Einstein equations for (h, v)-Berwald-Moor relativistic models

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Abstract. The paper determines basic relations between the metric canonically induced by the Berwald-Moor Finsler structure, the normalized flag Generalized Lagrange metric and the Pavlov poly-scalar product. Then, in the framework of vector bundles endowed with $(h, v)-$metrics, the extended Einstein equations are obtained for both the associated Generalized Lagrange and the Euclidean-Berwald-Moor models.

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1 Introduction

Let M be a 4-dimensional differential manifold of class $\mathcal{C}^\infty$, (TM, π, M) its tangent bundle and (x^i, y^i) local coordinates in TM. Let $F : TM \to R$, $F = F(y)$ be a locally Minkowski Finsler function ([8], [7]). Then we consider the induced fundamental metric tensor field

$$g^*_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = FF_{ij} + F_i F_j, \ \ i, j = \overline{1, 4}, \tag{1.1}$$

where we denote

$$F_i = \frac{\partial F}{\partial y^i}, \ F_{ij} = \frac{\partial^2 F}{\partial y^i \partial y^j}, \ F_{ijk} = \frac{\partial^3 F}{\partial y^i \partial y^j \partial y^k}, \text{ etc.} \tag{1.2}$$

For $M = \mathbb{R}^4$ and the Finsler function specialized to

$$F(y) = \sqrt[4]{|y^1 y^2 y^3 y^4|}, \ y^i \neq 0, i = \overline{1, 4}, \tag{1.3}$$

which is a particular case of the Shimada Finsler metric ([14, 15, 5, 4, 6])

$$F(x, y) = \sqrt[n]{a_{i_1 i_2 \ldots i_n}(x) y^{i_1} y^{i_2} \ldots y^{i_n}},$$

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where $a_{i_1 i_2 \dots i_n}$ is a $(0, n)$ tensor field on M, D.G. Pavlov has studied the "4-pseudoscalar product" ([11]) related to the Berwald-Moor metric (1.3),

$$(X, Y, Z, T) = G_{ijkl} X^i Y^j Z^k T^l, \tag{1.4}$$

where

$$G_{ijkl} = \frac{1}{4!} \frac{\partial^4 \mathcal{L}}{\partial y^i \partial y^j \partial y^k \partial y^l}, \quad \mathcal{L} = F^4. \tag{1.5}$$

In the folowing, we consider (1.4) and (1.5) for an arbitrary Finsler function $F = F(y)$. Starting from here, we will construct a generalized Lagrange space based on the tensor field (1.4).

First, we notice that the tensor field (1.5) satisfies the following conditions:

1. G_{ijkl} is totally symmetric w.r.t. the indices i, j, k, l;
2. $G_{ijk0} \equiv G_{ijkl} y^l = \dfrac{1}{4!} \dfrac{\partial^3 F^4}{\partial y^i \partial y^j \partial y^k}$ is 1-homogeneous in y;
3. $G_{ij00} \equiv G_{ijkl} y^k y^l = \dfrac{1}{12} \dfrac{\partial^2 F^4}{\partial y^i \partial y^j}$ is 2-homogeneous in y;
4. $G_{i000} \equiv G_{ijkl} y^j y^k y^l = \dfrac{1}{4} \dfrac{\partial F^4}{\partial y^i}$ is 3-homogeneous in y;
5. $G_{0000} \equiv G_{ijkl} y^i y^j y^k y^l = F^4$ is 4-homogeneous,

where the null index denotes the transvection with the directional argument y. The properties from above are direct consequences of the 1−homogeneity of F. The following relations are stragihtforward

$$\begin{cases} \mathcal{L}_i & = 4F^3 F_i \\ \mathcal{L}_{ij} & = 4(3F^2 F_i F_j + F^3 F_{ij}) \\ \mathcal{L}_{ijk} & = 4[6F F_i F_j F_k + 3F^2 \underset{ijk}{S}(F_i F_{jk}) + F^3 F_{ijk}] \\ \mathcal{L}_{ijkl} & = 4[6F_i F_j F_k F_l + F^3 F_{ijkl} + 6F S'(F_{ij} F_k F_l) + \\ & \quad +3F^2 \left[\underset{ijkl}{S}(F_{ij} F_{kl}) + \underset{ijkl}{S}(F_l F_{ijk}) \right], \end{cases}$$

where the lower index of F represents partial derivative with respect to the corresponding directional variable, and we denoted by S cyclic summation about the indices involved, and by S' distinct pairwise summation of 6 terms about the four indices. We define the pseudo-scalar product

$$\langle X, Y \rangle_y = \frac{1}{F^2}(X, Y, y, y), \quad X, Y \in \mathcal{X}(M),$$

where $y = y^i \dfrac{\partial}{\partial y^i}$ is the Liouville vector field ([7]) and the vector fields X, Y are considered at some point $x \in M$.

It is obvious that $\langle\ ,\ \rangle_y$ is bilinear in the two arguments and (e.g., for the Berwald-Moor metric) it satisfies the axioms of a pseudo-scalar product. We locally have

$$\langle X,Y\rangle_y = \frac{1}{F^2}G_{ijkl}X^iY^jy^ky^l = \frac{G_{ij00}}{F^2}X^iY^j,$$

and hence the coefficients of this pseudo-scalar product can be expressed as

$$g_{ij} = \frac{G_{ij00}}{F^2} = \frac{1}{12F^2}\frac{\partial^2 F^4}{\partial y^i \partial y^j} = \frac{1}{3}FF_{ij} + F_iF_j. \tag{1.6}$$

Then g_{ij} is a 2-covariant non-degenerate 0-homogeneous tensor field (called further *normalized flag metric*), which defines a *generalized Lagrange space* (M,g).

We note that though the associated to g Cartan tensor field

$$C_{ijk} = \frac{1}{2}\left[\frac{1}{3}(F_kF_{ij} + FF_{ijk}) + F_{ik}F_j + F_iF_{jk}\right] = \frac{1}{2}\left[\left(\frac{1}{3}FF_{ijk} + \underset{ijk}{S}\, F_iF_{jk}\right) - \frac{2}{3}F_kF_{ij}\right]$$

satisfies $C_{0jk} = C_{i0k} = C_{ij0} = 0$, it is still *non-symmetric* in its three indices. Hence the metric g_{ij} is *not* a Finsler fundamental tensor field, but a proper Generalized Lagrange metric. We remark that since F is $0-$homogeneous in y, it follows by using the Euler relations that $F_iy^i = F$ and $F_{ij}y^j = 0$. Then the absolute energy attached to g_{ij} is F^2, since

$$\mathcal{E} = g_{ij}y^iy^j = \left(\frac{1}{3}FF_{ij} + F_iF_j\right)y^iy^j = F^2. \tag{1.7}$$

Then the Lagrange metric associated to g via its energy is

$$\frac{1}{2}\frac{\partial^2\mathcal{E}}{\partial y^i\partial y^j} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = g^*_{ij}, \tag{1.8}$$

and then $(M, \mathcal{E} = F^2)$ is a *Lagrange space*.

From the homogeneity of F it also follows that

$$\frac{1}{2}\frac{\partial \mathcal{E}}{\partial y^i} = g_{ij}y^j. \tag{1.9}$$

Consequently, we have

Theorem 1. *a)* (M,g) *is a generalized Lagrange space with regular metric. The Finslerian metric* g^*_{ij} *provided by the energy* $\mathcal{E} = F^2$ *is related to the normalized flag metric* g *via:*

$$g^*{}_{ij} = g_{ij} + \frac{2}{3}FF_{ij}.$$

b) The families of metrics $\Sigma_\lambda :\ \tilde{g}_{ij} = g_{ij} + \lambda FF_{ij},\ \lambda \in \mathbb{R}$ *and* $\Sigma_\mu :\ \hat{g}_{ij} = \mu g_{ij} + (1-\mu)g^*_{ij},\ \mu \in \mathbb{R}$ *have the same energy* $\mathcal{E} = F^2$ *and include the metrics* g^* *and* g*, whence in particular*

$$\mathcal{E} = F^2 = g_{ij}y^iy^j = g^*_{ij}y^iy^j.$$

Proof. a) The relations (1.9) and (1.8) provide the first claim, while (1.1) and (1.6), the second. Using an argument similar to (1.7), b) follows. □

For the case when F is the Berwald-Moor metric, the matrices attached to g and to its dual have the particular form:

$$[g] = \frac{1}{12F^2}\begin{pmatrix} 0 & cd & bd & bc \\ cd & 0 & ad & ac \\ bd & ad & 0 & ab \\ bc & ac & ab & 0 \end{pmatrix}, \quad [g^{-1}] = 4F^2 \begin{pmatrix} -\frac{2a}{bcd} & \frac{1}{cd} & \frac{1}{db} & \frac{1}{cb} \\ \frac{1}{cd} & -\frac{2b}{cda} & \frac{1}{da} & \frac{1}{ca} \\ \frac{1}{db} & \frac{1}{da} & -\frac{2c}{dab} & \frac{1}{ba} \\ \frac{1}{cb} & \frac{1}{ca} & \frac{1}{ba} & -\frac{2d}{abc} \end{pmatrix}.$$

where we denoted $(a, b, c, d) = (y^1, y^2, y^3, y^4)$. Then one can easily check that $\det[g] = -3(abcd)^2/(12F^2)^4 < 0$ for $abcd \neq 0$. As well, the signature of $[g]$ is $(+, -, -, -)$, as clearly show its Maple 9.5 - derived eigenvalues, which are the roots

$$RootOf(\_Z^4 + (-a^2 * b^2 - a^2 * c^2 - c^2 * d^2 - b^2 * d^2 - a^2 * d^2 - b^2 * c^2) * \_Z^2 +$$
$$+(-2 * c^3 * d * b * a - 2 * a^3 * d * c * b - 2 * b^3 * d * c * a - 2 * c * d^3 * b * a) * \_Z -$$
$$-3 * c^2 * d^2 * a^2 * b^2).$$

The above construction in (1.5) can be generalized to an n-dimensional manifold M, as the "poly-pseudo-scalar product"

$$(X_1, X_2, ..., X_n) = G_{i_1 \dots i_n} X^{i_1} ... X^{i_n},$$

with

$$G_{i_1 \dots i_n} = \frac{1}{n!} \frac{\partial^n F^n}{\partial y^{i_1} ... \partial y^{i_n}}.$$

This relates to the generalized Lagrange geometry by defining the pseudo-scalar product

$$\langle X, Y \rangle = \frac{1}{F^{n-2}}(X, Y, y, \dots, y), \quad X, Y \in \mathcal{X}(M),$$

having the local components

$$g_{ij} = \frac{1}{F^{n-2}} G_{ij0\dots0} = \frac{1}{n(n-1)F^{n-2}} \frac{\partial^2 F^n}{\partial y^i \partial y^j}.$$

2 Links between g, g^* and G.

We shall first establish the relation between the generalized Lagrange metric g and the Finsler one g^*. For this purpose, we use a property of regular generalized Lagrange metrics ([7]):

$$g^*_{ij} = g_{ij} + \frac{\partial g_{ik}}{\partial y^j} y^k. \tag{2.1}$$

Taking into account that $\mathcal{E} = F^2$, we can write g in the more convenient form

$$g_{ij} = \frac{1}{12\mathcal{E}} \frac{\partial^2 \mathcal{E}^2}{\partial y^i \partial y^j}. \tag{2.2}$$

Using (1.2), (1.6) and ${g^*}_{i0} = FF_i$, ${g^*}_{00} = F^2$, where $g_{i0} = g_{ij} y^j$ etc., one easily infers the relation

(2.3)
$$g_{ij} = \frac{1}{3}\left(g^*_{ij} + 2\frac{g^*_{i0}\cdot g^*_{j0}}{g^*_{00}}\right).$$

We shall now express G_{ijkl} in terms of g^*. By a straightforward computation, we obtain

(2.4)
$$4!G_{ijkl} = 2\underset{ijkl}{S}\,\mathcal{E}_{ijk}\mathcal{E}_l + 2(\mathcal{E}_{ij}\mathcal{E}_{kl} + \mathcal{E}_{ik}\mathcal{E}_{jl} + \mathcal{E}_{il}\mathcal{E}_{jk}) + 2\mathcal{E}\mathcal{E}_{ijkl},$$

where the low indices of $\mathcal{E}$ mean derivation with the corresponding components of y. If in the above equality we replace

$$\mathcal{E} = g^*_{00},\quad \mathcal{E}_i = 2g^*_{i0},\ \mathcal{E}_{ij} = 2g^*_{ij},\quad \mathcal{E}_{ijk} = 2g^*_{ij,k} = 2\frac{\partial g^*_{ij}}{\partial y^k},$$

we obtain the components of the 4-scalar product in the alternative form

$$G_{ijkl} = \frac{1}{3!}\left[2\underset{ijkl}{S}(2g^*_{ij,k}g^*_{l0}) + 2(g^*_{ij}g^*_{kl} + g^*_{ik}g^*_{jl} + g^*_{il}g^*_{jk}) + g^*_{00}g^*_{ij,kl}\right].$$

Let us denote, for $X, Y, Z, T \in \mathcal{X}(M)$, $g^*_{XY} = g^*_{ij}X^iY^j$, and $G_{XYZT} = G_{ijkl}X^iY^jZ^kT^l$. Consequently, the basic multiple transvections of the 4- scalar product involved in the conformal properties of the Berwald-Moor space ([12]) are

$$(X,X,Y,Y) = G_{XXYY} = \tfrac{1}{3!}\Big[2\underset{X,X,Y,Y}{S} g^*_{XX,Y}g^*_{Y0} +$$
$$+2(g^*_{XX}g^*_{YY} + 2(g^*_{XY})^2) + g^*_{00}g^*_{XX,YY}\Big]$$

and we have as well

$$(X,X,X,Y) + (X,Y,Y,Y) = G_{XXXY} + G_{XYYY} =$$
$$= \tfrac{1}{3!}\left[2(g^*_{XX,X}g^*_{Y0} + g^*_{YY,Y}g^*_{X0}) + 6(g^*_{XY,X}g^*_{X0} + g^*_{YX,Y}g^*_{Y0}) +\right.$$
$$\left. +6g^*_{XY}(g^*_{XX} + g^*_{YY}) + g^*_{00}(g^*_{XX,XY} + g^*_{XY,YY})\right].$$

3 The Berwald-Moor case

For the sake of simplicity, we restrict ourselves to the case when $y^1y^2y^3y^4 > 0$. For F as in (1.3), we obtain $G_{ijkl} = 1/4!$, i.e., the 4-linear form defined in (1.5) on the space-time

$$(X,Y,Z,T) = \frac{1}{4!}X^{i_1}Y^{i_2}Z^{i_3}T^{i_4}\varepsilon_{i_1i_2i_3i_4},$$

where $\varepsilon_{i_1i_2i_3i_4}$ is 1 for i_1, i_2, i_3, i_4 different in pairs, and 0 else.

In the following, we maintain the convention to denote by i_1, i_2, i_3, i_4 the distinct values from 1 to 4 ($i_j \neq i_k$ for $j \neq k$). The absolute energy of M is then

$$\mathcal{E} = \sqrt{y^1y^2y^3y^4},$$

and the generalized Lagrange metric tensor given by (1.6) g_{ij}, which we call normalized flag Berwald-Moor metric, takes the form

$$g_{ii} = 0, \quad i = \overline{1,4}, \quad g_{i_1 i_2} = \frac{y^{i_3} y^{i_4}}{12\mathcal{E}}, \quad i_1 \neq i_2. \tag{3.1}$$

The inverse matrix g^{ij} has the components

$$g^{ii} = \frac{-8(y^i)^2}{\mathcal{E}}, \quad i = \overline{1,4}; \qquad g^{i_1 i_2} = \frac{4\mathcal{E}}{y^{i_3} y^{i_4}} = \frac{4 y^{i_1} y^{i_2}}{\mathcal{E}}, \quad i_1 \neq i_2.$$

For the associated Finsler metric, we have: $g^*_{i_1 i_2} = \dfrac{y^{i_3} y^{i_4}}{8\mathcal{E}}$, and $g^*_{ii} = -\dfrac{\mathcal{E}}{8(y^i)^2}$. It is worthy to notice that, for $i_1 \neq i_2$, we have $g_{i_1 i_2} = \dfrac{2}{3} g^*_{i_1 i_2}$. Let

$$C_{hjk} = g_{ih} C^i{}_{jk} = \frac{1}{2}\left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{kh}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right).$$

Then, for distinct i_1, i_2, i_3, we get

$$C_{i_1 i_2 i_3} = \frac{1}{3}\left(\frac{\partial g^*_{i_2 i_1}}{\partial y^{i_3}} + \frac{\partial g^*_{i_3 i_1}}{\partial y^{i_2}} - \frac{\partial g^*_{i_2 i_3}}{\partial y^{i_1}}\right) = \frac{1}{3}\left(\frac{1}{2}\mathcal{E}_{i_2 i_1 i_3} + \frac{1}{2}\mathcal{E}_{i_2 i_1 i_3} - \frac{1}{2}\mathcal{E}_{i_2 i_1 i_3}\right),$$

and hence $C_{i_1 i_2 i_3} = \frac{1}{6}\mathcal{E}_{i_2 i_1 i_3}$. In the same way, it follows that

$$C_{i_1 i_1 i_2} = 0 = C_{i_1 i_2 i_1}, \quad C_{i_2 i_1 i_1} = \frac{1}{3}\mathcal{E}_{i_1 i_1 i_2}, \quad C_{i_1 i_1 i_1} = 0.$$

We obtain now the coefficients $C^i{}_{jk} = g^{ih} C_{hjk}$ in terms of the energy $\mathcal{E}$ as:

$$\begin{cases} C^{i_1}{}_{i_2 i_3} = \dfrac{2}{3\mathcal{E}}(-2(y^{i_1})^2 \mathcal{E}_{i_1 i_2 i_3} + y^{i_1} y^{i_4} \mathcal{E}_{i_2 i_3 i_4}) \\ C^{i_1}{}_{i_1 i_2} = \dfrac{2}{3\mathcal{E}}(y^{i_1} y^{i_3} \mathcal{E}_{i_1 i_2 i_3} + y^{i_1} y^{i_4} \mathcal{E}_{i_1 i_2 i_4}) \\ C^{i_1}{}_{i_2 i_2} = \dfrac{4}{3\mathcal{E}}(-2(y^{i_1})^2 \mathcal{E}_{i_1 i_2 i_2} + y^{i_1} y^{i_3} \mathcal{E}_{i_2 i_2 i_3} + y^{i_1} y^{i_4} \mathcal{E}_{i_2 i_2 i_4}) \\ C^{i_1}{}_{i_1 i_1} = \dfrac{4}{3\mathcal{E}}(y^{i_1} y^{i_2} \mathcal{E}_{i_2 i_1 i_1} + y^{i_1} y^{i_3} \mathcal{E}_{i_3 i_1 i_1} + y^{i_1} y^{i_4} \mathcal{E}_{i_4 i_1 i_1}). \end{cases} \tag{3.2}$$

4 Einstein equations for Berwald-Moore type $(h, v)-$models

The considerations within the current section apply to any locally Minkowski Finsler function, including the Berwald-Moor fundamental function as a particular case. Due to the fact that F is locally Minkovski, it follows that the coefficients $N^i{}_j$ of the Kern nonlinear connection ([7]) vanish. As well, the canonical linear $d-$connection $C\Gamma(N) \equiv \{L^i_{jk}, C^i_{jk}\}$ for the Generalized Lagrange space (M, g) described by

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2} g^{ih}\left(\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right), \\ C^i{}_{jk} &= \frac{1}{2} g^{ih}\left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{kh}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right), \end{aligned} \tag{4.1}$$

has all its horizontal coefficients $L^i{}_{jk}$ zero and the components of its torsion vanish, except $hT(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}) = C^i{}_{jk}\frac{\delta}{\delta x^i}$. The coefficients of its curvature tensor are ([7]) $R^i_{j\ kh} = P^i_{j\ kh} = 0$, and

$$S^a_{bcd} = \dot{\partial}_{[d}C^a_{bc]} + C^a_{s[d}C^s_{bc]}, \tag{4.2}$$

where $\dot{\partial}_d$ is the partial w.r.t. y^d and we denoted $\tau_{[i\ldots j]} = \tau_{i\ldots j} - \tau_{j\ldots i}$.

In general, the Einstein equations for a (h, v)-metric (h, g) on TM have the form ([8])

$$\begin{cases} R_{ij} - \frac{1}{2}(R+S)h_{ij} = T^H_{ij} \\ P^1_{bj} = T^{M_1}_{bj}, \ \ P^2_{bj} = T^{M_2}_{jb}, \\ S_{ab} - \frac{1}{2}(R+S)g_{ab} = T^V_{ab}, \end{cases}$$

where $R_{ij}, P^1_{ij}, P^2_{ij}$ and S_{ab} are the Ricci $d-$tensors attached to the canonic connection, R, S are the scalars of curvature and $T^H_{ij}, T^{M_1}_{ij}, T^{M_2}_{ij}$ and T^V_{ij} are the energy-momentum d-tensor fields. Then, for the locally Minkovski model (M, g), given by the particular case when the $(h, v)-$metric (h, g) has $h = g = g(y)$, the following holds true:

Theorem 2. *The Einstein mixed tensors of the Generalized Lagrange model attached to the locally Minkowski model (M, g) identically vanish, and the Einstein equations are*

$$\begin{cases} -\frac{1}{2}Sg_{ij} = T^H_{ij}, \ \ 0 = T^{M_1}_{bj}, \ \ 0 = T^{M_2}_{jb} \\ E_{ab} \equiv S_{ab} - \frac{1}{2}Sg_{ab} = T^V_{ab}, \end{cases} \tag{4.3}$$

where the vertical Einstein tensor has the specific form

$$E_{ab} = S^p_{rst}\delta^t_p(\delta^r_a\delta^s_b - \frac{1}{2}g^{rs}g_{ab}), \tag{4.4}$$

with S^p_{rst} given by (4.2) and C^a_{bc} by (4.1), and where g^{rs} is the dual of g_{ab}.

In the case when the $(h, v)-$metric has its horizontal part Euclidean, of coefficients $h_{ij},\ i, j = \overline{1, n}$, then the canonic linear $d-$connection $C\Gamma(N) \equiv \{L^i_{jk}, L^a_{bk}, C^i_{ja}, C^a_{bc}\}$ has the first three sets of coefficients zero and all its torsion components vanish; the same holds true for the curvature, except the set $S_b{}^a{}_{cd}$ given in (4.2). In this case we have

Theorem 3. *The Einstein equations for the (h, v) Einstein-locally Minkowski metric $(h_{ij}, g_{ij}(y))$ write*

$$\begin{cases} -\frac{1}{2}Sh_{ij} = T^H_{ij}, \ \ 0 = T^{M_1}_{bj}, \ \ 0 = T^{M_2}_{jb} \\ E_{ab} \equiv S_{ab} - \frac{1}{2}Sg_{ab} = T^V_{ab}, \end{cases} \tag{4.5}$$

with (4.2) and (4.4) satisfied.

We note that in the case when g is of Berwald-Moor type (3.1), the equations (4.3) and (4.5) have the vertical coefficients C^a_{bc} involved in (4.4)-(4.2) specialized by (3.2).

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