X$  we can consider the set  $\rho_T(x)$  of elements  $\lambda_0 \in \mathbf{C}$  such that there exists an analytic function  $\lambda \rightarrow x(\lambda)$  defined in a neighborhood of  $\lambda_0$  with values in  $X$  which verifies  $(\lambda I - T)x(\lambda) \equiv x$ ;  $x(\lambda)$  is unique,  $\rho_T(x)$  is open and  $\rho(T) \subset \rho_T(x)$ . Take  $\sigma_T(x) = \mathbf{C} \setminus \rho_T(x)$  and  $X_T(F) = \{x \in X \mid \sigma_T(x) \subset F\}$  where  $F \subset \mathbf{C}$  is closed.  $\rho_T(x)$  is named the local *resolvent set* of  $x$  with respect to  $T$  and  $\sigma_T(x)$  the spectrum of  $x$  with respect to  $T$ .

If  $T \in B(X)$  and  $Y$  is an invariant (closed) subspace of  $T$ , let us denote by  $T|Y$  the restriction of  $T$  to  $Y$ . In what follows by subspace of  $X$ , we mean a closed linear manifold of  $X$ . Recall that  $Y$  is a *spectral maximal space* of  $T$  if it is an invariant subspace of  $T$  such that for any other invariant subspace  $Z$  of  $T$ , the inclusion  $\sigma(T|Z) \subset \sigma(T|Y)$  implies  $Z \subset Y$ , ([3]).

An open set  $\Omega \subset \mathbf{C}$  is of *analytic uniqueness* of  $T \in B(X)$  if for any open set  $w \subset \Omega$  and any analytic function  $f_0 : w \rightarrow X$  satisfying the equation  $(\lambda I - T)f_0(\lambda) \equiv 0$ , there follows  $f_0(\lambda) \equiv 0$  in  $w$ . For  $T \in B(X)$  there exists a unique *maximal open*  $\Omega_T$  of *analytic uniqueness* ([8], 2.1). We shall denote  $S_T = \mathbf{C} \setminus \Omega_T$  and call this the *analytic residuum* of  $T$ . For  $x \in X$ , a point  $\lambda \in \delta_T(x)$ , if in a neighbourhood  $V_\lambda$  of  $\lambda$  there exists at least an analytic function  $f_x$  (called  $T$ -associated with  $x$ ) such that

$$(\mu I - T)f_x(\mu) = x$$

for  $\mu \in V_\lambda$ . We shall put  $\gamma_T(x) = \mathbf{C}\delta_T(x)$ ,  $\rho_T(x) = \delta_T \cap \Omega_T$ ,  $\sigma_T(x) = \mathbf{C}\rho_T(x) = \gamma_T \cup S_T$  and

$$X_T(F) = \{x; x \in X, \sigma_T(x) \subset F\}$$

where  $S_T \subset F \subset \mathbf{C}$ , ([8]).  $T \in B(X)$  has the single-valued extension property if and only if  $S_T(x) = \delta_T(x)$  a unique analytic function  $x(\lambda)$ ,  $T$ -associated with  $x$ , for any  $x \in X$ . We recall that if  $T \in B(X)$ ,  $S_T \neq \emptyset$  and  $X_T(F)$  is closed for  $F \subset \mathbf{C}$  closed ( $F \supset S_T$ ) then  $X_T(F)$  is a spectral maximal space of  $T$  ([8], Proposition 2.4 and 3.4).

DEFINITION 1. A family of open sets  $\{G_S\} \cup \{G_i\}_i^n$  is an  $S$ -covering of the closed set  $\sigma \subset \mathbf{C}$  if

$$G_S \cup \left( \bigcup_{i=1}^n G_i \right) \supset \sigma \cup S \quad \text{and} \quad \bar{G}_i \cap S = \emptyset$$

( $i = 1, 2, \dots, n$ ), ([8]).

DEFINITION 2. Let  $T \in B(X)$  and  $S \subset \sigma(T)$  be a compact set.  $T$  is called  $S$ -decomposable (see also [2]) if for any finite, open  $S$ -covering  $\{G_S\} \cup \{G_i\}_i^n$  of  $\sigma(T)$ , there exists a system  $\{Y_S\} \cup \{Y_i\}_i^n$  of spectral maximal spaces of  $T$  such that

$$(i) \quad \sigma(T|Y_S) \subset G_S, \quad \sigma(T|Y_i) \subset G_i \quad (1 \leq i \leq n),$$

$$(ii) \quad X = Y_S + \sum_{i=1}^n Y_i.$$

$T$  is *strongly  $S$ -decomposable* operator if (ii) is replaced by

$$(ii') \quad Z = Z \cap Y_S + \sum_{i=1}^n Z \cap Y_i.$$

If  $S = \emptyset$  we have decomposable (strongly) operators where  $Z$  is any spectral maximal space of  $T$ .

LEMMA 3. Let  $T \in B(X)$  be a strongly  $S$ -decomposable operator and  $Y$  a spectral maximal space of  $T$  with  $\sigma(T|Y) \supset S$ . If  $\dot{Z}$  is spectral maximal space of  $\dot{T}$  ( $\dot{T}$  being the operator induced by  $T$  in  $\dot{X} = X/Y$ ), then  $Z = \varphi^{-1}(\dot{Z})$  is a spectral maximal space of  $T$ , where  $\varphi: X \rightarrow \dot{X}$  is the canonical map.

*Proof.* We have  $S_{\dot{T}} = \emptyset$  (see 1.1.9, [2]). If  $Z \supset Y$  and  $Z$  is an invariant to  $T$  linear (closed) subspace of  $X$ ,  $Y$  is also a spectral maximal space of  $T$  (see [2], 1.2]) hence  $S \subset \sigma(T|Y)\sigma(T|Z)$ , that is  $X_T(\sigma(T|Z)) \supset Y$  is a spectral maximal space of  $T$ . By Lemma 2.6.6 of [2], there follows

$$\overline{\sigma T|X_T(\sigma(T|Z))} = \overline{\sigma(T|X_t(\sigma(T|Z))) \setminus \sigma(Y|Y)}.$$

But  $\overline{\sigma T|X_T(\sigma(T|Z))}$  and  $\sigma(T|Y) = \overline{\sigma T|\dot{Z}} \cup \sigma(T|Y)$ , since  $Y$  is a spectral maximal space of  $T|Z$ ; consequently we have

$$\overline{\sigma T|X_T(\sigma(T|Z))} = \overline{((\sigma(T|Z)) \cup \sigma(T|Y)) \setminus \sigma(T|Y)} \subset \overline{(T|Z)}.$$

From the equalities  $\overline{T|X_T(\sigma(T|Z))} = \dot{T}|\overline{T|X_T(\sigma(T|Z))}$ ,  $\overline{(T|Z)} = \dot{T}|\dot{Z}$  we obtain  $\varphi(X_T(\sigma(T|Z))) \subset \varphi(Z)$ , hence  $X_T(\sigma(T|Z)) \subset Z$ ; then  $Z = X_T(\sigma(T|Z))$ , and hence  $Z$  is a spectral maximal space of  $T$ .

**THEOREM 4.** *Let  $T \in B(X)$  be a strongly  $S$ -decomposable operator and  $Y$  a spectral maximal space of  $T$  with  $\sigma(T|Z) \supset S$ . Then  $\dot{T}$  is a strongly  $S_1$ -decomposable operator, where  $S_1 = S \cap \sigma(\dot{T})$ , and  $\dot{T}$  is the operator induced by  $T$  in  $\dot{X} = X/Y$ .*

*Proof.* Let  $\{G_{S_i}\} \cup \{G_i\}_1^n$  be an open  $S_1$ -covering of  $\sigma(\dot{T})$  and  $G_S = G_{S_1} \cup \rho(\dot{T})$ ; we can suppose that  $G_i \cap D = \emptyset$  ( $i = 1, 2, \dots, n$ ). Then  $\{G_{S_i}\} \cup \{G_i\}_1^n$  is a  $S$ -covering of  $\sigma(T)$ . Let  $\{Y_S\} \cup \{Y_i\}_1^n$  be the corresponding system of spectral maximal spaces of  $T$ , such that

$$\sigma(T|Y_S) \subset G_S, \quad \sigma(T|Y_i) \subset G_i, \quad (i = 1, 2, \dots, n)$$

and

$$X = Y_S + \sum_{i=1}^n Y_i.$$

We shall set  $\sigma_S = \sigma(T|Y_S) \cup \sigma(T|Y)$ ,  $\sigma_i = \sigma(T|Y_i) \cup \sigma(T|Y)$ , ( $i = 1, 2, \dots, n$ );  $Z_S = X_T(\sigma_S)$ , ( $i = 1, 2, \dots, n$ ),  $Z_i = X_T(\sigma_i)$ , ( $i = 1, 2, \dots, n$ ) are spectral maximal spaces of  $T$  (we have  $\sigma_S \supset S$ ,  $\sigma_i \supset S$ , see Theorem 2.1.3, [2]) and  $Y \subset Z_S, Y \sigma Y_i$ . Consequently  $\dot{Z}_S, \dot{Z}_i$  are spectral maximal spaces of  $\dot{T}$  ([4], 3.2) and by Lemma 2.6.6 from [2], we obtain

$$\sigma(\dot{T}|\dot{Z}_S) = \overline{\sigma(T|Z_S)} = \overline{\sigma(T|Z_S) \setminus \sigma(T|Y)},$$

and analogously

$$\sigma(\dot{T}|\dot{Z}_i) = \overline{\sigma(T|Z_i)} \subset \sigma(T|Y_i) \subset G_i \quad (i = 1, 2, \dots, n).$$

If  $\dot{Z}$  is an arbitrary spectral maximal space of  $\dot{T}$ , then  $Z = \varphi^{-1}(\dot{Z})$  is a spectral maximal space of  $T$  (where  $\varphi$  is the canonical map; see the preceding lemma). Hence

$$Y_S \cap Z + Y_1 \cap Z + \dots + Y_n \cap Z = Z.$$

But from the inclusions  $\dot{Y}_S \subset \dot{Z}_S, \dot{Y}_i \subset \dot{Z}_i, \varphi(Y_S \cap Z) \subset \dot{Y}_S \cap \dot{Z}, \varphi(Y_i \cap Z) \subset \dot{Y}_i \cap \dot{Z}$ , ( $i = 1, 2, \dots, n$ ) we get

$$\dot{Z} = \varphi(Y_S \cap Z) + \varphi(Y_1 \cap Z) + \dots + \varphi(Y_n \cap Z) \subset \dot{Z}_S \cap \dot{Z} + \dot{Z}_1 \cap \dot{Z} + \dots + \dot{Z}_n \cap \dot{Z} \subset \dot{Z},$$

and hence  $\dot{T}$  is strongly  $S_1$ -decomposable.

**COROLLARY 5.** *Let  $T \in B(X)$  be a strongly  $S$ -decomposable operator and  $Y$  a spectral maximal space of  $T$  such that  $\sigma(T) \cap S = \emptyset$ ; then  $\dot{T}$  is a strongly decomposable operator.*

*Proof.* It follows by the preceding Theorem, since  $S_1 = \emptyset$ .

**PROPOSITION 6.** *Let  $T_\alpha \in B_\alpha(X)$  two strongly  $S_\alpha$ -decomposable operators ( $\alpha = 1, 2$ ); then  $T = T_1 \oplus T_2$  is a strongly  $S$ -decomposable operator, where  $S = S_1 \cup S_2$ .*

*Proof.* By Proposition 2.6.2 and [Theorem 2.2.3, [2]] it follows that it will suffice to show that  $T$  satisfies strongly the condition  $\beta_S$  (see Definition, [2]). Let  $Y$  be a spectral maximal space of  $T$  and  $\mathbf{G} = \{G_{S'}\} \cup \{G_i\}_1^n$  an open  $S'$ -covering of  $\sigma(T|Y)$ , where  $S' = S \cap \sigma(T|Y)$ . Then, in accordance with Proposition 2.1.7 from [2], we have  $Y = Y_1 \oplus Y_2$ , where  $Y_\alpha$  is a spectral maximal space of  $T_\alpha$  ( $\alpha = 1, 2$ ). If  $y \in Y$ , then  $y = y^1 \oplus y^2$ , with  $y^\alpha \in Y_\alpha$ , ( $\alpha = 1, 2$ ); since  $T_\alpha$  ( $\alpha = 1, 2$ ) are strongly  $S$ -decomposable it follows that  $T_\alpha|Y_\alpha$  verifies the condition  $\beta_S$ , where  $S'_\alpha = S_\alpha \cap \sigma(T_\alpha|Y_\alpha)$ , ( $\alpha = 1, 2$ ). Consequently

$$y^\alpha = y_{S'}^\alpha + y_1^\alpha + \dots + y_n^\alpha \quad (\alpha = 1, 2)$$

and

$$\begin{aligned}\gamma_T(y_{S'}^\alpha) &= \gamma_{T_\alpha|Y_\alpha}(y_{S'}^\alpha) \subset G_{S'} \quad (\alpha = 1, 2), \\ \gamma_{T_\alpha}(y_i^\alpha) &= \gamma_{T_\alpha|Y_\alpha}(y_i^\alpha) \subset G_i \quad (\alpha = 1, 2; i = 1, 2, \dots, n).\end{aligned}$$

This yields

$$\begin{aligned}y &= y^1 \oplus y^2 = (y_{S'_1}^1 + y_1^1 + \dots + y_n^1) + (y_{S'_2}^2 + y_1^2 + \dots + y_n^2) = \\ &= (y_{S'_1}^1 \oplus y_{S'_2}^2) + (y_1^1 \oplus y_1^2) + \dots + (y_n^1 \oplus y_n^2) = y_{S'} + y_1 + \dots + y_n\end{aligned}$$

and

$$\begin{aligned}\gamma_T(y_{S'}) &= \gamma_{T|Y}(y_{S'}) = \gamma_{T_1|Y_1}(y_{S'_1}^1) \cup \gamma_{T_2|Y_2}(y_{S'_2}^2) \subset G_{S'}, \\ \gamma_T(y_i) &= \gamma_{T|Y}(y_i) = \gamma_{T_1|Y_1}(y_i^1) \cup \gamma_{T_2|Y_2}(y_i^2) \subset G_i, \quad (1 \leq i \leq n)\end{aligned}$$

hence  $T$  satisfies strongly the condition  $\beta_S$ .

**DEFINITION 7.** A  $S$ -decomposable operator  $T \in B(X)$  is said to be *almost strongly  $S$ -decomposable* if for any spectral maximal space  $Y$  of  $T$  such that  $\sigma(T|Y) \cap S = \emptyset$  or  $\sigma(T|Y) \supset S$ , we have that restriction  $T|Y$  is a decomposable respectively  $S$ -decomposable operator.

**REMARK 8.** The need to state the definition is justified by the following: being given a  $S$ -decomposable (strongly  $S$ -decomposable) operator, we know about the existence of the spectral maximal spaces  $Y$  of  $T$ , that have the property that  $\sigma(T|Y) \cap S = \emptyset$  or  $\sigma(T|Y) \supset S$ ; these are the spaces which result from the relations  $Y \oplus X_T(S) = X_T(\sigma(T|Y) \cup S)$  or  $Y = X_T(\sigma(T|Y))$ . However, we know nothing about the existence of the spectral maximal spaces  $Y$  of  $T$  that have the property that  $\sigma(T|Y) \cap S = S'$  is a separated part of  $S$  (open and closed in  $S$ ). Obviously strongly  $S$ -decomposable operators are almost strongly  $S$ -decomposable. It seems that strong  $S$ -decomposability (unlike the strong decomposability) has no such favourable demeanour as the one of the  $S$ -decomposability (considering the properties from Definition 2.2.1 and Proposition 2.2.17, [2]).

**PROPOSITION 9.** Let  $T = T_1 \oplus T_2 \in B(X_1 \oplus X_2)$  be a strongly  $S$ -decomposable operator; then  $T_\alpha$ ,  $(\alpha = 1, 2)$  are almost strongly  $S_\alpha$ -decomposable, where  $S_\alpha = S \cap \sigma(T_\alpha)$ ,  $(\alpha = 1, 2)$ .

*Proof.* It will suffice to prove that if  $F \subset \sigma(T_1)$  and  $F \cap S_1 = \emptyset$  or  $F \supset S_1$ , then we also have  $F \cap S = \emptyset$  or, respectively,  $(F \cup S) \cap \sigma(T_1) \supset S_1$ . If  $F \cap S_1 = \emptyset$ , we also have

$$F \cap S = (F \cap S) \cap \sigma(T_1) = F \cap (S \cap \sigma(T_1)) = F \cap \sigma(T_1) = \emptyset,$$

hence when  $\sigma(T_1|Y) \cap S_1 = \emptyset$  we also have  $\sigma(T_1|Y) \cap S = \emptyset$  (where  $Y$  is a spectral maximal space of  $T_1$ ). But it also follows that

$$\begin{aligned}X_{T_1 \oplus T_2}(\sigma(T_1|Y_1) \cup S) &= X_{T_1}(\sigma(T_1|Y_1) \cup S) \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S) = \\ &= [Y_1 + X_{T_1}(S)] \oplus [Y_2 + X_{T_2}(S)] = X_{T_1 \oplus T_2}(S) + Y\end{aligned}$$

and we can easily verify that  $Y = Y_1 \oplus Y_2$ .  $T \in T|Y_1 \oplus Y_2$  being decomposable, by Proposition 2.2.6 from [2], there follows that  $T_1|Y_1$  is decomposable. Let now  $Y_1$  be a maximal space of  $T_1$  such that  $\sigma(T_1|Y_1) \supset S_1$ . Then we have

$$X_{T_1 \oplus T_2}(\sigma(T_1|Y_1) \cup S) = X_{T_1}(\sigma(T_1|Y_1) \cup S) \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S) =$$

$$= X_{T_1}([\sigma(T_1|Y_1) \cup S]) \cap \sigma(T_1) \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S) = Y_1 \oplus X_{T_2}(\sigma(T_1|Y_1) \cup S)$$

whence it results  $T_1|Y_1$  is  $S_1$ -decomposable. Analogously, we verify that  $T_2$  is almost strongly  $S_2$ -decomposable.

**THEOREM 10.** *Let  $T = T_1 \oplus T_2 \in B(X_1 \oplus X_2)$  be a strongly decomposable operator. Then  $T_1$  and  $T_2$  are strongly decomposable.*

*Proof.* This follows by Propositions 4 and 7.

**PROPOSITION 11.** *Let  $T \in B(X)$  be a strong  $S$ -decomposable operator and  $P \in B(X)$  a projection commuting with  $T$ . Then  $T|PX$  is almost strong  $S$ -decomposable, where  $S_1 = \sigma(T|PX) \cap S$ .*

*Proof.* We have  $X = X_1 \oplus X_2$ ,  $T = T_1 \oplus T_2$ , where  $X_1 = PX$ ,  $X_2 = (I - P)X$ ,  $T_1 = T|X_1$ ,  $T_2 = T|X_2$  and by Proposition 7 we have that  $T|PX$  is almost strong  $S_1$ -decomposable.

**COROLLARY 12.** *Let  $T \in B(X)$  be a strong decomposable operator and  $P \in B(X)$  a projection. Then  $T|PX$  is strong decomposable.*

*Proof.* This follows from the preceding proposition.

**PROPOSITION 13.** *Let  $T \in B(X)$  be a strong  $S$ -decomposable operator and let  $\sigma$  be a separated part of  $\sigma(T)$ . Then  $T|E(\sigma, T)X$  is strong  $S_1$ -decomposable, where  $S_1 = S \cap \sigma$  (for  $E(\sigma, T)$  see Corollary 2.2.8 from [2]).*

*Proof.*  $X_1 = E(\sigma, T)X$  is a spectral maximal space of  $T$ . Let  $Y_1$  be a spectral maximal space of  $T|X_1$ . Then by [2, Proposition 1.2] this is also a spectral maximal space of  $T$ , hence  $T|_1$  is  $S'_1$ -decomposable, where  $S'_1 = \sigma(T|Y_1) \cap S$ . But  $\sigma(T|Y_1) \cap S_1 = \sigma(T|Y_1) \cap (\sigma \cap S) = (\sigma(T|Y_1) \cap \sigma) \cap S = S'_1$ , hence  $(T|X_1)|Y_1$  is  $S'_1$ -decomposable, that is  $T|E(\sigma, T)X$  is strong  $S_1$ -decomposable.

**PROPOSITION 14.** *Let  $T \in B(X)$  be a strong  $S$ -decomposable operator and let  $f : G \rightarrow \mathbf{C}$  ( $G \supset \sigma(T)$  open and connected) be an analytic function, injective on  $\sigma(T)$ . Then  $f(T)$  is almost strong  $S_1$ -decomposable.*

*Proof.* From the equalities  $X_{f(T)}(F) = X_T(f^{-1}(F))$  (where  $F \supset S_1 = f(S)$ ) and

$$X_{f(T)}(F \cup S_1) = X_T(f^{-1}(F) \cup S) = Y_F \oplus X_T(S) = Y_F \oplus X_{f(T)}(S_1)$$

(where  $F \cap S_1 = \emptyset$ ) and by Proposition 2.2.9 from [2], it follows that the spectral maximal spaces  $Y$  of  $f(T)$  that have the property  $\sigma(f(T)|Y) \supset S_1$  or  $\sigma(f(T)|Y) \cap S_1 = \emptyset$  are also spectral maximal spaces of  $T$ . One further performs the proof as for Proposition 2.2.9 from [2], since a  $S_1$ -covering of  $\sigma(f(T))$  is easily transformed through  $f^{-1}$  into a  $S$ -covering of  $\sigma(T)$ .

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Ion Bacalu  
University Politehnica of Bucharest,  
Department of Mathematics II,  
Splaiul Independenţei 313,  
RO-060042 Bucharest, Romania