Some inequalities satisfied by periodical solutions of multi-time Hamilton equations

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Abstract. The objective of this paper is to find some inequalities satisfied by periodical solutions of multi-time Hamilton systems, when the Hamiltonian is convex. To our knowledge, this subject of first-order field theory is still open.

Section 1 recall well-known facts regarding the equivalence between Euler-Lagrange equations and Hamilton equations and analyses the action that produces multi-time Hamilton equations, emphasizing the role of the polysymplectic structure. Section 2 extends two inequalities of [21] from a cube to parallelipiped and proves two inequalityes concerning multiple periodical solutions of multi-time Hamilton equations.

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1 Multi-time Hamilton equations and polysymplectic structure

The paper studies the solutions with multiple periodicity of the Hamilton multi-time equations.

A function $u = \left(u^1, ..., u^n\right)$ with many variables $\left(t^1, ..., t^p\right)$, is multiple periodical with the period $T = \left(T^1, ..., T^p\right) \in R^p$ if

$$u\left(t^1 + k_1 T^1, ..., t^p + k_p T^p\right) = u\left(t^1, ..., t^p\right),$$

where $k_1, ..., k_p$ are integers. We consider the function u defined on the parallelepiped $T_0 = \left[0, T^1\right] \times \left[0, T^2\right] \times ... \times \left[0, T^p\right] \subset R^p$, with values in R^n. We will denote by $T = \left(T^1, ..., T^p\right) \in R^p$. The existence of the weak gradient of the function u assures the multiple periodicity of the function u. We use the Hilbert space H_T^1 attached to the Sobolev space $W_T^{1,2}$ of the functions $u \in L^2\left(T_0, R^n\right)$ which have a weak gradient $\dfrac{\partial u}{\partial t} \in L^2\left(T_0, R^n\right)$. The Wirtinger inequality from this paper has a specific form because of the multidimensional character of the definition domain T_0. The inequalities

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from theorems 3 and 4 constitute generalizations of some theorems of [5], from the particular case $p = 1$ to an arbitrary p.

The Euclidean structure on R^n is based on the scalar product $(u,v) = \delta_{ij} u^i v^j$, and the norm $|u| = \sqrt{\delta_{ij} u^i u^j}$. The Hilbert space H^1_T is endowed with the scalar product

$$\langle u,v\rangle = \int_{T_0} \left(\delta_{ij} u^i (t) v^j (t) + \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha} (t) \frac{\partial v^j}{\partial t^\beta} (t) \right) dt^1 \wedge ... \wedge dt^p,$$

and the corresponding norm $\sqrt{\langle u,u\rangle} = \|u\|$.

1.1 Multi-time Hamilton equations

We consider the multi-time variable $t = \left(t^1,...,t^p\right) \in T_0 \subset R^p$, the functions $x^i : R^p \to R$, $\left(t^1,...,t^p\right) \to x^i \left(t^1,...,t^p\right)$, $i = 1,...,n$, and the partial velocities $x^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha}$, $\alpha = 1,...,p$.

Definition 1 The PDEs

$$\frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^i_\alpha} = \frac{\partial L}{\partial x^i}, \quad i = 1,...,n, \quad \alpha = 1,...,p$$

(second order PDEs system on the n-dimensional space) are called Euler-Lagrange equations for the Lagrangian

$$L : R^{p+n+np} \to R, \quad \left(t^\alpha, x^i, x^i_\alpha\right) \to L\left(t^\alpha, x^i, x^i_\alpha\right)$$

The Hamilton equations in the multi-time case are obtained using the partial derivatives (polymomenta)

$$p^\alpha_k = \frac{\partial L}{\partial x^k_\alpha} \tag{1}$$

and the Hamiltonian $H = p^\alpha_k x^k_\alpha - L$. If L satisfies some regularity conditions, then the system (1) defines a C^1 bijective transformation $x^i_\alpha \to p^\alpha_i$, called the Legendre transformation for the multi-time case. By this transformation we have

$$\frac{\partial H}{\partial p^\alpha_i} = x^i_\alpha + p^\beta_k \frac{\partial x^k_\beta}{\partial p^\alpha_i} - \frac{\partial L}{\partial x^k_\beta} \frac{\partial x^k_\beta}{\partial p^\alpha_i} = x^i_\alpha$$

$$\frac{\partial H}{\partial x^i} = p^\alpha_k \frac{\partial x^k_\alpha}{\partial x^i} - \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial x^k_\alpha} \frac{\partial x^k_\alpha}{\partial x^i} = -\frac{\partial L}{\partial x^i}.$$

Consequently, the $np + n$ Hamilton equations

$$\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p^\alpha_i},$$

$$\frac{\partial p^\alpha_i}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}$$

(summation after α), $i = 1,...,n$, $\alpha = 1,...,p$ are first order PDEs on the space R^{n+pn}, equivalent to the Euler-Lagrange equations on R^n.

There are different point of views to study these equations which appear in first-order field theory (see [1]-[3], [8]-[10], [12]-[22]). In our context, we need of Hilbert-Sobolev space methods for PDEs ([4], [6], [11]).

Let us write the multi-time Hamilton equations in the form

$$\delta^\alpha_\beta \delta^i_j \frac{\partial p^\beta_i}{\partial t^\alpha} + \frac{\partial H}{\partial x^j} = 0, \quad -\delta^\alpha_\beta \delta^i_j \frac{\partial x^j}{\partial t^\alpha} + \frac{\partial H}{\partial p^\beta_i} = 0, i, j = 1, ..., n; \alpha, \beta = 1, ..., p$$

or

$$(\delta \otimes J) \frac{\partial u}{\partial t} = -\nabla H, \tag{2}$$

where

$$\delta \otimes J = \begin{pmatrix} 0 & \delta^\alpha_\beta \delta^i_j \\ -\delta^\alpha_\beta \delta^i_j & 0 \end{pmatrix}, \frac{\partial u}{\partial t} = \begin{pmatrix} \dfrac{\partial x^j}{\partial t^\alpha} \\ \dfrac{\partial p^\beta_i}{\partial t^\alpha} \end{pmatrix}, \nabla H = \begin{pmatrix} \dfrac{\partial H}{\partial x^j} \\ \dfrac{\partial H}{\partial p^\beta_j} \end{pmatrix}.$$

The operator $\delta \otimes J$ is a polysymplectic structure acting on R^{np+np^2} with values in R^{n+np}. The (1,2)-block $\delta^\alpha_\beta \delta^i_j$ acts linearly by δ^i_j and tracely by δ^α_β. The (2,1)-block $-\delta^\alpha_\beta \delta^i_j$ acts lineraly both by δ^i_j and δ^α_β. The operator $\delta \otimes J$ induces a multisymplectic PDE operator $(\delta \otimes J)\dfrac{\partial}{\partial t}$ which work as follows

$$(\delta \otimes J)\frac{\partial}{\partial t} \begin{pmatrix} x \\ p \end{pmatrix} : \begin{pmatrix} 0 & \delta^\alpha_\beta \delta^i_j \\ -\delta^\alpha_\beta \delta^i_j & 0 \end{pmatrix} \frac{\partial}{\partial t^\alpha} \begin{pmatrix} x^j \\ p^\beta_i \end{pmatrix} = \begin{pmatrix} \dfrac{\partial p^\alpha_j}{\partial t^\alpha} \\ -\dfrac{\partial x^j}{\partial t^\beta} \end{pmatrix}.$$

Repeating we obtain the square

$$(\delta \otimes J)\frac{\partial}{\partial t} \begin{pmatrix} \mathrm{div} p \\ -\dfrac{\partial x}{\partial t} \end{pmatrix} : \begin{pmatrix} 0 & \delta^\alpha_\beta \delta^i_j \\ -\delta^\alpha_\beta \delta^i_j & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial^2 p^\gamma_i}{\partial t^\alpha \partial t^\gamma} \\ -\dfrac{\partial^2 x^j}{\partial t^\alpha \partial t \beta} \end{pmatrix} = \begin{pmatrix} -\Delta x^i \\ -\dfrac{\partial^2 p^\gamma_i}{\partial t^\beta \partial t^\gamma} \end{pmatrix}.$$

1.2 The action that produces multi-time Hamilton equations

We consider a Hamiltonian $H : T_0 \times R^n \times R^{np} \to R, (t, u) \to H(t, u)$ whose restriction $H(t, \cdot)$ is C^1 and convex.

Theorem 1. [21] *Let $u = (x, p)$. The action Ψ, whose Euler-Lagrange equations are the Hamilton equations, is*

$$\Psi(u) = \int_{T_0} \mathcal{L}\left(t, u, \frac{\partial u}{\partial t}\right) dt^1 \wedge ... \wedge dt^p,$$

$$\mathcal{L}\left(t, u, \frac{\partial u}{\partial t}\right) == -\frac{1}{2} G\left(\delta \otimes J \frac{\partial u}{\partial t}, u\right) - H(t, u),$$

where the scalar product is represented by the matrix

$$G = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta^{\beta\alpha}\delta_{ij} \end{pmatrix}$$

(standard Riemannian metric from R^{n+np}).

Proof. Indeed, the Euler-Lagrange equations produced by

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial p_i^\alpha}{\partial t^\alpha}x^i - \frac{\partial x^i}{\partial t^\alpha}p_i^\alpha\right) - H\left(t, x, p\right) =$$

$$= -\frac{1}{2}\left(\frac{\partial p_i^\alpha}{\partial t^\alpha}, -\frac{\partial x^j}{\partial t^\beta}\right)\begin{pmatrix} \delta_{ij} & o \\ o & \delta^{\alpha\beta}\delta_{ij} \end{pmatrix}\begin{pmatrix} x^i \\ p_i^\alpha \end{pmatrix} - H\left(t, x\left(t\right), p\left(t\right)\right)$$

can be rewritten

$$\frac{1}{2}\frac{\partial p_i^\alpha}{\partial t^\alpha} = -\frac{1}{2}\frac{\partial p_i^\alpha}{\partial t^\alpha} - \frac{\partial H}{\partial x^i}\text{ , i.e. , }\frac{\partial p_i^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}$$

and

$$\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p_i^\alpha}.$$

2 Basic inequalities

2.1 Wirtinger multi-time inequality

In $L^2\left(T_0, R^n\right)$ we use the scalar product

$$\langle u, v\rangle = \int_{T_0}\left(\delta_{ij}u^i v^j\right)dt^1 \wedge ... \wedge dt^p$$

and the norm $\|u\|_{L^2} = \sqrt{\langle u, u\rangle}$. Similarly, in $L^2\left(T_0, C^n\right)$ we use the scalar product

$$\langle u, v\rangle = \int_{T_0}\left(\delta_{ij}u^i \overline{v}^j\right)dt^1 \wedge ... \wedge dt^p$$

and the norm $\|u\|_{L^2} = \sqrt{\langle u, \overline{u}\rangle}$.

Let us extend the Theorem 4.4 from [21] to the parallelipiped T_0.

Theorem 2 *Any function u from H_T^1 with mean zero satisfies the inequality*

$$\int_{T_0}\left|u\left(t\right)\right|^2 dt^1 \wedge ... \wedge dt^p \leq \frac{\left(\max\limits_i\left\{T^i\right\}\right)^2}{4\pi^2}\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p.$$

Proof. We express the function u as the sum of a multiple Fourier series

$$u\left(t\right) = \left(u^1\left(t^1, ..., t^p\right), ..., u^n\left(t^1, ..., t^p\right)\right)$$

$$= \left(\sum_{j_1^1,...,j_1^p \in Z^p} C^1_{j_1^1,...,j_1^p} e^{i\left(\frac{2\pi}{T^1}\cdot j_1^1 t^1 + ... + \frac{2\pi}{T^p}\cdot j_1^p t^p\right)}, \right.$$

$$\left. ..., \sum_{j_n^1,...,j_n^p \in Z^p} C^n_{j_n^1,...,j_n^p} e^{i\left(\frac{2\pi}{T^1}\cdot j_n^1 t^1 + ... + \frac{2\pi}{T^p}\cdot j_n^p t^p\right)} \right).$$

We calculate the square of the norm

$$(u,u) = \int_{T_0} \left(u(t), \overline{u(t)} \right) dt^1 \wedge ... \wedge dt^p =$$

$$= \sum_{j_1^1,...,j_1^p \in Z^p} \left(C^1_{j_1^1,...,j_1^p} \right)^2 \int_0^{T^1} e^{i\frac{2\pi}{T^1}\left(j_1^1 - j_1^1\right)t^1} dt^1 ... \int_0^{T^p} e^{i\frac{2\pi}{T^p}\left(j_1^p - j_1^{p1}\right)t^p} dt^p + ...$$

$$... + \sum_{j_n^1,...,j_n^p \in Z^p} \left(C^1_{j_n^1,...,j_n^p} \right)^2 \int_0^{T^1} e^{i\frac{2\pi}{T^1}\left(j_n^1 - j_n^1\right)t^1} dt^1 ... \int_0^{T^p} e^{i\frac{2\pi}{T^p}\left(j_n^p - j_n^p\right)t^p} dt^p$$

$$= \sum_{j_1^1,...,j_1^p \in Z^p} \left(C^1_{j_1^1,...,j_1^p} \right)^2 T^1 ... T^p + ... + \sum_{j_1^1,...,j_1^p \in Z^p} \left(C^n_{j_n^1,...,j_n^p} \right)^2 T^1 ... T^p.$$

If we denote

$$C_{k_1,...,k_p} = \left(C^1_{k_1,...,k_p}, ..., C^n_{k_1,...,k_p} \right)$$

and

$$u = \sum_{(k_1,...,k_p) \in Z^p} C_{k_1,...,k_p} e^{i2\pi\left(\frac{k_1}{T^1} t_1 + ... + \frac{k_p}{T^p} t_p\right)},$$

we find

$$(u,u) = \sum_{(k_1,...,k_p) \in Z^p} \left| C_{k_1,...,k_p} \right|^2 T^1 ... T^p.$$

Similarly, we consider the scalar product

$$\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right) = \int_{T_0} \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha} \overline{\frac{\partial v^j}{\partial t^\beta}} dt^1 \wedge ... \wedge dt^p.$$

It follows the square of the norm

$$\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t} \right)$$

$$= \sum_{(k_1,\dots,k_p)\in Z^p} T^1 ... T^p \left[\left(C^1_{k_1...k_p}\right)^2 \left(\frac{2\pi k_1}{T^1}\right)^2 + ... + \left(C^1_{k_1...k_p}\right) \left(\frac{2\pi k_p}{T^p}\right)^2 + ... \right.$$

$$\left. ... + \left(C^n_{k_1...k_p}\right)^2 \left(\frac{2\pi k_1}{T^1}\right)^2 + ... + \left(C^n_{k_1...k_p}\right)^2 \left(\frac{2\pi k_p}{T^p}\right)^2 \right]$$

$$= \sum_{(k_1,\dots,k_p)\in Z^p} T^1 ... T^p \left[\left|C_{k_1...k_p}\right|^2 \left(\frac{2\pi k_1}{T^1}\right)^2 + ... + \left|C_{k_1...k_p}\right|^2 \left(\frac{2\pi k_p}{T^p}\right)^2 \right]$$

$$= \sum_{(k_1,\dots,k_p)\in Z^p} T^1 ... T^p \left[\left|C_{k_1...k_p}\right|^2 4\pi^2 \left(\left(\frac{k_1}{T^1}\right)^2 + ... + \left(\frac{k_p}{T^p}\right)^2 \right) \right]$$

$$\geq \frac{4\pi^2}{\left(\max_i \left\{T^i\right\}\right)^2} \sum_{(k_1,\dots,k_p)\in Z^p} T^1 ... T^p \left|C_{k_1...k_p}\right|^2 \left(k_1^2 + ... + k_1^2\right)$$

$$\geq \frac{4\pi^2}{\left(\max_i \left\{T^i\right\}\right)^2} \sum_{(k_1,\dots,k_p)\in Z^p} T^1 ... T^p \left|C_{k_1...k_p}\right|^2$$

$$\geq \frac{4\pi^2}{\left(\max_i \left\{T^i\right\}\right)^2} \int_{T_0} \left|u\left(t\right)\right|^2 dt^1 \wedge ... \wedge dt^p.$$

Consequently

$$\int_{T_0} \left|u\left(t\right)\right|^2 dt^1 \wedge ... \wedge dt^p \leq \frac{\left(\max_i \left\{T^i\right\}\right)^2}{4\pi^2} \int_{T_0} \left|\frac{\partial u}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p,$$

and this ends the proof.

2.2 An estimate of the quadratic form $\int_{T_0} \left(\delta \otimes J \frac{\partial u}{\partial t}, u\left(t\right)\right) dt^1 \wedge ... \wedge dt^p$

Let us extend the Theorem 4.5 from [21] to the parallelipiped T_0.

Theorem 3 *For any* $u \in H^1_T$ *we have*

$$\int_{T_0} \left(\delta \otimes J \frac{\partial u}{\partial t}, u\left(t\right)\right) dt^1 \wedge ... \wedge dt^p$$

$$\geq -\frac{\sqrt{p}\max_i\left\{T^i\right\}}{2\pi}\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p.$$

Proof. We denote $\widetilde{u}(t) = u(t) - \int_{T_0} u(t)\, dt^1 \wedge ... \wedge dt^p$.. By using the Cauchy-Schwarz inequality and the multiple periodicity of u we obtain the inequality

$$\int_{T_0}\left(\delta\otimes J\frac{\partial u}{\partial t}, u(t)\right)dt^1\wedge ...\wedge dt^p$$

$$= \int_{T_0}\left(\delta\otimes J\frac{\partial u}{\partial t}, \widetilde{u}(t) + \int_{T_0} u(t)\,dt^1\wedge ...\wedge dt^p\right)dt^1\wedge ...\wedge dt^p$$

$$= \int_{T_0}\left(\delta\otimes J\frac{\partial u}{\partial t}, \widetilde{u}(t)\right)dt^1\wedge ...\wedge dt^p$$

$$+ \int_{T_0}\left(\delta\otimes J\frac{\partial u}{\partial t}, \int_{T_0} u(t)\,dt^1\wedge ...\wedge dt^p\right)dt^1\wedge ...\wedge dt^p$$

$$= \int_{T_0}\left(\delta\otimes J\frac{\partial u}{\partial t}, \widetilde{u}(t)\right)dt^1\wedge ...\wedge dt^p$$

$$\geq -\left(\int_{T_0}\left|\delta\otimes J\frac{\partial u}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p\right)^{\frac{1}{2}}\left(\int_{T_0}\left|\tilde{u}(t)\right|^2 dt^1\wedge ...\wedge dt^p\right)^{\frac{1}{2}}$$

From the inequality given by the Theorem 2 we have

$$\int_{T_0}\left|\widetilde{u}(t)\right|^2 dt^1\wedge ...\wedge dt^p \leq \frac{\left(\max_i\left\{T^i\right\}\right)^2}{4\pi^2}\int_{T_0}\left|\frac{\partial\widetilde{u}}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p$$

$$= \frac{\left(\max_i\left\{T^i\right\}\right)^2}{4\pi^2}\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p.$$

Because $\left|\delta\otimes J\frac{\partial u}{\partial t}\right|^2 \leq p\left|\frac{\partial u}{\partial t}\right|^2$, we obtain

$$\int_{T_0}\left(\delta\otimes J\frac{\partial u}{\partial t}, u(t)\right)dt^1\wedge ...\wedge dt^p$$

$$\geq -\left(\int_{T_0}\left|\delta\otimes J\frac{\partial u}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p\right)^{\frac{1}{2}}\frac{\max_i\left\{T^i\right\}}{2\pi}$$

$$\cdot\left(\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p\right)^{\frac{1}{2}}$$

$$\geq -\sqrt{p}\;\frac{\max_i\left\{T^i\right\}}{2\pi}\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p.$$

2.3 Inequalities satisfied by periodical solutions of multi-time Hamilton equations

Let us find properties of solutions of $(\delta \otimes J)\dfrac{\partial u}{\partial t}+\nabla H(t,u(t))=0$ a.e. on T_0 satisfying the boundary conditions

$$u\mid_{S^+}=u\mid_{S^-},$$

were S^+ and S^- are opposite sides of the parallelipiped T_0. Practically, we refer to bounds for such solutions.

Theorem 4 *We consider the Hamiltonian*

$$H: T_0 \times R^{n+np} \to R, (t,u) \to H(t,u)$$

like a measurable function in t for any $u \in R^{n+np}$, and C^1 *convex in u for any*

$$t \in T_0 = \left[0, T^1\right] \times ... \times \left[0, T^p\right] \subset R^p.$$

If there exists the constants

$$\alpha \in \left(0, \frac{\pi}{\sqrt{p}\max\limits_i \left\{T^i\right\}}\right), \beta \geq 0, \gamma \geq 0, \delta \geq 0$$

such that

$$\delta\left|u\right| + \beta \leq H(t,u) \leq \frac{\alpha}{2}\left|u\right|^2 + \gamma$$

for all $t \in T_0$ and $u \in R^{n+np}$, then, any multiple periodical solution

$$u = \left(x_i, p_i^\alpha\right), i = 1, ..., n, \alpha = 1, ..., p,$$

of the equation

$$\delta \otimes J\frac{\partial u}{\partial t} + \nabla H(t, u(t)) = 0, \tag{3}$$

verifies the inequalities

$$\int_{T_0} \left|\frac{\partial u}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p \leq \frac{2\alpha(\beta+\gamma)\pi T^1 ... T^p}{\pi - \alpha\max\limits_i\left\{T^i\right\}\sqrt{p}} \tag{4}$$

$$\int_{T_0} \left|u(t)\right| dt^1 \wedge ... \wedge dt^p \leq \frac{\pi T^1 ... T^p(\beta+\gamma)}{\delta\left(\pi - \alpha\max\limits_i\left\{T^i\right\}\sqrt{p}\right)}. \tag{5}$$

Proof. From the inequality

$$\delta\left|u\right|^2 - \beta \leq H(t,u) \leq \frac{\alpha}{2}\left|u\right|^2 + \gamma$$

we obtain

$$-\beta \leq H(t,u) \leq \alpha 2^{-1}\left|u\right|^2 + \gamma.$$

By applying [5, Proposition 2.2], considering $F(u) = H(t,u)$, $p = q = 2$, $v = \nabla H(t,u)$ we obtain

$$\frac{1}{2\alpha}\left|\nabla H\left(t,u\right)\right|^{2}\leq\left(\nabla H\left(t,u\right),u\right)+\beta+\gamma.$$

Because u is the solution of the equation (3), we have $\nabla H\left(t,u\right)=-\delta\otimes J\frac{\partial u}{\partial t}$ and the previous inequality becomes

$$\frac{1}{2\alpha}\left|-\delta\otimes J\frac{\partial u}{\partial t}\right|^{2}\leq\left(-\delta\otimes J\frac{\partial u}{\partial t},u\right)+\beta+\gamma. \tag{6}$$

In the hypothesis' conditions, by integration of the inequality (6) we have

$$\frac{1}{2\alpha}\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^{2}dt^1\wedge...\wedge dt^p$$
$$+\int_{T_0}\left(\delta\otimes J\frac{\partial u}{\partial t},u\right)dt^1\wedge...\wedge dt^p\leq\left(\beta+\gamma\right)T^1...T^p.$$

By using the inequality from Theorem 3, we have

$$\frac{1}{2\alpha}\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^{2}dt^1\wedge...\wedge dt^p$$
$$-\frac{\sqrt{p}\max_i\left\{T^i\right\}}{2\pi}\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^{2}dt^1\wedge...\wedge dt^p\leq\left(\beta+\gamma\right)T^1...T^p.$$

So

$$\left(\frac{1}{2\alpha}-\frac{\sqrt{p}\max\limits_i\left\{T^i\right\}}{2\pi}\right)\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^{2}dt^1\wedge...\wedge dt^p$$
$$\leq\left(\beta+\gamma\right)T^1...T^p$$

and, as consequence

$$\int_{T_0}\left|\frac{\partial u}{\partial t}\right|^{2}dt^1\wedge...\wedge dt^p\leq\frac{2\pi\alpha\left(\beta+\gamma\right)T^1...T^p}{\pi-\alpha\max\limits_i\left\{T^i\right\}\sqrt{p}}.$$

By integration, the inequality

$$\delta\left|u\right|-\beta\leq H\left(t,u\right)$$

produces

$$\delta\int_{T_0}\left|u\left(t\right)\right|dt^1\wedge...\wedge dt^p-\beta T^1...T^p\leq\int_{T_0}H\left(t,u\right)dt^1\wedge...\wedge dt^p.$$

Because $H\left(t,u\right)$ is convex in u,

$$H\left(t,u\right)-H\left(t,0\right)\leq\left(\nabla H\left(t,u\left(t\right)\right),u\left(t\right)\right),$$

we obtain

$$\int_{T_0} H\left(t, u\left(t\right)\right) dt^1 \wedge ... \wedge dt^p$$

$$\leq \int_{T_0} \left[H\left(t, 0\right) + \left(\nabla H\left(t, u\left(t\right)\right), u\left(t\right)\right)\right] dt^1 \wedge ... \wedge dt^p$$

$$\leq \gamma T^1 ... T^p - \int_{T_0} \left(\delta \otimes J \frac{\partial u}{\partial t}, u\left(t\right)\right) dt^1 \wedge ... \wedge dt^p$$

$$\leq \gamma T^1 ... T^p + \frac{\sqrt{p} \max_i \left\{T^i\right\}}{2\pi} \int_{T_0} \left|\frac{\partial u}{\partial t}\right|^2 dt^1 \wedge ... \wedge dt^p$$

$$\leq \gamma T^1 ... T^p + \frac{\sqrt{p} \max_i \left\{T^i\right\} 2\pi\alpha \left(\beta + \gamma\right) T^1 ... T^p}{2\pi \left(\pi - \alpha \max_i \left\{T^i\right\} \sqrt{p}\right)}.$$

By consequence

$$\int_{T_0} \left|u\left(t\right)\right| dt^1 \wedge ... \wedge dt^p$$

$$\leq \frac{1}{\delta} \left(\beta T^1 ... T^p + \gamma T^1 ... T^p + \frac{\sqrt{p} \max_i \left\{T^i\right\} \alpha \left(\beta + \gamma\right)}{\pi - \alpha \max_i \left\{T^i\right\} \sqrt{p}} T^1 ... T^p\right),$$

meaning that

$$\int_{T_0} \left|u\left(t\right)\right| dt^1 \wedge ... \wedge dt^p \leq \frac{\left(\beta + \gamma\right) T^1 ... T^p \pi}{\delta \left(\pi - \alpha \max_i \left\{T^i\right\} \sqrt{p}\right)}$$

and the proof ends.

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