

# Complex timelike isothermic surfaces and their geometric transformations

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*Dedicated to the memory of Radu Rosca (1908-2005)*

**Abstract.** We construct an explicit action of a rational map with two simple poles on the space of solutions of the  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system which is associated with time-like isothermic surfaces in  $\mathbb{R}^{n-j, j}$  whose second fundamental forms are diagonalizable over  $\mathbb{C}$ . We show that these actions correspond to the Ribaucour and Darboux transformations for these surfaces.

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## 1 Introduction

It is well known that there exists a connection between submanifold theory and integrable systems. Perhaps the most famous example of this is the relation between the pseudo-spherical surface, their Bäcklund transformations and the integrable Sin-Gordon equation.

Several papers study this connection for particular submanifolds which have an integrable system description. Recently, a new integrable system was defined, the  $U/K$ -system, and a systematic study of the submanifold geometry associated to it was begun. The concept of  $U/K$ -system, for  $U/K$  a symmetric space, was introduced by Terng in [12] and comes from putting the  $n$  first flows of ZS-AKNS together. Terng started the project of finding submanifolds in certain symmetric space whose Gauss-Codazzi-Ricci equations are equivalent to the  $U/K$ -system and indentifying the geometric transformations corresponding to the dressing action of certain simple element in the loop group formalism.

This project has been carried out for some symmetric spaces. For instance, isothermic surfaces in  $\mathbb{R}^m$ , submanifolds with constant sectional curvatures and submanifolds admitting special principal curvature coordinates are associated to  $O(m+n)/O(m) \times O(n)$  and  $O(m+n, 1)/O(m) \times O(n, 1)$ -systems (see [3]). Similarly, the flat timelike submanifolds in  $S_{2q}^{2n-1}(1)$  are associated to  $O(2n-2q, 2q)/O(n-q, q) \times O(n-q, q)$ -system (see [14]). Moreover in these cases, the dressing actions of simple elements on the space of solutions of those  $U/K$ -systems correspond to Ribaucour, Darboux and Backlund transformations of these submanifolds.

Isothermic surfaces in  $\mathbb{R}^3$  and their geometric transforms were the objects of intensive research at the end of the nineteenth century. Recently there has been a resurgence of interest in isothermic surfaces because of their integrable system description, initiated by Cieslinski-Goldstein-Sym in [4]. In this work, Cieslinski-Goldstein-Sym write down a zero-curvature formulation of the Gauss-Codazzi equations of an isothermic surface in  $\mathbb{R}^3$ , and so the methods of integrable systems theory can be applied to them and their geometric transformations. The work begun in [4] is taken up in [1], [2], [10] which emphasize the relation between isothermic surfaces and the theory of curved flats. In addition [3] establishes that isothermic surfaces in  $\mathbb{R}^3$  are also associated to the Grassmanian system  $O(4, 1)/O(3) \times O(1, 1)$ -system and that in the loop group formalism, the dressing action of certain simple elements on the space of solutions of the  $O(4, 1)/O(3) \times O(1, 1)$ -system, corresponds to Darboux transformations of isothermic surfaces. On other hand, from another point of view, the non-linear system associated to isothermic surfaces is interesting in its own right, because it displays some unconventional soliton features and, physically, could be applied in the theory of infinitesimal deformations of membranes (see [4]).

Changing the ambient space to the pseudo-riemannian space  $\mathbb{R}^{n-j, j}$  for any signature  $j$ , recent works of the authors treat spacelike isothermic surfaces and timelike isothermic surfaces in  $\mathbb{R}^{n-j, j}$  ([7], [8]). In these works, it is established that these surfaces are also associated with integrable systems, just as is the case with isothermic surfaces in  $\mathbb{R}^m$ . In particular, in [7], [8], it is shown that the spacelike isothermic surfaces and timelike isothermic surfaces are associated to  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system, depending on a choice one of three different maximal abelian subalgebras. These correspond to the three distinct geometries: spacelike isothermic surfaces, timelike isothermic surface whose second fundamental forms are all diagonalizable over  $\mathbb{R}$  (referred to as real timelike isothermic surface), and the last one, corresponding to the timelike isothermic surfaces all of whose second fundamental forms are diagonalizable over  $\mathbb{C}$  (called complex timelike isothermic surface).

The aim of this paper is the study of Ribaucour and Darboux transformations for complex timelike isothermic surfaces in  $\mathbb{R}^{n-j, j}$  with the loop group formalism, using the  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system. We show in this note that the dressing action of certain simple elements, correspond to Ribaucour and Darboux transformations, correctly defined, for complex timelike isothermic surfaces in  $\mathbb{R}^{n-j, j}$ . The paper is organized as follows: First we construct a dressing action on the space of solutions of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system. Next, we establish the definition of Ribaucour and Darboux transforms for complex timelike isothermic surfaces in  $\mathbb{R}^{n-j, j}$ . Finally, we show that the dressing action gives rise to Ribaucour and Darboux transforms for complex timelike isothermic surfaces. The result has been proved for timelike real isothermic surfaces in ([15]).

## 2 Preliminaries

We define the (indefinite) inner product in  $\mathbb{R}^{m-k, k}$  to be

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + \dots + u_{m-k} v_{m-k} - u_{m-k+1} v_{m-k+1} - \dots - u_m v_m$$

for  $\vec{u} = (u_1, \dots, u_m)$  and  $\vec{v} = (v_1, \dots, v_m)$ . A surface  $M^{1,1}$  immersed  $\mathbb{R}^{n-j,j}$  is called time-like if each tangent plane inherits an inner product from  $\mathbb{R}^{n-j,j}$  which is equivalent to the standard inner product on  $\mathbb{R}^{1,1}$ .

Timelike surfaces have symmetric shape operators which can be put into one of three canonical forms on a fixed tangent space with respect to an orthonormal basis:

$$a) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad b) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad c) \begin{pmatrix} \nu \pm a/2 & -a/2 \\ a/2 & \nu \mp a/2 \end{pmatrix}.$$

We allow with  $b = 0$  in the second case and  $a = 0$  in the third, so that umbilic points fall into all three categories. In the first case the shape operator is diagonalized over  $\mathbb{R}$ , in the second over  $\mathbb{C}$  with conjugate eigenvalues  $a \pm ib$ . In this paper we are working locally and assume that, in a neighborhood of a point, each shape operator falls into the second case.

There are many equivalent definitions of isothermic for positive definite surfaces. One is that there is an isothermal coordinate system which diagonalizes all the shape operators. The corresponding definition of isothermic for timelike surfaces must take into account the algebraic type of the shape operator. In the case where all shape operators are diagonalizable over  $\mathbb{C}$ , we need the appropriate definition of a complex isothermic surface, i.e., one that has an isothermal coordinate system with respect to which all the shape operators are diagonalized over  $\mathbb{C}$ , ([8]).

**Definition 1. (Complex isothermic surface)** Let  $\mathcal{O}$  be a domain in  $\mathbb{R}^{1,1}$ . An immersion  $X : \mathcal{O} \rightarrow \mathbb{R}^{n-j,j}$  is called a complex timelike isothermic surface if it has flat normal bundle and the two fundamental forms are:

$$I = \pm e^{2v}(-dx_1^2 + dx_2^2), \quad II = \sum_{i=2}^{n-1} e^v(g_{i1}(dx_2^2 - dx_1^2) - 2g_{i2}dx_1dx_2)e_i,$$

with respect to some parallel normal frame  $\{e_i\}$ .

It is known that given any complex isothermic surface there is a dual isothermic surface with parallel normal space([11]), hence we establish the following definition:

**Definition 2. (Complex isothermic timelike dual pair in  $\mathbb{R}^{n-j,j}$  of type  $O(1,1)$ ).** Let  $\mathcal{O}$  be a domain in  $\mathbb{R}^{1,1}$  and  $X_i : \mathcal{O} \rightarrow \mathbb{R}^{n-j,j}$  an immersion with flat and non-degenerate normal bundle for  $i = 1, 2$ .  $(X_1, X_2)$  is called a complex isothermic time-like dual pair in  $\mathbb{R}^{n-j,j}$  of type  $O(1,1)$  if:

- (i) The normal plane of  $X_1(x)$  is parallel to the normal plane of  $X_2(x)$  and  $x \in \mathcal{O}$ ,
- (ii) there exists a common parallel normal frame  $\{e_2, \dots, e_{n-1}\}$ , where  $\{e_i\}_2^{n-j}$  and  $\{e_i\}_{n-j+1}^{n-1}$  are space-like and time-like vectors resp.
- (iii)  $x \in \mathcal{O}$  is a isothermal coordinate system with respect to  $\{e_2, \dots, e_{n-1}\}$ , for each  $X_k$ , such that the fundamental forms of  $X_k$  are diagonalizable over  $\mathbb{C}$ . Namely,

$$(2.1) \quad \begin{cases} I_1 = b^{-2}(dx_1^2 - dx_2^2), \\ II_1 = -b^{-1} \sum_{i=1}^{n-2} [g_{i,2}(dx_2^2 - dx_1^2) + 2g_{i,1}dx_1dx_2]e_{i+1}, \\ I_2 = b^2(-dx_1^2 + dx_2^2), \\ II_2 = b \sum_{i=1}^{n-2} [g_{i,1}(dx_2^2 - dx_1^2) - 2g_{i,2}dx_1dx_2]e_{i+1}, \end{cases}$$

where  $B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$  is in  $O(1,1)$  and a  $\mathcal{M}_{(n-2) \times 2}$ -valued map  $G = (g_{ij})$ .

A result established in [8] and which we will use henceforth, shows that there is a correspondence between the solutions  $(F, G, B)$  of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system II (see below) and a dual pair of complex isothermic timelike surfaces in  $\mathbb{R}^{n-j, j}$  of type  $O(1, 1)$ . This makes clear that they should be considered essentially as a single unit. An example which shows this correspondence is the Lorentzian helicoid and the Lorentzian sphere in  $\mathbb{R}^{2, 1}$ , which by the results in [8], is a solution of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system II.

We finish this section establishing the complex systems:

The PDE for the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system for

$$\xi = \begin{pmatrix} \xi_1 & \xi_2 \\ r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \\ \xi_2 & -\xi_1 \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathcal{M}_{n \times 2},$$

is given by

$$(2.2) \quad \begin{cases} -r_{i,2x_2} - r_{i,1x_1} & = 2(r_{i,2}\xi_1 - r_{i,1}\xi_2), \quad i = 1, \dots, n-2 \\ -r_{i,1x_2} + r_{i,2x_1} & = -2(r_{i,1}\xi_1 + r_{i,2}\xi_2), \quad i = 1, \dots, n-2 \\ (-2\xi_1)_{x_2} + (2\xi_2)_{x_1} & = \sum_{i=1}^{n-2} \sigma_i (r_{i,1}^2 + r_{i,2}^2) \\ (2\xi_2)_{x_2} - (2\xi_1)_{x_1} & = 0, \end{cases}$$

where  $\sigma_i = 1$  for  $i = 1, \dots, n-j-1$  and  $\sigma_i = -1$  for  $i = n-j, \dots, n-2$ . This complex system comes from calculating  $d\theta_\lambda = -\theta_\lambda \wedge \theta_\lambda$ , where  $\theta_\lambda$  is the Lax connection obtained using the maximal subalgebra  $\mathcal{A} = \text{span}\{a_1, a_2\}$  with

$$a_1 = e_{1,n+1} + e_{n,n+2} + e_{n+1,n} - e_{n+2,1}$$

$$a_2 = -e_{1,n+2} + e_{n,n+1} + e_{n+1,1} + e_{n+2,n}.$$

Here  $e_{i,j}$  is the matrix with 1 in the  $ij^{\text{th}}$  place. More explicitly  $\theta_\lambda = \begin{pmatrix} \omega & M \\ N & P \end{pmatrix}$ , where  $\omega \in \mathcal{M}_{n \times n}$ ,  $M \in \mathcal{M}_{n \times 2}$ ,  $N \in \mathcal{M}_{2 \times n}$  and  $P \in \mathcal{M}_{2 \times 2}$ . They being resp.

$$(2.3) \quad \omega = \begin{pmatrix} 0 & \vec{a} & \vec{b} & c \\ -\vec{a}^t & 0 & 0 & \vec{d}^t \\ \vec{b}^t & 0 & 0 & \vec{e}^t \\ c & \vec{d} & -\vec{e} & 0 \end{pmatrix}, \quad M = \lambda \begin{pmatrix} dx_1 & -dx_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ dx_2 & dx_1 \end{pmatrix}$$

$$(2.4) \quad N = \lambda \begin{pmatrix} dx_2 & 0 & \dots & 0 & dx_1 \\ -dx_1 & 0 & \dots & 0 & dx_2 \end{pmatrix}, \quad P = \begin{pmatrix} 2(\xi_2 dx_1 + \xi_1 dx_2) & 0 \\ 0 & -2(\xi_1 dx_2 + \xi_2 dx_1) \end{pmatrix}$$

with

$$\begin{aligned}
\vec{a} &= (a_1, \dots, a_{n-j-1}) \text{ and } a_k = r_{k,1}dx_2 - r_{k,2}dx_1, \text{ for } 1 \leq k \leq n-j-1, \\
\vec{b} &= (b_{n-j}, \dots, b_{n-2}) \text{ and } b_q = -r_{q,1}dx_2 + r_{q,2}dx_1, \text{ for } n-j \leq q \leq n-2, \\
c &= -2\xi_1dx_1 - 2\xi_2dx_2 \\
\vec{d} &= (d_1, \dots, d_{n-j-1}) \text{ and } d_k = -r_{k,1}dx_1 - r_{k,2}dx_2, \text{ for } 1 \leq k \leq n-j-1, \\
\vec{e} &= (e_q, \dots, e_{n-2}) \text{ and } e_q = -r_{q,1}dx_1 - r_{q,2}dx_2, \text{ for } n-j \leq q \leq n-2.
\end{aligned}$$

Finally, the PDE for a solution to the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system II  $(u, r_{1,1}, r_{1,2}, \dots, r_{n-2,1}, r_{n-2,2})$  becomes:

$$(2.5) \quad \begin{cases} -r_{i,2}x_2 - r_{i,1}x_1 &= 2(r_{i,2}u_{x_2} - r_{i,1}u_{x_1}), \\ -r_{i,1}x_2 + r_{i,2}x_1 &= -2(r_{i,1}u_{x_2} + r_{i,2}u_{x_1}), \\ -2u_{x_2x_2} + 2u_{x_1x_1} &= \sum_{i=1}^{n-2} \sigma_i (r_{i,1}^2 + r_{i,2}^2), \end{cases}$$

where  $\sigma_i = 1$  for  $i = 1, \dots, n-j-1$ ,  $\sigma_i = -1$  for  $i = n-j, \dots, n-2$  and  $B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}$ . Specifically, complex system II (2.5), comes from setting  $d\theta_\lambda^{II} = -\theta_\lambda^{II} \wedge \theta_\lambda^{II}$ , where  $\theta_\lambda^{II}$  is the connection associated to system (2.5).

### 3 The explicit action

In this section we establish the explicit action of a certain simple element on the space of local solutions of the complex systems.

Let  $O(n-j+1, j+1) \otimes \mathbb{C} = O(n-j+1, j+1; \mathbb{C}) = U_{\mathbb{C}}$  be defined by

$$O(n-j+1, j+1; \mathbb{C}) = \{X \in GL(n+2, \mathbb{C}) \mid X^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} X = \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}\},$$

where  $J' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . From the two involutions  $\tau, \sigma$  which determined the symmetric space  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ , the  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -reality condition is:

$$(3.1) \quad \begin{cases} \overline{g(\bar{\lambda})} = g(\lambda) \\ I_{n,2} g(-\lambda) I_{n,2} = g(\lambda) \\ g(\lambda)^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} g(\lambda) = \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}, \end{cases}$$

for a map  $g : \mathbb{C} \rightarrow U_{\mathbb{C}}$ .

We recall that a frame for a solution  $\xi$  of the  $U/K$ -system (II) is a trivialization of the corresponding Lax connection  $\theta_\lambda$  ( $\theta_\lambda^{II}$ ) that satisfies the  $U/K$ -reality condition.

Let

$$G_+ = \{g : \mathbb{C} \rightarrow U_{\mathbb{C}} \mid g \text{ is holomorphic and satisfies (3.1)}\}$$

$$G_- = \{g : S^2 \rightarrow U_{\mathbb{C}} \mid g \text{ is meromorphic, } g(\infty) = I \text{ and satisfies (3.1)}\}.$$

Next we find certain simple elements in  $G_-$  explicitly. Let  $W = (w_1, \dots, w_n)^t \in \mathbb{R}^{n-j,j}$ ,  $Z = (z_1, z_2)^t \in \mathbb{R}^{1,1}$  be unit vectors. In  $\mathbb{R}^{1,1}$  we use the inner product  $(u_1, u_2) \cdot (v_1, v_2) = u_1 v_2 + u_2 v_1$  which is equivalent to the standard one. Let  $\mathbb{C}^{n+2}$  be equipped with the bi-linear form:

$$\langle U, V \rangle_2 = \sum_{i=1}^{n-j} \bar{u}_i v_i - \sum_{i=n-j+1}^n \bar{u}_i v_i + \bar{u}_{n+1} v_{n+2} + \bar{u}_{n+2} v_{n+1}.$$

Let  $\pi$  the orthogonal projection of  $\mathbb{C}^{n+2}$  onto the span of  $\begin{pmatrix} W \\ iZ \end{pmatrix}$  with respect to  $\langle \cdot, \cdot \rangle_2$ , i.e.,

$$(3.2) \quad \pi = \frac{1}{2} \begin{pmatrix} WW^t & -iWZ^t \\ iZZ^t & ZZ^t \end{pmatrix} \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}.$$

Then  $\bar{\pi}$  is the projection onto  $\mathbb{C} \begin{pmatrix} -W \\ iZ \end{pmatrix}$ , which is perpendicular to  $\begin{pmatrix} W \\ iZ \end{pmatrix}$ . So  $\bar{\pi}\pi = \pi\bar{\pi} = 0$ . Let  $s \in \mathbb{R}$ ,  $s \neq 0$ , and define  $g_{s,\pi}(\lambda) = (\pi + \frac{\lambda-is}{\lambda+is}(I - \pi))(\bar{\pi} + \frac{\lambda+is}{\lambda-is}(I - \bar{\pi}))$ . Substituting (3.2) in  $g_{s,\pi}$ , we get

$$(3.3) \quad g_{s,\pi}(\lambda) = \frac{1}{\lambda^2 + s^2} [\lambda^2 I + s^2 \begin{pmatrix} I - 2WW^t I_{n-j,j} & 0 \\ 0 & I - 2ZZ^t J' \end{pmatrix} + 2s\lambda \begin{pmatrix} 0 & WZ^t J' \\ -ZZ^t I_{n-j,j} & 0 \end{pmatrix}].$$

One can see that  $g_{s,\pi}(\lambda)$  (3.3) satisfies the reality condition (3.1), so that  $g_{s,\pi} \in G_-$ .

Following the ideas from [3], we can get an explicit construction of the action of  $g_{s,\pi}$  on the space of solutions of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system.

**Theorem 1.** *Let  $\xi : \mathbb{R}^2 \rightarrow \mathcal{M}_{n \times 2}$  be a solution of the  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system (2.2), and  $E(x, \lambda)$  a frame of  $\xi$  such that  $E(x, \lambda)$  is holomorphic for  $\lambda \in \mathbb{C}$ . Let  $W$  and  $Z$  be unit vectors in  $\mathbb{R}^{n-j,j}$ ,  $\mathbb{R}^{1,1}$  respectively,  $\pi$  the orthogonal projection onto  $\mathbb{C} \begin{pmatrix} W \\ iZ \end{pmatrix}$  with respect to  $\langle \cdot, \cdot \rangle_2$  and  $g_{s,\pi}$  the map defined by (3.3). Let*

$\tilde{\pi}(x)$  *denotes the orthogonal projection onto  $\mathbb{C} \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix}(x)$  with respect to  $\langle \cdot, \cdot \rangle_2$ , where*

$$(3.4) \quad \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix}(x) = E(x, -is)^{-1} \begin{pmatrix} W \\ iZ \end{pmatrix}.$$

Let  $\widehat{W} = \frac{\widetilde{W}}{\|\widetilde{W}\|_{n-j,j}}$  and  $\widehat{Z} = \frac{\widetilde{Z}}{\|\widetilde{Z}\|_{1,1}}$ ,  $\widetilde{E}(x, \lambda) = g_{s,\pi}(\lambda)E(x, \lambda)g_{s,\tilde{\pi}(x)}(\lambda)^{-1}$ ,

$$(3.5) \quad \widetilde{\xi} = \xi - 2s(\widehat{W}\widehat{Z}^t J')_*,$$

where  $(\vartheta_*)$  is the projection onto the span of  $\{a_1, a_2\}^\perp$ . Then

(1)  $\tilde{\xi}$  is a solution of system (2.2),  $\tilde{E}$  is a frame for  $\tilde{\xi}$  and  $\tilde{E}(x, \lambda)$  is holomorphic in  $\lambda \in \mathbb{C}$ .

(2)  $(\tilde{W}(x), \tilde{Z}(x))$  is a solution of the system:

$$(3.6) \quad \begin{cases} (\tilde{w}_1)_{x_1} = r_{1,2}\tilde{w}_2 - \dots - r_{n-2,2}\tilde{w}_{n-1} + 2\xi_1\tilde{w}_n - s\tilde{z}_1 \\ (\tilde{w}_1)_{x_2} = -r_{1,1}\tilde{w}_2 - \dots + r_{n-2,1}\tilde{w}_{n-1} + 2\xi_2\tilde{w}_n + s\tilde{z}_2 \\ (\tilde{w}_n)_{x_1} = r_{1,1}\tilde{w}_2 - \dots - r_{n-2,1}\tilde{w}_{n-1} + 2\xi_1\tilde{w}_1 - s\tilde{z}_2 \\ (\tilde{w}_n)_{x_2} = r_{1,2}\tilde{w}_2 - \dots - r_{n-2,2}\tilde{w}_{n-1} + 2\xi_2\tilde{w}_1 - s\tilde{z}_1 \\ (\tilde{w}_j)_{x_i} = -\epsilon_i r_{j-1,\bar{i}}\tilde{w}_1 + r_{j-1,i}\tilde{w}_n \\ (\tilde{z}_i)_{x_i} = s\tilde{w}_n - 2\epsilon_i\xi_i\tilde{z}_i \\ (\tilde{z}_i)_{x_k} = \epsilon_i(s\tilde{w}_1 - 2\xi_i\tilde{z}_i), \end{cases}$$

for  $j = 2, \dots, n-1$ ,  $i = 1, 2$  where  $\epsilon_i = 1$  if  $i = 1$ ,  $\epsilon_i = -1$  if  $i = 2$ , and  $\bar{1} = 2$ ,  $\bar{2} = 1$ . In the last equation  $i \neq k$ .

We note that for

$$u^t = \begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n-11} & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n-12} & u_{n2} \end{pmatrix}$$

$$u_*^t = \begin{pmatrix} \frac{u_{11}-u_{n2}}{2} & u_{21} & \cdots & u_{n-11} & \frac{u_{n1}+u_{12}}{2} \\ \frac{u_{n1}+u_{12}}{2} & u_{22} & \cdots & u_{n-12} & -\frac{u_{11}-u_{n2}}{2} \end{pmatrix}.$$

The proof of Theorem 1 uses the following lemma.

**Lemma 1.** *With the same conditions as in Theorem 1 above, we get*

(i)  $\tilde{W}(x) \in \mathbb{R}^{n-j,j}$ ,  $\tilde{Z}(x) \in \mathbb{R}^{1,1}$ .

(ii)  $\|\tilde{W}(x)\|_{n-j,j} = \|\tilde{Z}(x)\|_{1,1} \quad \forall x$  and  $g_{s,\bar{\pi}}$  satisfies the reality conditions (3.1), i.e.  $g_{s,\bar{\pi}} \in G_-$ .

(iii)  $\tilde{E}(x, \lambda)$  is holomorphic in  $\lambda \in \mathbb{C}$ .

The statement and proof are similar to the Lemma (9.4) of [3] and we omit it, with the following comment. We need to know that we can choose  $\tilde{W}$  and  $\tilde{Z}$  which are not null vectors. Since  $\|\tilde{W}(x)\|_{n-j,j} = \|\tilde{Z}(x)\|_{1,1}$ , we want  $\tilde{z}_1\tilde{z}_2 \neq 0$ . We only need to do this locally. Fix  $x_o$  and look at  $E(x_o, -is_o)^{-1}$ . This is a matrix whose last two rows we denote by  $\vec{r}_{n+1}$  and  $\vec{r}_{n+2}$ . We must pick a real vector

$$\vec{s} = (\vec{W}, i\vec{Z}) \in (\vec{r}_{n+1}^\perp)^c \cap (\vec{r}_{n+2}^\perp)^c \cap (S^{n-j-1,j+1} \times S^{0,1}),$$

where  $S^{p-1,k} \subset \mathbb{R}^{p,k}$  is the set of unit vectors. We see that this is a non-empty intersection, since the complement of two hyperplanes must intersect the product.

In the course of the proof we find the following expression for  $\tilde{E}(x, \lambda)$ :

$$\tilde{E}(x, \lambda) = \left( \pi + \frac{\lambda - is}{\lambda + is}(I - \pi) \right) \left( \bar{\pi} + \frac{\lambda + is}{\lambda - is}(I - \bar{\pi}) \right) E(x, \lambda) \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}$$

$$\left( \bar{\pi}^t + \frac{\lambda - is}{\lambda + is}(I - \bar{\pi}^t) \right) \left( \bar{\pi}^t + \frac{\lambda + is}{\lambda - is}(I - \bar{\pi}^t) \right) \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}.$$

Using this expression one can show the holomorphicity of  $\tilde{E}(x, \lambda)$  for  $\lambda \in \mathbb{C}$ .

*Proof of Theorem 1:*

We find that

$$(3.7) \quad \tilde{E}^{-1}d\tilde{E} = g_{s,\tilde{\pi}}E^{-1}dEg_{s,\tilde{\pi}}^{-1} - dg_{s,\tilde{\pi}}g_{s,\tilde{\pi}}^{-1}.$$

But  $\theta_\lambda = E^{-1}dE = \sum_{i=1}^2(a_i\lambda + [a_i, v])dx_i$ ,  $\tilde{E}(x, \lambda)$  is holomorphic in  $\lambda \in \mathbb{C}$  and  $g_{s,\tilde{\pi}}(\lambda)$  is holomorphic at  $\lambda = \infty$ . So  $\tilde{E}^{-1}d\tilde{E}$  must be of the form:  $\sum_{i=1}^2(a_i\lambda + \mu_i)dx_i$ . Now we write

$$g_{s,\tilde{\pi}}(\lambda) = I + \frac{m_1(x)}{\lambda} + \frac{m_2(x)}{\lambda^2} + \dots$$

so that  $m_1(x) = 2s \begin{pmatrix} 0 & \widehat{W}\widehat{Z}^t J' \\ -\widehat{Z}\widehat{W}^t I_{n-j,j} & 0 \end{pmatrix}$  and  $m_1(x) \in \mathcal{P}$ . Now multiplying (3.7) by  $g_{s,\tilde{\pi}}$  on the right side and equating the constant terms, we have

$$\mu_i = [a_i, v - m_1] = [a_i, v - p_o(m_1)],$$

where  $p_o$  is the projection from  $\mathcal{P}$  onto  $\mathcal{P} \cap \mathcal{A}^\perp$ . Therefore  $\tilde{v} = v - p_o(m_1)$  is a solution of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system (2.2). More specifically,

if we write  $v = \begin{pmatrix} 0 & \xi \\ -J'\xi^t I_{n-j,j} & 0 \end{pmatrix}$  and  $\tilde{v} = \begin{pmatrix} 0 & \tilde{\xi} \\ -J'\tilde{\xi}^t I_{n-j,j} & 0 \end{pmatrix}$ ,  $\tilde{\xi} = \xi - 2s(\widehat{W}\widehat{Z}^t J')_*$

is a new solution of  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system (2.2).

(2) follows from taking the differential of (3.4), that is

$$d \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix} (x) = -\theta_{-is} \begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix} (x). \quad \blacksquare$$

Now let

$$(3.8) \quad \tilde{E}^\sharp(x, \lambda) = E(x, \lambda)g_{s,\tilde{\pi}}^{-1},$$

i.e.,

$$(3.9) \quad \begin{aligned} \tilde{E}^\sharp(x, \lambda) = E(x, \lambda) & \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} \frac{1}{\lambda^2 + s^2} [\lambda^2 I + s^2 \begin{pmatrix} I - 2I_{n-j,j}\widehat{W}\widehat{W}^t & 0 \\ 0 & I - 2J'\widehat{Z}\widehat{Z}^t \end{pmatrix} + \\ & + 2s\lambda \begin{pmatrix} 0 & -I_{n-j,j}\widehat{W}\widehat{Z}^t \\ J'\widehat{Z}\widehat{W}^t & 0 \end{pmatrix}] \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}. \end{aligned}$$

A direct computation gives  $\tilde{E}^\sharp$  is a frame for  $\tilde{\xi}$  and  $\tilde{E}^\sharp(x, \cdot)$  is not in  $G_+$ . The reality condition (3.1) implies that both  $E(x, 0)$  and  $\tilde{E}^\sharp(x, 0)$  are in  $O(n-j, j) \times O(1, 1)$ , so we write

$$E(x, 0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}, \quad \tilde{E}^\sharp(x, 0) = \begin{pmatrix} \tilde{A}^\sharp(x) & 0 \\ 0 & \tilde{B}^\sharp(x) \end{pmatrix},$$

for some  $A, B, \tilde{A}^\sharp(x), \tilde{B}^\sharp(x)$ . Then taking  $\lambda = 0$  in (3.8), one gets

$$(3.10) \quad \tilde{A}^\sharp = A(I - 2\widehat{W}\widehat{W}^t I_{n-j,j}), \quad \tilde{B}^\sharp = B(I - 2\widehat{Z}\widehat{Z}^t J').$$

**Corollary 1.** *Suppose  $\tilde{E}$  is a frame of the solution  $\xi$  of the system (2.2) such that  $E(x, \lambda)$  is holomorphic for  $\lambda \in \mathbb{C}$ .*

(i) *If  $E(0, \lambda) = I$ , then  $\tilde{\xi}$  obtained in Theorem 1, is the dressing action  $g_{s, \pi} \# \xi$ , and  $\tilde{E}$  is the frame of  $\tilde{\xi}$  with  $\tilde{E}(0, \lambda) = I$ .*

(ii) *Let  $g_+(\lambda) = E(0, \lambda)$  and  $\tilde{\xi}$  the new solution of (2.2) obtained in Theorem 1. Then  $g_+ \in G_+$  and  $\tilde{\xi} = \tilde{g}_- \# \xi$ , where  $\tilde{g}_-$  is obtained by factoring  $g_{s, \pi} g_+ = \tilde{g}_+ \tilde{g}_-$  with  $\tilde{g}_\pm \in G_\pm$ .*

Denoting the entries of  $\xi$  by:  $F = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix}$  and  $G = \begin{pmatrix} r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \end{pmatrix}$ , the

new solution  $\tilde{\xi}$  given by Theorem 1 is :

$$(3.11) \quad \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} - 2s(\widehat{W} \widehat{Z}^t J')^{**}.$$

The \*\* denotes the projection as above, with the last row moved to the second row. So  $(F, G, B)$  and  $(\tilde{F}, \tilde{G}, \tilde{B}^\#)$  are solutions of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system II (2.5). In components  $F = (f_{ij})_{2 \times 2}$ ,  $G = (r_{ij})_{(n-2) \times 2}$ ,  $\tilde{F} = (\tilde{f}_{ij})_{2 \times 2}$ ,  $\tilde{G} = (\tilde{r}_{ij})_{(n-2) \times 2}$ , the formula (3.11) for  $\tilde{\xi}$  is

$$(3.12) \quad \begin{cases} \tilde{f}_{11} = -\tilde{f}_{22} = f_{11} - s(\widehat{w}_1 \widehat{z}_2 - \widehat{w}_n \widehat{z}_1), \\ \tilde{f}_{12} = \tilde{f}_{21} = f_{12} - s(\widehat{w}_1 \widehat{z}_1 + \widehat{w}_n \widehat{z}_2), \\ \tilde{r}_{i1} = r_{i1} - 2s\widehat{w}_{1+i} \widehat{z}_2 \\ \tilde{r}_{i2} = r_{i2} - 2s\widehat{w}_{1+i} \widehat{z}_1. \end{cases}$$

Let  $\tilde{E}^\#$  frame of  $\tilde{\xi}$ ,  $E^{II}$  of  $(F, G, B)$  and  $\tilde{E}^{\#II}$  of  $(\tilde{F}, \tilde{G}, \tilde{B}^\#)$ . Then they are related by:

$$\tilde{E}^{\#II}(x, \lambda) = \tilde{E}^\#(x, \lambda) \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \widehat{B}^{\#t} \end{pmatrix} \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix},$$

$$(3.13) \quad \tilde{E}^{\#II}(x, \lambda) = E^{II}(x, \lambda) \left[ I - \frac{2}{\lambda^2 + s^2} \begin{pmatrix} s^2 \widehat{W} \widehat{W}^t I_{n-j,j} & -s\lambda \widehat{W} \widehat{Z}^t B^t J' \\ -s\lambda B \widehat{Z} \widehat{W}^t I_{n-j,j} & \lambda^2 B \widehat{Z} \widehat{Z}^t B^t J' \end{pmatrix} \right].$$

Henceforth we use the following notation:

$$(\tilde{\xi}, \tilde{E}^\#) = g_{s, \pi} \cdot (\xi, E), \quad \tilde{A}^\# = g_{s, \pi} \cdot A, \quad \tilde{B}^\# = g_{s, \pi} \cdot B, \quad (\tilde{F}, \tilde{G}, \tilde{B}^\#, \tilde{E}^{\#II}) = g_{s, \pi} \cdot (F, G, B, E^{II}).$$

## 4 Ribaucour Transformation

Now our interest is to show that the action of the element  $g_{s, \pi}$  on the space of local solutions of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system II (2.5), which was established in Section 3, corresponds to Ribaucour and Darboux transformations

correctly defined. To do this, we must adjust the definition of Ribaucour transformation given in [6], and the definition of Darboux transformation for surfaces in euclidean space  $\mathbb{R}^n$  for our time-like surfaces in  $\mathbb{R}^{n-j,j}$  whose shape operators have conjugate eigenvalues. We start this section with the definition of Ribaucour transformation.

For  $x \in \mathbb{R}^{n-j,j}$  and  $v \in (T\mathbb{R}^{n-j,j})_x$ , where let  $\gamma_{x,v}(t) = x + tv$  denote the geodesic starting at  $x$  in the direction of  $v$ .

**Definition 3.** Let  $M^m$  and  $\widetilde{M}^m$  be Lorentzian submanifolds whose shape operators are all diagonalizable over  $\mathbb{R}$  or  $\mathbb{C}$  immersed in the pseudo-riemannian space  $\mathbb{R}^{n-j,j}$ ,  $0 < j < n$ . A sphere congruence is a vector bundle isomorphism  $P : \mathcal{V}(M) \rightarrow \mathcal{V}(\widetilde{M})$  that covers a diffeomorphism  $\phi : M \rightarrow \widetilde{M}$  with the following conditions:

(1) If  $\xi$  is a parallel normal vector field of  $M$ , then  $P \circ \xi \circ \phi^{-1}$  is a parallel normal field of  $\widetilde{M}$ .

(2) For any nonzero vector  $\xi \in \mathcal{V}_x(M)$ , the geodesics  $\gamma_{x,\xi}$  and  $\gamma_{\phi(x),P(\xi)}$  intersect at a point that is the same parameter value  $t$  away from  $x$  and  $\phi(x)$ .

For the following definition we assume that each shape operator is diagonalized over the real or complex numbers. We note that there are submanifolds for which this does not hold.

**Definition 4.** A sphere congruence  $P : \mathcal{V}(M) \rightarrow \mathcal{V}(\widetilde{M})$  that covers a diffeomorphism  $\phi : M \rightarrow \widetilde{M}$  is called a Ribaucour transformation if it satisfies the following additional properties:

(1) If  $e$  is an eigenvector of the shape operator  $A_\xi$  of  $M$ , corresponding to a real eigenvalue then  $\phi_*(e)$  is an eigenvector of the shape operator  $A_{P(\xi)}$  of  $\widetilde{M}$  corresponding to a real eigenvalue.

If  $e_1 + ie_2$  is an eigenvector of  $A_\xi$  on  $(TM)^\mathbb{C}$  corresponding to the complex eigenvalue  $a + ib$  (so that  $e_1 - ie_2$  corresponds to the eigenvalue  $a - ib$ ), then  $\phi_*(e_1) + i\phi_*(e_2)$  is an eigenvector corresponding to a complex eigenvalue for  $A_{P(\xi)}$ .

(2) The geodesics  $\gamma_{x,e}$  and  $\gamma_{\phi(x),\phi_*(e)}$  intersect at a point that is equidistant to  $x$  and  $\phi(x)$  for real eigenvectors  $e$  and  $\gamma_{x,e_j}$  and  $\gamma_{\phi(x),\phi_*(e_j)}$  meet for the real and imaginary parts of complex eigenvectors  $e_1 + ie_2$ , i.e., for  $j = 1, 2$ .

**Theorem 2.** Let  $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$  solution of (2.2),  $E$  frame of  $\xi$ ,  $E(x, 0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}$ ,  $(F, G, B)$  a solution corresponding to complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system II, and

$$(\widetilde{F}, \widetilde{G}, \widetilde{B}^\sharp, \widetilde{E}^{\sharp II}) = g_{s,\pi} \cdot (F, G, B, E^{II}), \quad \widetilde{A}^\sharp = g_{s,\pi} \cdot A.$$

Let  $e_i, \widetilde{e}_i$  denote the  $i$ -th column of  $A$  and  $\widetilde{A}^\sharp$  resp. Then we have

(i)

$$\frac{\partial E}{\partial \lambda}(x, 0)E^{-1}(x, 0) = \begin{pmatrix} 0 & X \\ -J'X^t I_{n-j,j} & 0 \end{pmatrix}, \quad \frac{\partial \widetilde{E}^\sharp}{\partial \lambda}(x, 0)\widetilde{E}^{\sharp -1}(x, 0) = \begin{pmatrix} 0 & \widetilde{X} \\ -J'\widetilde{X}^t I_{n-j,j} & 0 \end{pmatrix}$$

for some  $X$  and  $\widetilde{X}$ .

(ii)  $X = (X_1, X_2)$  and  $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$  are complex isothermic time-like dual pairs in  $\mathbb{R}^{n-j,j}$  of type  $O(1, 1)$  such that  $\{e_2, \dots, e_{n-1}\}$  and  $\{\widetilde{e}_2, \dots, \widetilde{e}_{n-1}\}$  are parallel normal

frames for  $X_j$  and  $\tilde{X}_j$  respectively for  $j = 1, 2$ , where  $\{e_\alpha\}_{\alpha=2}^{n-j}$  and  $\{e_\alpha\}_{\alpha=n-j+1}^{n-1}$  are space-like and time-like vectors resp.

(iii) The solutions of the complex  $O(n-j+1, j+1)/O(n-j, j) \times O(1, 1)$ -system II corresponding to  $X$  and  $\tilde{X}$  are  $(F, G, B)$  and  $(\tilde{F}, \tilde{G}, \tilde{B}^\sharp)$  resp.

(iv) The bundle morphism  $P(e_k(x)) = \tilde{e}_k(x)$   $k = 2, \dots, n-1$ , is a Ribaucour Transformation covering the map  $X_j(x) \mapsto \tilde{X}_j(x)$  for each  $j = 1, 2$ .

(v) There exist smooth functions  $\psi_{ik}$  such that  $X_i + \psi_{ik}e_k = \tilde{X}_i + \psi_{ik}\tilde{e}_k$  for  $1 \leq i \leq 2$  and  $1 \leq k \leq n$ .

For the proof we will need the following result which can be proved as Corollary (6.11) in ([3]).

**Proposition 1.** Let  $E(x, \lambda)$  be a frame for the solution  $\xi$  of system (2.2), and  $Y(x) = \frac{\partial E}{\partial \lambda}(x, 0)E^{-1}(x, 0)$ . Then we have

(i)

$$Y = \begin{pmatrix} 0 & X \\ -J'X^t I_{n-j,j} & 0 \end{pmatrix} \text{ for some } X \in \mathcal{M}_{n \times 2}.$$

(ii)  $X = (X_1, X_2)$  is complex isothermic time-like dual pair in  $\mathbb{R}^{n-j,j}$  of type  $O(1, 1)$

(iii)

$$(4.1) \quad dX = A \begin{pmatrix} dx_1 & 0 & \dots & 0 & dx_2 \\ -dx_2 & 0 & \dots & 0 & dx_1 \end{pmatrix}^t B^{-1}.$$

*Proof of Theorem 2:*

Following the same lines as that of Theorem 10.6 in [3], we arrive at the formula  $\tilde{X} = X + \frac{2}{s}A\widehat{W}\widehat{Z}^t B^t J'$ . Letting  $\eta = \sum_{j=1}^n \widehat{w}_j e_j$ , we see that the  $i$ -th column of  $\tilde{X}$  is given by

$$(4.2) \quad \tilde{X}_i = X_i + \frac{2}{s} \sum_{j=1}^2 (\widehat{z}_j b_{\bar{i},j}) \eta, \text{ where } \bar{1} = 2 \text{ and } \bar{2} = 1.$$

Next from the relation  $\tilde{A}^\sharp = A(I - 2\widehat{W}\widehat{W}^t I_{n-j,j})$  we get  $\tilde{e}_i = e_i - 2\widehat{w}_i \eta \epsilon_i$  with  $\epsilon_i = 1$ ,  $i = 1, \dots, n-j$  and  $\epsilon_i = -1$ ,  $i = n-j+1, \dots, n$ . Now using this last relation, we have

$$(4.3) \quad X_i + \psi_{ik}e_k = \tilde{X}_i + \psi_{ik}\tilde{e}_k,$$

where

$$\psi_{ik} = \frac{\epsilon_k}{s\widehat{w}_k} \sum_{j=1}^2 \widehat{z}_j b_{\bar{i},j}, \quad \text{for } i = 1, 2, \quad k = 1, 2, \dots, n.$$

To see that the condition on the eigenvectors holds, we note that the shape operators can be calculated using the  $r_{ij}$  and the algebraic form is preserved. In our case both shape operators are diagonalized over  $\mathbb{C}$ . So we conclude that  $P$  is a Ribaucour transformation. ■

## 5 Darboux Transformations for complex timelike isothermic surfaces in $\mathbb{R}^{n-j,j}$

Here, we are interested in considering Darboux transformations for timelike isothermic surfaces in  $\mathbb{R}^{n-j,j}$ . In fact, in our next result we show that the transformation constructed in Theorem 2 is a Darboux transformation.

Let  $M, \widetilde{M}$  be two time-like surfaces in  $\mathbb{R}^{n-j,j}$  with flat and non-degenerate normal bundle, shape operators that are diagonalizable over  $\mathbb{C}$  and  $P : \mathcal{V}(M) \rightarrow \mathcal{V}(\widetilde{M})$  a Ribaucour transformation that covers the map  $\phi : M \rightarrow \widetilde{M}$ . If, in addition,  $\phi$  is a sign-reversing conformal diffeomorphism then  $P$  is called a Darboux transformation. By a sign-reversing conformal diffeomorphism we mean that the time-like and space like vectors are interchanged and the conformal coordinates remain conformal. With this, we have:

**Theorem 3.** *Let  $(X_1, X_2)$  be a complex isothermic time-like dual pair in  $\mathbb{R}^{n-j,j}$  of type  $O(1, 1)$  corresponding to the solution  $(u, G)$  of the system (2.5), and let  $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$  the corresponding solution of the system (2.2), where*

$$F = \begin{pmatrix} u_{x_2} & u_{x_1} \\ u_{x_1} & -u_{x_2} \end{pmatrix}, B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}.$$

Let also  $s \in \mathbb{R}$  be different of zero,  $\pi$  a projection on  $\mathbb{C} \begin{pmatrix} W \\ iZ \end{pmatrix}$ ,  $g_{s,\pi}$  the rational element defined in (3.3), and  $\widehat{W}, \widehat{Z}$  as in Theorem 1, for the solution  $\xi$  of the system (2.2). Let  $(\widetilde{E}^\sharp, \widetilde{A}^\sharp, \widetilde{B}^\sharp) = g_{s,\pi} \cdot (E^{II}, A, B)$ . Write  $A = (e_1, \dots, e_n)$  and  $\widetilde{A}^\sharp = (\widetilde{e}_1, \dots, \widetilde{e}_n)$ . Set

$$(5.1) \quad \begin{cases} \widetilde{X}_1 = X_1 + \frac{2}{s} \widehat{z}_2 e^{-2u} \sum_{i=1}^n \widehat{w}_i e_i, \\ \widetilde{X}_2 = X_2 + \frac{2}{s} \widehat{z}_1 e^{2u} \sum_{i=1}^n \widehat{w}_i e_i, \end{cases}$$

Then

(i)  $(\widetilde{u}, \widetilde{G})$  is the solution of system (2.5), corresponding to  $(\widetilde{X}_1, \widetilde{X}_2)$ , where  $e^{4\widetilde{u}} = \frac{4\widehat{z}_2^4}{e^{4u}}$  and  $\widetilde{G} = (\widetilde{r}_{ij})$  is defined by (3.12).

(ii) The fundamental forms of pair  $(\widetilde{X}_1, \widetilde{X}_2)$  are respectively

$$\begin{cases} \widetilde{I}_1 = e^{4\widetilde{u}}(-dx_1^2 + dx_2^2) \\ \widetilde{II}_1 = e^{2\widetilde{u}} \sum_{i=1}^{n-2} [\widetilde{r}_{i,1}(dx_2^2 - dx_1^2) - 2\widetilde{r}_{i,2}dx_1dx_2] \widetilde{e}_{i+1}. \\ \widetilde{I}_2 = e^{-4\widetilde{u}}(dx_1^2 - dx_2^2) \\ \widetilde{II}_2 = -e^{-2\widetilde{u}} \sum_{i=1}^{n-2} \widetilde{r}_{i,2}(dx_2^2 - dx_1^2) + 2\widetilde{r}_{i,1}dx_1dx_2] \widetilde{e}_{i+1}. \end{cases}$$

(iii) The bundle morphism  $P(e_k(x)) = \widetilde{e}_k(x)$ ,  $k = 2, \dots, n-1$  covering the map  $X_i \rightarrow \widetilde{X}_i$  is a Darboux transformation for each  $i = 1, 2$ .

*Proof.* For (i) and (ii) we just observe that

$$d\widetilde{X} = \widetilde{A}^\sharp \begin{pmatrix} dx_1 & 0 & \dots & 0 & dx_2 \\ -dx_2 & 0 & \dots & 0 & dx_1 \end{pmatrix}^t \widetilde{B}^\sharp^{-1},$$

and calculate.

For (iii) we observe that the map  $\phi : X_i \rightarrow \tilde{X}_i$  is sign-reversing conformal because the coordinates  $(x_1, x_2)$  are isothermic for  $X_i$  and  $\tilde{X}_i$  but time-like and space-like vectors are interchanged. The rest of the properties of Darboux transformation were proved above. ■

**Example 1.** Let  $n = 3, j = 1$ , so that we have the  $O(3, 2)/O(2, 1) \times O(1, 1)$ -system. Let  $(u, r_{11}, r_{12}) = (0, 0, 0)$  be a trivial solution of (2.5), then  $F = 0, G = 0, B = I$ . So a complex isothermic time-like dual pair in  $\mathbb{R}^{2,1}$  of type  $O(1, 1)$  corresponding to trivial solution is:

$$X = \begin{pmatrix} x & -y \\ 0 & 0 \\ y & x \end{pmatrix}$$

The frame  $E(x, y, \lambda)$  is the following, where we let

$$u = \frac{-x + y}{\sqrt{2}}, v = \frac{x + y}{\sqrt{2}} :$$

and write the columns of  $E(x, y, \lambda)$  as  $c_1, \dots, c_5$

$$(5.2) \quad c_1 = \begin{pmatrix} \cos(u \lambda) \cosh(v \lambda) \\ 0 \\ \frac{\sin(u \lambda) \sinh(v \lambda)}{\cosh(v \lambda) \sin(u \lambda) + \cos(u \lambda) \sinh(v \lambda)} \\ \frac{\cosh(v \lambda) \sin(u \lambda) - \cos(u \lambda) \sinh(v \lambda)}{\sqrt{2}} \end{pmatrix}, c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, c_3 = \begin{pmatrix} -(\sin(u \lambda) \sinh(v \lambda)) \\ 0 \\ \cos(u \lambda) \cosh(v \lambda) \\ \frac{-(\cosh(v \lambda) \sin(u \lambda)) + \cos(u \lambda) \sinh(v \lambda)}{\sqrt{2}} \\ \frac{\cosh(v \lambda) \sin(u \lambda) + \cos(u \lambda) \sinh(v \lambda)}{\sqrt{2}} \end{pmatrix}$$

$$(5.3) \quad c_4 = \begin{pmatrix} \frac{-(\cosh(v \lambda) \sin(u \lambda)) + \cos(u \lambda) \sinh(v \lambda)}{\sqrt{2}} \\ 0 \\ \frac{\cosh(v \lambda) \sin(u \lambda) + \cos(u \lambda) \sinh(v \lambda)}{\sqrt{2}} \\ \cos(u \lambda) \cosh(v \lambda) \\ \sin(u \lambda) \sinh(v \lambda) \end{pmatrix}, c_5 = \begin{pmatrix} -\left( \frac{\cosh(v \lambda) \sin(u \lambda) + \cos(u \lambda) \sinh(v \lambda)}{\sqrt{2}} \right) \\ 0 \\ \frac{-(\cosh(v \lambda) \sin(u \lambda)) + \cos(u \lambda) \sinh(v \lambda)}{\sqrt{2}} \\ -(\sin(u \lambda) \sinh(v \lambda)) \\ \cos(u \lambda) \cosh(v \lambda) \end{pmatrix}$$

Then from Theorem 1, we obtain

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} \frac{\cosh(s u) (2 w_1 \cos(s v) + \sqrt{2} (-z_1 + z_2) \sin(s v)) + (\sqrt{2} (z_1 + z_2) \cos(s v) + 2 w_3 \sin(s v)) \sinh(s u)}{2} \\ w_2 \\ \frac{\cosh(s u) (2 w_3 \cos(s v) - \sqrt{2} (z_1 + z_2) \sin(s v)) + (-\sqrt{2} (z_1 - z_2) \cos(s v) - 2 w_1 \sin(s v)) \sinh(s u)}{2} \\ \frac{\cosh(s u) (2 z_1 \cos(s v) + \sqrt{2} (w_1 + w_3) \sin(s v)) + (\sqrt{2} (w_1 - w_3) \cos(s v) + 2 z_2 \sin(s v)) \sinh(s u)}{2} \\ \frac{\cosh(s u) (2 z_2 \cos(s v) + \sqrt{2} (-w_1 + w_3) \sin(s v)) + (\sqrt{2} (w_1 + w_3) \cos(s v) - 2 z_1 \sin(s v)) \sinh(s u)}{2} \end{pmatrix}.$$

From equation (5.1), we get that the isothermic timelike dual pair in  $\mathbb{R}^{2,1}$  of type  $O(1, 1)$  constructed by applying the Darboux transformation to the trivial solution is:

$$\tilde{X}_1 = X_1 + \frac{2}{s} \hat{z}_2 \sum_{i=1}^3 \hat{w}_i e_i, \quad \tilde{X}_2 = X_2 + \frac{2}{s} \hat{z}_1 \sum_{i=1}^3 \hat{w}_i e_i,$$

If we make the choice  $w_1 = 1/\sqrt{2} = w_2 = z_1 = z_2, w_3 = 0$  we get:

$$\tilde{X}_1 = \begin{pmatrix} \frac{-u+v}{\sqrt{2}} + \frac{2 \cos(v) (\sqrt{2} \cosh(u)+2 \sinh(u)) (\cosh(u) (\sqrt{2} \cos(v)-\sin(v))+(\cos(v)-\sqrt{2} \sin(v)) \sinh(u))}{1+3 \cos(2v) \cosh(2u)+2\sqrt{2} \cos(2v) \sinh(2u)} \\ \frac{2 (\cosh(u) (2 \cos(v)-\sqrt{2} \sin(v))+(\sqrt{2} \cos(v)-2 \sin(v)) \sinh(u))}{1+3 \cos(2v) \cosh(2u)+2\sqrt{2} \cos(2v) \sinh(2u)} \\ \frac{u+v}{\sqrt{2}} + \frac{2 \sin(v) (2 \cosh(u)+\sqrt{2} \sinh(u)) (\cosh(u) (-\sqrt{2} \cos(v)+\sin(v))+(-\cos(v)+\sqrt{2} \sin(v)) \sinh(u))}{1+3 \cos(2v) \cosh(2u)+2\sqrt{2} \cos(2v) \sinh(2u)} \end{pmatrix}$$

$$\tilde{X}_2 = \begin{pmatrix} -\left(\frac{u+v}{\sqrt{2}}\right) + \frac{2 \cos(v) (\sqrt{2} \cosh(u)+2 \sinh(u)) (\cosh(u) (\sqrt{2} \cos(v)+\sin(v))+(\cos(v)+\sqrt{2} \sin(v)) \sinh(u))}{1+3 \cos(2v) \cosh(2u)+2\sqrt{2} \cos(2v) \sinh(2u)} \\ \frac{2 (\cosh(u) (2 \cos(v)+\sqrt{2} \sin(v))+(\sqrt{2} \cos(v)+2 \sin(v)) \sinh(u))}{1+3 \cos(2v) \cosh(2u)+2\sqrt{2} \cos(2v) \sinh(2u)} \\ \frac{-u+v}{\sqrt{2}} - \frac{2 \sin(v) (2 \cosh(u)+\sqrt{2} \sinh(u)) (\cosh(u) (\sqrt{2} \cos(v)+\sin(v))+(\cos(v)+\sqrt{2} \sin(v)) \sinh(u))}{1+3 \cos(2v) \cosh(2u)+2\sqrt{2} \cos(2v) \sinh(2u)} \end{pmatrix}$$

Using the equation (4.1) for  $\tilde{X}_1, \tilde{X}_2$  we see that

$$d\tilde{X}_1 = \frac{-2\hat{z}_2^2}{b} (-dx_2\tilde{e}_1 + dx_1\tilde{e}_3), \quad d\tilde{X}_2 = -2b\hat{z}_1^2(dx_1\tilde{e}_1 + dx_2\tilde{e}_3)$$

so that

$$\tilde{I}_1 = \frac{4\hat{z}_2^4}{b^2} (dx_2^2 - dx_1^2), \quad \tilde{I}_2 = 4b^2\hat{z}_1^4 (dx_1^2 - dx_2^2).$$

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