Natural tensor fields of type $(0, 2)$ on the tangent and cotangent bundles of a Fedosov manifold

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Abstract. To any (0,2)-tensor field on the tangent and cotangent bundles of a Fedosov manifold, we associate a global matrix function 'mutatis mutandis' as in the semi-Riemannian case. Based on this fact, natural (0,2)-tensor fields on these bundles are defined and characterized.

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1 Introduction

Let M be a manifold of dimension $2n$, $\omega \in \Omega^2(M)$ a non-degenerate closed $2-$form on M and ∇ a free of torsion linear connection compatible with ω; i.e., $X\omega(Y,Z) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$ for any vector fields on M, $X, Y, Z \in \mathfrak{X}(M)$.

The triple (M, ω, ∇) is called a Fedosov manifold. For a detailed study of these manifolds we refer to [3]. Fedosov manifolds constitute a natural generalization of Kähler manifolds. In fact, let $<,>$ be a semi-Riemannian metric on M with Levi-Civita connection ∇ and J an almost complex structure on M which satisfies $< J(X), J(Y) >=< X, Y >$ and $J(\nabla_X Y) = \nabla_X JY$ for any $X, Y \in \mathfrak{X}(M)$; i.e., $(M, <,>, J)$ is a Kähler manifold.

By defining $\omega(X, Y) =< J(X), Y >$, it follows that (M, ω, ∇) is a Fedosov manifold.

In contrast, there are Fedosov manifolds which do not admit Kähler structure ([2]).

In [1], we lifted to suitable bundles (0,2)-tensor fields defined on tangent and cotangent bundles over manifolds endowed with semi-Riemannian metrics so as to look at them as global matrix functions. These matrix representations allowed us to define and classify natural (0,2)-tensor fields with respect to semi-Riemannian metrics. The main result that lets us characterize these tensor fields is Theorem 2.1 of [1]. In this paper, the main result is Theorem 2.1. We apply this result to characterize natural (0,2)-tensor fields on tangent (Proposition 3.1) and cotangent (Proposition 4.1) bundles over Fedosov manifolds.

Throughout, all geometric objects are assumed to be differentiable, i.e. C^∞.

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2 The main result

For any integer $n \geqslant 1$, let $S = (s_{ij}) \in \mathbb{R}^{2n\times 2n}$ be the matrix

$$S = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$$

where $I_n \in \mathbb{R}^{n\times n}$ is the unit matrix. Hence,

$$s_{ij} = \begin{cases} -1 & \text{if } i-j=n \\ 1 & \text{if } j-i=n \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{G} = S_p(2n)$ be the real symplectic group ; i.e., $a \in \mathcal{G}$ if and only if $a.S.a^t = S$.

Theorem 2.1. *Let $A : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n\times 2n}$ be a differentiable map which satisfies*

$$A(x) = a.A(x.a).a^t$$

for any $a \in \mathcal{G}$ and $x \in \mathbb{R}^{2n}$. Then, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$A(x) = \alpha.S + \beta.(x.S)^t.(x.S)$$

where $y^t y = (y_{ij}) \in \mathbb{R}^{2n\times 2n}$ is the matrix defined by $y_{ij} = y_i.y_j$, if $y = (y_1, \ldots, y_{2n})$.

We will prove this theorem using the following two results

Proposition 2.2. *If $x, y \in \mathbb{R}^{2n}$ are non-zero vectors, there exists $a \in \mathcal{G}$ such that $y = x.a$.*

Proof. Let $e_1, \ldots, e_{2n} \in \mathbb{R}^{2n}$ be the canonical basis. We need only to check the case when $x = e_1$.

It is well known (see [4]) that there exists a symplectic basis $v_1, \ldots, v_{2n}$ of $\mathbb{R}^{2n}$; i.e.,

(2.2.1) $$v_i S v_j^t = s_{ij} \quad , \quad 1 \leqslant i, j \leqslant 2n$$

such that $v_1 = y$.

Let us define $a \in GL(2n, \mathbb{R})$ by $e_i.a = v_i$ if $1 \leqslant i \leqslant 2n$, hence from (2.2.1) it follows that $a \in \mathcal{G}$. □

Proposition 2.3. *Let $\mathcal{G}_1$ be the stabilizer of e_1 in $\mathcal{G}$; i.e., $\mathcal{G}_1 = \{a \in \mathcal{G} / e_1.a = e_1\}$. The centralizer Z of $\mathcal{G}_1$ in $\mathbb{R}^{2n\times 2n}$ is the set*

$$Z = \{\alpha.I_{2n} + \beta.e_{n+1}^t.e_1 / \alpha, \beta \in \mathbb{R}\}$$

Proof. Let $\sigma \in Z$. Hence, for any $a \in \mathcal{G}_1$ we have

(2.2.2) $$a.\sigma = \sigma.a$$

Let $\mathcal{D} \subset GL(2n, \mathbb{R})$ be the set of diagonal matrices $d = (d_{ij})$ such that $d_{11} = 1$ and $d_{(n+i)(n+i)} = d_{ii}^{-1}$ for $1 \leqslant i \leqslant n$.

Let $\mathcal{S}$ be the set of matrices $a \in \mathbb{R}^{2n\times 2n}$ such that

$$a = \left(\begin{array}{c|c} I_n & 0 \\ \hline \text{s} & I_n \end{array}\right)$$

where $s \in \mathbb{R}^{n\times n}$ is s symmetric matrix.

Clearly $\mathcal{D} \subset \mathcal{G}_1$ and $\mathcal{S} \subset \mathcal{G}_1$. Writing $\sigma \in Z$ in the block form

$$\sigma = \left(\begin{array}{c|c} \sigma_1 & \sigma_2 \\ \hline \sigma_4 & \sigma_3 \end{array}\right)$$

where $\sigma_i \in \mathbb{R}^{n\times n}$, $1 \leqslant i \leqslant 4$, condition (2.2.2) applied to any $a \in \mathcal{S}$ implies that $\sigma_2 = 0$ and $\sigma_1 = \sigma_3 = \alpha.I_n$ for some $\alpha \in \mathbb{R}$.

Now, condition (2.2.2) applied to any $a \in \mathcal{D}$ implies that $\sigma_4 = (a_{ij})$ satisfies $a_{ij} = 0$ if $(i,j) \neq (1,1)$. Writing $\beta = a_{11}$, one gets

$$\sigma = \alpha.I_{2n} + \beta.e^t_{n+1}.e_1 \tag{2.2.3}$$

Conversely, if σ is of the form (2.2.3), it is clear that $\sigma \in Z$ if and only if $e^t_{n+1}.e_1 \in Z$.

Let $a \in \mathcal{G}_1$, then

$$\begin{aligned} a.e^t_{n+1}.e_1 &= a.(e_1.S)^t.e_1 = a.S^t.e^t_1.e_1 = -a.S.e^t_1.e_1 \\ &= -S(a^{-1})^t.e^t_1.e_1 = -S(e_1.a^{-1})^t.e_1 \\ &= -S.e^t_1.e_1 = e^t_{n+1}.e_1 = e^t_{n+1}.e_1.a \end{aligned}$$

□

Proof of Theorem 2.1. Let $A : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n\times 2n}$ be a differentiable function satisfying

$$A(x) = a.A(x.a).a^t \tag{2.2.4}$$

for any $a \in \mathcal{G}$ and $x \in \mathbb{R}^{2n}$.

Let $x \in \mathbb{R}^{2n}$ be a non zero vector. According to Proposition 2.2, there exists $b \in \mathcal{G}$ such that $x.b = e_1$; hence,

$$A(x) = b.A(e_1).b^t \tag{2.2.5}$$

Equality (2.2.4) applied to any $a \in \mathcal{G}_1$ implies that $A(e_1) = a.A(e_1).a^t$. Since $a.S.a^t = S$, it follows that $a^t = S^{-1}.a^{-1}.S$; and consequently

$$A(e_1).S^{-1}.a = a.A(e_1).S^{-1} \tag{2.2.6}$$

Equality (2.2.6) shows that $A(e_1).S^{-1} \in Z$; hence, by Proposition 2.3, there exist $\alpha, \beta \in \mathbb{R}$ such that $A(e_1).S^{-1} = \alpha.I_{2n} + \beta.e^t_{n+1}.e_1$; or, equivalently

$$A(e_1) = \alpha.S + \beta.e^t_{n+1}.e_1.S \tag{2.2.7}$$

Since $e_1.S = e_{n+1}$, from (2.2.5) and (2.2.6) one gets

$$\begin{aligned} A(x) &= b.(\alpha.S + \beta.e^t_{n+1}.e_1.S).b^t = \alpha.b.S.b^t + \beta.b.e^t_{n+1}.e_1.S.b^t \\ &= \alpha.S + \beta.b(e_1.S)^t.e_1.b^{-1}.b.S.b^t \\ &= \alpha.S + \beta.b.S^t.e^t_1.e_1.b^{-1}.S \\ &= \alpha.S + \beta(e_1.b^{-1}.S)^t.e_1.b^{-1}.S \end{aligned}$$

Since $x.b = e_1$, it follows that

$$A(x) = \alpha.S + \beta.(x.S)^t.(x.S) \tag{2.2.8}$$

Continuity of A implies that (2.2.8) holds for any $x \in \mathbb{R}^{2n}$.

□

3 Natural (0,2)-tensor fields on tangent bundles

Let (M, ω, ∇) be a Fedosov manifold of dimension $2n$ and $\pi : TM \longrightarrow M$ be the tangent bundle over M.

If $\mathcal{L}(M)$ denotes the frame bundle over M, let

$$\mathcal{S}(M) = \{(p, u_1, \ldots, u_{2n}) \in \mathcal{L}(M)/\omega(p)(u_i, u_j) = s_{ij}\}$$

be the symplectic frame bundle over M and $\psi : \mathbf{N} = \mathcal{S}(M) \times \mathbb{R}^{2n} \longrightarrow TM$ the map defined by $\psi(p, u, \xi) = \sum_{i=1}^{2n} \xi^i.u_i$ where $(p, u) = (p, u_1, \ldots, u_{2n})$ and $\xi = (\xi^1, \ldots, \xi^{2n})$. The family of maps $R_a : \mathbf{N} \longrightarrow \mathbf{N}$, $a \in \mathcal{G}$, given by

$$R_a(p, u, \xi) = (p, ua, \xi.(a^t)^{-1})$$

where

$$ua = \left(\sum_{i=1}^{2n} a_1^i.u_i, \ldots, \sum_{i=1}^{2n} a_{2n}^i.u_i\right) \quad , \quad a = \begin{pmatrix} a_1^1 & \cdots & a_{2n}^1 \\ \vdots & & \vdots \\ a_1^{2n} & \cdots & a_{2n}^{2n} \end{pmatrix}$$

define the action of $\mathcal{G}$ on $\mathbf{N}$. Clearly $\psi \circ R_a = \psi$.

Let $K : TTM \longrightarrow TM$ be the connection map induced by ∇ and for any $p \in M$ and any $v \in M_p$, let $\pi_{*v} : (TM)_v \longrightarrow M_p$ be the differential map of π at v, and $K_v : (TM)_v \longrightarrow M_p$ the restriction of K to $(TM)_v$.

Since the linear map $\pi_{*v} \times K_v : (TM)_v \longrightarrow M_p \times M_p$ defined by $\pi_{*v} \times K_v(b) = (\pi_{*v}(b), K_v)$ is an isomorphism that maps isomorphically the horizontal subspace H_v (= kernel of K_v) onto $M_p \times (0_p)$ and the vertical subspace V_v (= kernel of π_{*v}) onto $(0_p) \times M_p$, where 0_p denotes the zero vector, we define –as in [1]– the differentiable mappings $e_i, e_{2n+i} : \mathbf{N} \longrightarrow TTM$ for $1 \leqslant i \leqslant 2n$ by

$$e_i(p, u, \xi) = (\pi_{*v} \times K_v)^{-1}(u_i, 0_p) \qquad \text{and} \qquad e_{2n+i}(p, u, \xi) = (\pi_{*v} \times K_v)^{-1}(0_p, u_i)$$

where $v = \psi(p, u, \xi)$.

Since $(TM)_v = H_v \oplus V_v$, any vector field X on TM may be written in the form $X = X^h + X^v$, where

$$X^h(v) = (\pi_{*v} \times K_v)^{-1}(\pi_{*v}(X(v)), 0_p) \quad , \quad X^v(v) = (\pi_{*v} \times K_v)^{-1}(0_p, K_v(X(v)))$$

if $v \in M_p$. Hence, the mappings e_i, e_{2n+i} let us view X as the function ${}^{\nabla}X = (x^1, \ldots, x^{4n}) : \mathbf{N} \longrightarrow \mathbb{R}^{4n}$ where $x^\ell : \mathbf{N} \longrightarrow \mathbb{R}$ are determined –for $v = \psi(p, u, \xi)$– by

$$(3.3.1)\qquad \begin{array}{rcl} x^i(p,u,\xi) &=& \omega(p)(\pi_{*v}(X(v)),u_{n+i}) \\ x^{n+i}(p,u,\xi) &=& \omega(p)(\pi_{*v}(X(v)),u_i) \end{array}$$

and

$$(3.3.2)\qquad \begin{array}{rcl} x^{2n+i}(p,u,\xi) &=& \omega(p)(K_v(X(v)),u_{n+i}) \\ x^{3n+i}(p,u,\xi) &=& -\omega(p)(K_v(X(v)),u_i) \end{array}$$

for $1 \leqslant i \leqslant n$.

¿From (3.3.1) and (3.3.2) one gets that

$$(3.3.3)\qquad {}^{\nabla}X \circ R_a = {}^{\nabla}X \ . \begin{pmatrix} (a^t)^{-1} & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}$$

for any $a \in \mathcal{G}$.

As in [1], for any $(0,2)-$tensor field G on TM we define the differentiable function

$${}^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix} : \mathbf{N} \longrightarrow \mathbb{R}^{4n\times 4n}$$

as follows: if $(p,u,\xi) \in \mathbf{N}$ and $v = \psi(p,u,\xi)$, let ${}^{\nabla}G(p,u,\xi)$ be the matrix of the bilinear form $G_v : (TM)_v \times (TM)_v \longrightarrow \mathbb{R}$ induced by G on $(TM)_v$ with respect to the basis $\{e_1(p,u,\xi),\ldots,e_{4n}(p,u,\xi)\}$. Hence, for any pair of vector fields X,Y on TM one gets

$$(3.3.4)\qquad G(X,Y)\circ\psi = {}^{\nabla}X.{}^{\nabla}G.({}^{\nabla}Y)^t$$

Equalities (3.3.3) and (3.3.4) imply that each $A_i : \mathbf{N} \longrightarrow \mathbb{R}^{2n\times 2n}$ satisfies the following $\mathcal{G}-$invariance property

$$(3.3.5)\qquad A_i \circ R_a = a^t.A_i.a \qquad (i=1,2,3,4)$$

We shall call ${}^{\nabla}G$ the matrix of G with respect to (ω,∇). Hence, we get a one to one correspondence "${}^{\nabla}G \longleftrightarrow T$" between $(0,2)-$tensor fields on TM and differentiable functions $T = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix} : \mathbf{N} \longrightarrow \mathbb{R}^{4n\times 4n}$ where each A_i satisfies (3.3.5). The differentiability of G –for T given– follows from (3.3.4) and the fact that ψ is a submersion.

Just as we did in [1], we define G to be natural with respect to (ω,∇) if ${}^{\nabla}G$ only depends on ξ.

Proposition 3.1. *Let G be a $(0,2)-$tensor field on TM and ${}^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$ the matrix of G with respect to (ω,∇). Then G is natural with respect to (ω,∇) if there exist real numbers $\alpha_i,\beta_i \in \mathbb{R}$ $(i=1,2,3,4)$ such that*

$$A_i(p,u,\xi) = \alpha_i.S + \beta_i.(\xi.S)^t.(\xi.S)$$

or, equivalently, if for any vector fields X,Y on TM, the following equalities are satisfied

$$\begin{aligned}
G(X^h,Y^h)(v) &= \alpha_1 \ .\ \omega(p)(\pi_{*v}(X(v)),\pi_{*v}(Y(v))) \\
&\quad +\beta_1 \ .\ \omega(p)(v,\pi_{*v}(X(v)))\ .\ \omega(p)(v,\pi_{*v}(Y(v))) \\
G(X^h,Y^v)(v) &= \alpha_2 \ .\ \omega(p)(\pi_{*v}(X(v)),K_v(Y(v))) \\
&\quad +\beta_2 \ .\ \omega(p)(v,\pi_{*v}(X(v)))\ .\ \omega(p)(v,K_v(Y(v))) \\
G(X^v,Y^h)(v) &= \alpha_4 \ .\ \omega(p)(K_v(X(v)),\pi_{*v}(Y(v))) \\
&\quad +\beta_4 \ .\ \omega(p)(v,K_v(X(v)))\ .\ \omega(p)(v,\pi_{*v}(Y(v))) \\
G(X^v,Y^v)(v) &= \alpha_3 \ .\ \omega(p)(K_v(X(v)),K_v(Y(v))) \\
&\quad +\beta_3 \ .\ \omega(p)(v,K_v(X(v)))\ .\ \omega(p)(v,K_v(Y(v)))
\end{aligned}$$

Proof. According to (3.3.5), if G is natural, each matrix function A_i can be viewed as a function $B : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n\times 2n}$ which satisfies $B(\xi.(a^{-1})^t) = a^t.B(\xi).a$ for any $\xi \in \mathbb{R}^{2n}$ and any $a \in \mathcal{G}$; or, equivalently, $B(\xi) = (a^{-1})^t.B(\xi.(a^{-1})^t).a^{-1}$.

Since $b \in \mathcal{G}$ implies that $b^t \in \mathcal{G}$, it follows that $B(\xi) = aB(\xi.a)a^t$ for any $a \in \mathcal{G}$. Consequently, by Theorem 2.1, there exist $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$ such that

$$A_i(p,u,\xi) = \alpha_i.S + \beta_i(\xi.S)^t,(\xi.S)$$

The expression of G applied to vector fields is now a consequence of (3.3.1), (3.3.2) and (3.3.4). □

4 Natural $(0,2)-$tensor fields on cotangent bundles

For any $p \in M$, let M_p^* be the dual space of M_p and let $\pi : T^*M \longrightarrow M$ be the cotangent bundle of M.

For any $(p,u) \in \mathcal{S}(M)$, we denote with (p,u^*) the dual basis and $\mathcal{S}^*(M)$ the bundle consisting of all those ordered dual basis. Set $\mathcal{N} = \mathcal{S}^*(M) \times \mathbb{R}^{2n}$ and let $\psi : \mathcal{N} \longrightarrow T^*M$ be the map defined by

$$\psi(p,u^*,\xi) = \sum_{i=1}^{2n} \xi_i.u^i$$

if $u^* = \{u^1,\dots,u^{2n}\}$ and $\xi = (\xi_1,\dots,\xi_{2n})$.

The family of maps $R_a : \mathcal{N} \longrightarrow \mathcal{N}$, $a \in \mathcal{G}$, given by

$$R_a(p,u^*,\xi) = (p,(ua)^*,\xi.a)$$

defines the action of $\mathcal{G}$ on $\mathcal{N}$. Clearly, $\psi \circ R_a = \psi$. Let $K^* : T(T^*M) \longrightarrow T^*M$ be the dual connection map. We'll recall that for any $p \in M$ and any co-vector $w \in M_p^*$, the restriction $K_w^* : (T^*M)_w \longrightarrow M_p^*$ of K^* to $(T^*M)_w$ is a surjective linear map characterized by the fact that for any $1-$form θ on M such that $\theta(p) = w$ and any vector $v \in M_p$, it satisfies $K_w^*(\theta_{*p}(v)) = \nabla_v\theta$ where $\theta_{*p} : M_p \longrightarrow (T^*M)_w$ denotes the differential map of θ at p.

Since the linear map $\pi_{*w} \times K_w^* : (T^*M)_w \longrightarrow M_p \times M_p^*$ defined by $\pi_{*w} \times K_w^*(b) = (\pi_{*w}(b), K_w^*(b))$ is an isomorphism that maps the horizontal subspace H_w (= kernel of K_w^*) onto $M_p \times (0_p)$ and the vertical subspace V_w (= kernel of π_{*w}), where 0_p denotes indistinctly the zero vector and the zero co-vector, we define –as in [1]– the differentiable mappings $e_i, e_{2n+i} : \mathcal{N} \longrightarrow T(T^*M)$ for $1 \leqslant i \leqslant 2n$ by

$$e_i(p,u^*,\xi) = (\pi_{*w} \times K^*_w)^{-1}(u_i, 0_p) \qquad \text{and} \qquad e_{2n+i}(p,u^*,\xi) = (\pi_{*w} \times K^*_w)^{-1}(0_p, u^i)$$

where $w - \psi(p,u^*,\xi)$.

Since $(T^*M)_w = H_w \oplus V_w$, any vector field X on T^*M may be written in the form $X = X^h + X^v$, where

$$X^h(w) = (\pi_{*w} \times K^*_w)^{-1}(\pi_{*w}(X(w)), 0_p) \quad , \quad X^v(w) = (\pi_{*w} \times K^*_w)^{-1}(0_p, K^*_w(X(w)))$$

if $w \in M^*_p$. Hence, the mappings e_i, e_{2n+i} let us view X as the function ${}^\nabla X = (x^1, \ldots, x^{4n}) : \mathcal{N} \longrightarrow \mathbb{R}^{4n}$, where $x^\ell : \mathcal{N} \longrightarrow \mathbb{R}$ are determined –for $w = \psi(p,u^*,\xi)$– by

$$\begin{array}{rcl} x^i(p,u^*,\xi) & = & u^i(\pi_{*w}(X(w))) \\ x^{2n+i}(p,u^*,\xi) & = & K^*_w(X(v))(u_i) \end{array} \tag{4.4.1}$$

for $1 \leqslant i \leqslant 2n$.

¿From (4.4.1), one gets that

$${}^\nabla X \circ R_a = {}^\nabla X. \begin{pmatrix} (a^t)^{-1} & 0 \\ 0 & a \end{pmatrix} \tag{4.4.2}$$

for any $a \in \mathcal{G}$.

As in [1], for any $(0,2)-$tensor field G on T^*M, we define the differentiable function

$${}^\nabla G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix} : \mathcal{N} \longrightarrow \mathbb{R}^{4n \times 4n}$$

as follows: if $(p,u^*,\xi) \in \mathcal{N}$ and $w = \psi(p,u^*,\xi)$, let ${}^\nabla G(p,u^*,\xi)$ be the matrix of the bilinear form $G_w : (T^*M)_w \times (T^*M)_w \longrightarrow \mathbb{R}$ induced by G on $(T^*M)_w$ with respect to the basis $\{e_1(p,u^*,\xi), \ldots, e_{4n}(p,u^*,\xi)\}$.

Hence, for any pair of vector fields X, Y on T^*M one gets

$$G(X,Y) \circ \psi = {}^\nabla X. {}^\nabla G. ({}^\nabla Y)^t \tag{4.4.3}$$

Equalities (4.4.2) and (4.4.3) imply that each $A_i : \mathcal{N} \longrightarrow \mathbb{R}^{2n \times 2n}$ satisfies the following $\mathcal{G}-$invariance property

$$\begin{array}{rcl} A_1 \circ R_a & = & a^t.A_1.a \\ A_2 \circ R_a & = & a^t.A_2.(a^t)^{-1} \\ A_3 \circ R_a & = & a^{-1}.A_3.(a^{-1})^t \\ A_4 \circ R_a & = & a^{-1}.A_4.a \end{array} \tag{4.4.4}$$

We shall call ${}^\nabla G$ the matrix of G with respect to (ω, ∇). Hence, we get a one to one correspondence "${}^\nabla G \longleftrightarrow T''$ between $(0,2)-$tensor fields on T^*M and differentiable functions $T = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix} : \mathcal{N} \longrightarrow \mathbb{R}^{4n \times 4n}$ where A_i satisfies (4.4.4). The differentiability of G –for T given– follows from (4.4.3) and the fact that ψ is a submersion.

We define G to be natural with respect to (ω, ∇) if ${}^\nabla G$ only depends on ξ.

Proposition 4.1. *Let G be a $(0,2)-$tensor field on T^*M and* $^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$ *the matrix of G with respect to (ω, ∇). Then, G is natural wigh respect to (ω, ∇) if there exist real numbers $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$ such that*

$$A_1(p, u^*, \xi) = \alpha_1.S + \beta_1.(\xi^t).\xi \tag{4.4.5}$$

$$A_2(p, u^*, \xi) = \alpha_2.I_{2n} + \beta_2.(\xi^t).(\xi.S) \tag{4.4.6}$$

$$A_3(p, u^*, \xi) = \alpha_3.S + \beta_3.(\xi.S)^t.(\xi.S) \tag{4.4.7}$$

$$A_4(p, u^*, \xi) = \alpha_4.I_{2n} + \beta_4.(\xi.S)^t.\xi \tag{4.4.8}$$

Proof. By setting $B_i(\xi) = A_i(p, u^*, \xi)$, from (4.4.4) it follows that the functions $B_i : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n \times 2n}$ satisfy

$$B_1(\xi.a) = a^t.B_1(\xi).a \tag{4.4.9}$$

$$B_2(\xi.a) = a^t.B_2(\xi).(a^t)^{-1} \tag{4.4.10}$$

$$B_3(\xi.a) = a^{-1}.B_3(\xi).(a^{-1})^t \tag{4.4.11}$$

$$B_4(\xi.a) = a^{-1}.B_4(\xi).a \tag{4.4.12}$$

Since $a^{-1} = -S.a^t.S$ if $a \in \mathcal{G}$, equalities (4.4.9) to (4.4.12) imply that the matrix functions $S.B_1.S$, $S.B_2$, B_3 and $B_4.S$ satisfy Theorem 2.1. This implies equalities (4.4.5) to (4.4.8). □

Remark 4.1. Let θ be the canonical $1-$form on T^*M which is defined for any vector field X on T^*M an any co-vector $w \in T^*M$ by

$$\theta(X)(w) = w(\pi_{*w}(X(w))) \tag{4.4.13}$$

On the other hand, for any $p \in M$, let $L_p : M_p \longrightarrow M_p^*$ be the isomorphism induced by ω; i.e.,

$$L_p(v)(u) = \omega(p)(v, u) \qquad \text{for any } v, u \in M_p$$

Hence, ω induces a $(2,0)-$tensor field ω^* on M by defining

$$\omega^*(p)(w, \gamma) - \omega(p)(L_p^{-1}(w), L_p^{-1}(\gamma)) \tag{4.4.14}$$

for any $w, \gamma \in M_p^*$.

In terms of θ, ω and ω^*, one gets

Corollary 4.2. *Let G be a $(0,2)-$tensor field on T^*M. Then, G is natural if there exist real numbers $\alpha_i, \beta_i \in \mathbb{R}$ such that for any vector fields X, Y on T^*M, the following equalities hold*

$$\begin{aligned}
G(X^h, Y^h)(w) &= \alpha_1 \,.\, \omega(p)(\pi_{*w}(X(w)), \pi_{*w}(Y(w))) \\
&\quad + \beta_1 \,.\, \theta(X)(w) \,.\, \theta(Y)(w) \\
G(X^h, Y^v)(w) &= \alpha_2 \,.\, K_w^*(Y(w))(\pi_{*w}(X(w))) \\
&\quad + \beta_2 \,.\, \theta(X)(w) \,.\, \omega^*(p)(w, K_w^*(Y(w))) \\
G(X^v, Y^h)(w) &= \alpha_4 \,.\, K_w^*(X(w))(\pi_{*w}(Y(w))) \\
&\quad + \beta_4 \,.\, \theta(Y)(w) \,.\, \omega^*(p)(w, K_w^*(X(w))) \\
G(X^v, Y^v)(w) &= \alpha_3 \,.\, \omega^*(p)(K_w^*(X(w)), K_w^*(Y(w))) \\
&\quad + \beta_3 \,.\, \omega^*(p)(w, K_w^*(X(w))) \,.\, \omega^*(p)(w, K_w^*(Y(w)))
\end{aligned}$$

if $w \in M_p^*$

Remark 4.2. Let $\phi : TM \longrightarrow T^*M$ be the diffeomorphism induced by ω; i.e., $\phi(v)(u) = \omega(p)(v,u)$ if $v, u \in M_p$.

Since the diagram

$$\begin{array}{ccc} (TM)_v & \xrightarrow{K_v} & M_p \\ \phi_{*v}\downarrow & & \downarrow L_P \\ (T^*M)_w & \underset{K_w^*}{\longrightarrow} & M_p^* \end{array}$$

commutes, where $w = \phi(v)$. From Proposition 3.1 and Corollary 4.2, it follows that naturatily of $(0,2)-$tensor fields on TM and T^*M is preserved under the pull-back of ϕ.

Remark 4.3. Assume that $(M, <,>, J)$ is a Semi-Riemannian Kähler manifold. As we pointed out in the Introduction, (M, ω, J) is then a Fedosov manifold. ¿From Proposition 3.1 and Proposition 3.1 of [1], it follows —after a straightforward computation— that the only $(0,2)-$tensor field on TM which is natural with respect to $(M, <,>)$ and (M, ω) is the null tensor. Consequently, by Remark above, this is also true for $(0,2)-$tensor fields on T^*M.

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