

Induced invariant Finsler metrics on quotient groups

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Abstract. In this paper we show that every invariant Finsler metric on Lie group G , induces an invariant Finsler metric on quotient group G/H in the natural way, where H is a closed normal Lie subgroup of G .

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1 Introduction.

The study of invariant structures on homogeneous manifolds is an important problem in geometry. K. Nomizu obtained many interesting properties of invariant Riemannian metrics on homogeneous space G/H . He introduced *reductive* homogeneous spaces and studied invariant Riemannian metrics and the existence and properties of invariant affine connections on reductive homogeneous spaces (See [4] and [6]). Also some curvature properties of invariant Riemannian metrics on Lie groups and homogeneous spaces have studied by J. Milnor and H. Samelson (See [5] and [7]). So it is important to study invariant Finsler metrics which are a generalization of invariant Riemannian metrics.

Some properties of invariant Finsler metrics on reductive homogeneous manifolds are studied in [2] and [3] by S. Deng and Z. Hou. The authors of these papers obtained a necessary and sufficient condition for homogeneous manifolds to have invariant Finsler metrics. Then they studied bi-invariant Finsler metrics on Lie groups and obtained a necessary and sufficient condition for a Lie group to have bi-invariant Finsler metrics. In this paper we show that every invariant Finsler metric on a Lie group G induces an invariant Finsler metric on quotient group G/H in the natural way, where H is a closed normal Lie subgroup of G .

Note. In this article we **do not assume** the quotient groups G/H are reductive.

2 Preliminaries.

Definition 2.1. A Minkowski norm on \mathbb{R}^n is a nonnegative function $\mathbb{F} : \mathbb{R}^n \rightarrow [0, \infty)$ which has the following properties:

- (i) \mathbb{F} is \mathcal{C}^∞ on $\mathbb{R}^n \setminus \{0\}$.
- (ii) $\mathbb{F}(\lambda y) = \lambda \mathbb{F}(y)$ for all $\lambda > 0$ and $y \in \mathbb{R}^n$
- (iii) The $n \times n$ matrix (g_{ij}) , where $g_{ij}(y) := [\frac{1}{2}\mathbb{F}^2]_{y^i y^j}(y)$, is positive-definite at all $y \neq 0$.

Definition 2.2. Let M be an n -dimensional smooth manifold. Also let TM be the tangent bundle of M . A function $F : TM \rightarrow [0, \infty)$ is called a Finsler metric if it has the following properties:

- (i) F is \mathcal{C}^∞ on the slit tangent bundle $TM \setminus 0$.
- (ii) For each $x \in M$, $\mathbb{F}_x := F|_{T_x M}$ is a Minkowski norm on $T_x M$.

If the Minkowski norm satisfies $\mathbb{F}(-y) = \mathbb{F}(y)$, then one has the absolutely homogeneity $F(\lambda y) = |\lambda|F(y)$, for any $\lambda \in \mathbb{R}$. Every absolutely homogeneous Minkowski norm is a norm in the sense of functional analysis. Every Riemannian manifold (M, g) by defining

$$F(x, y) := \sqrt{g_x(y, y)} \quad x \in M, y \in T_x M$$

is a Finsler manifold (For more details about Finsler geometry see [1]).

We also use the following notations:

- $R_g : G \rightarrow G$, right translation, $R_g(h) = hg$.
- $L_g : G \rightarrow G$, left translation, $L_g(h) = gh$.
- $\nu : G \rightarrow G$, inversion, $\nu(g) = g^{-1}$.
- $e \in G$, the unit element.

We use F for Finsler metrics on Lie group G , \mathbb{F} for Minkowski norms on a specific tangent space $T_x M$ or a real vector space \mathbb{R}^n and \mathcal{F} for Finsler metrics on quotient group G/H . Also if $f : M \rightarrow N$ is a smooth function between manifolds and $x \in M$, we denote by $T_x f : T_x M \rightarrow T_{f(x)} N$ the derivative of f at x . If $f : M \rightarrow N$ is a local diffeomorphism then $T_x f$ is an isomorphism of vector spaces, yielding for each vector field $Y \in \mathcal{X}(N)$ on N a vector field $f^* Y \in \mathcal{X}(M)$ defined by $(f^* Y)(x) = (T_x f)^{-1} Y(f(x))$.

3 Induced Invariant Finsler Metrics on Quotient Groups.

Let G be a compact connected Lie group, H a closed subgroup of G , $M = G/H$ the homogeneous space which consists of left cosets of zH , $z \in G$, and $p : G \rightarrow M$ be the natural projection of G onto M . The group G admits a bi-invariant Riemannian metric. Now we can obtain a Riemannian metric on M in the following way which is invariant under the customary action of G on M :

Let $x \in M$, $X \in T_x M$, $z \in G$ and $Z \in T_z G$, such that $p(z) = x$, $T_z p(Z) = X$ and let Z be orthogonal to the coset $z.H$ (a submanifold of G) at z . Now we define $|X| := |Z|$ (see[7]).

But in Finsler geometry we have no orthogonality for tangent vectors so we can't use the above way. In this article we try to replace the orthogonality condition, by other conditions such that by define $\mathcal{F}(X) := F(Z)$, have a bi-invariant Finsler metric on M .

From now G is an arbitrary finite dimensional Lie group (no necessarily compact or connected).

For construct a left or right invariant Finsler metric on a Lie group G , it is sufficient to have a Minkowski norm on $T_e G$ such as \mathbb{F}_0 , then define

$$F : TG \rightarrow [0, \infty)$$

$$F(x, y) = \mathbb{F}_0(T_x L_{x^{-1}} y) \quad x \in G, y \in T_x G$$

for left invariant Finsler metrics, and

$$F(x, y) = \mathbb{F}_0(T_x R_{x^{-1}} y)$$

for right invariant Finsler metrics.

Lemma 3.1. *Assume that G is any Lie group and H any closed subgroup, and denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of right invariant vector fields of G and H , respectively. Let V be a vector subspace complementary to \mathfrak{h} in \mathfrak{g} , that is, $\mathfrak{g} = V \oplus \mathfrak{h}$, and $M := G/H$ be the quotient manifold consists of left cosets zH , $z \in G$. Then $\pi : V \rightarrow \mathcal{X}(M)$ defined unambiguously by $\pi(X)(p(z)) = T_z p(X(z))$ is a linear function, where $p : G \rightarrow M := G/H$ is the natural projection.*

Proof: Since π defined by Tp so π is linear. Let $\{X_1, \dots, X_k, X_{k+1}, \dots, X_n\}$ be a basis of Lie algebra of the Lie group G (consists of right invariant vector fields) such that $\{X_1, \dots, X_k\}$ is a basis of the Lie algebra of closed Lie subgroup H . So $\{X_{k+1}, \dots, X_n\}$ is a basis of vector space V . We must show π is welldefined. Assume that $z_1, z_2 \in G$ and $p(z_1) = p(z_2) = x$, therefore $z_1 H = z_2 H$, so $z_1^{-1} z_2 \in H$. Also for $h \in H$ we have $p \circ R_h = p$ because for any $g \in G$

$$p \circ R_h(g) = p(gh) = ghH = gH = p(g).$$

So $p \circ R_{z_1^{-1} z_2} = p$.

Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ be a real valued differentiable function, then by attention to the

fact that (for $i = 1, \dots, n$) X_i is right invariant we have

$$\begin{aligned}
(\pi(X_i)(p(z_2)))f &= (T_{z_2}p(X_i(z_2)))f \\
&= (T_{z_2}p(T_{z_1}R_{z_1^{-1}z_2}(X_i(z_1))))f \\
&= X_i(z_1)(f \circ p \circ R_{z_1^{-1}z_2}) \\
&= X_i(z_1)(f \circ p) \\
&= (T_{z_1}p(X_i(z_1)))f \\
&= (\pi(X_i)(p(z_1)))f.
\end{aligned}$$

So for any $X \in V$ such that $X = \sum_{i=k+1}^n \lambda_i X_i$ we have

$$(\pi(X)(p(z_2)))f = (\pi(X)(p(z_1)))f$$

Therefore the definition of π is welldefined. \square

Lemma 3.2. *Consider the assumptions of Lemma 3.1 and also suppose that H is a closed normal Lie subgroup of G . Then*

$$T_z p : V(z) \rightarrow T_{p(z)} M$$

is an isomorphism, where $V(z) = \text{span}\{X_{k+1}(z), \dots, X_n(z)\}$.

Proof: For $i = 1, \dots, k$ we have $X_i(e) \in T_e H$, and also

$$R_z : H \rightarrow Hz = zH$$

is a diffeomorphism, so $X_i(z) = T_e R_z X_i(e) \in T_z zH$. Therefore $X_i(z) \in \ker(T_z p : T_z G \rightarrow T_{p(z)} M)$. But we know that $T_z p : T_z G / T_z zH \rightarrow T_{p(z)} M$ is an isomorphism of vector spaces and $T_z G / T_z zH \simeq V(z)$, so

$$T_z p : V(z) \rightarrow T_{p(z)} M$$

is an isomorphism of vector spaces. \square

Theorem 3.3. *Assume that G is any n -dimensional Lie group, H any closed normal Lie subgroup, $M = G/H$ the quotient group and $p : G \rightarrow M$ is the natural projection. If F is a right invariant Finsler metric on G , then there is a Finsler metric on M induced by F such that is invariant under the natural right action of G on M .*

Proof: Suppose that \mathfrak{g} and \mathfrak{h} are the algebras of right invariant vector fields of G and H , respectively, and $\{X_1, \dots, X_k, X_{k+1}, \dots, X_n\}$ is a basis of \mathfrak{g} such that $\{X_1, \dots, X_k\}$ is a basis of \mathfrak{h} . Let V be a vector subspace complementary to \mathfrak{h} in \mathfrak{g} , that is, $\mathfrak{g} = V \oplus \mathfrak{h}$. Assume that $x \in M$ and $X \in T_x M$ is a tangent vector at x . Let $z \in G$ and $Z \in V(z)$ such that $p(z) = x$ and $T_z p(Z) = X$. (By Lemma 3.2 for any fixed z such that $p(z) = x$, there is a unique $Z \in V(z)$ such that $T_z p(Z) = X$) In this situation we define

$$\mathcal{F}(X) := F(Z)$$

At the first we show that this definition is welldefined.

For this, we must to show that the definition of \mathcal{F} is independent of choice of z .

Assume $z_1, z_2 \in G, Z_1 \in V(z_1), Z_2 \in V(z_2)$ and $p(z_1) = p(z_2) = x$,
 $T_{z_1}p(Z_1) = T_{z_2}p(Z_2) = X, Z_1 = \sum_{i=k+1}^n \lambda_i X_i(z_1), Z_2 = \sum_{i=k+1}^n \mu_i X_i(z_2)$.
 Now we can write

$$\begin{aligned}
 T_{z_1}p(T_{z_2}R_{z_2^{-1}z_1}(Z_2)) &= T_{z_1}p(T_{z_2}R_{z_2^{-1}z_1}(\sum_{i=k+1}^n \mu_i X_i(z_2))) \\
 &= T_{z_1}p(\sum_{i=k+1}^n \mu_i T_{z_2}R_{z_2^{-1}z_1}(X_i(z_2))) \\
 &= T_{z_1}p(\sum_{i=k+1}^n \mu_i (X_i(z_1))) \\
 &= \sum_{i=k+1}^n \mu_i T_{z_1}p(X_i(z_1)) \\
 &= \sum_{i=k+1}^n \mu_i (\pi(X_i)(p(z_1))) \\
 &= \sum_{i=k+1}^n \mu_i (\pi(X_i)(p(z_2))) \\
 &= \sum_{i=k+1}^n \mu_i T_{z_2}p(X_i(z_2)) \\
 &= T_{z_2}p(\sum_{i=k+1}^n \mu_i (X_i(z_2))) \\
 &= T_{z_2}p(Z_2) = X
 \end{aligned}$$

But since $T_{z_1}p : V(z_1) \rightarrow T_x M$ is an isomorphism of vector spaces, we have $T_{z_2}R_{z_2^{-1}z_1}(Z_2) = Z_1$ (This also shows that for $i = k+1, \dots, n$ we have $\lambda_i = \mu_i$).
 So

$$(3.1) \quad F(T_{z_2}R_{z_2^{-1}z_1}(Z_2)) = F(Z_1).$$

But F is a right invariant Finsler metric on G , so for any $g_1, g_2 \in G$ and $X_{g_1} \in T_{g_1}G$ we have

$$F(T_{g_1}R_{g_2}(X_{g_1})) = F(X_{g_1}),$$

therefore

$$(3.2) \quad F(Z_2) = F(T_{z_2}R_{z_2^{-1}z_1}(Z_2)).$$

By equations 3.1 and 3.2 we have $F(Z_1) = F(Z_2)$.

It means the definition of $\mathcal{F}(X)$ is independent of choice of z , so \mathcal{F} is welldefined.
 \mathcal{F} has all two conditions of Finsler metrics, because $\mathcal{F} = F \circ (Tp|_V)^{-1}$. Also \mathcal{F} is

right invariant under right action of G on M , because F is right invariant on G . \square

Remark 3.4. *If we want to have a similar theorem as Theorem 3.3, for left invariant Finsler metrics, it suffices to replace the word “right” by “left” in Lemma 3.2 and Theorem 3.3, and to use the fact that $zH = Hz$ by the normality of H .*

Theorem 3.5. *Assume that G is any n -dimensional Lie group, H any closed normal Lie subgroup, $M = G/H$ the quotient group and $p : G \rightarrow M$ is the natural projection. If F is a bi-invariant Finsler metric on G , then there is a Finsler metric on M induced by F such that is invariant under the natural right and left actions of G on M .*

Proof: Suppose that \mathfrak{g} , \mathfrak{h} , $\{X_1, \dots, X_k, X_{k+1}, \dots, X_n\}$ and V are the same objects in the proof of Theorem 3.3. Let $z \in G$ and $Z \in V(z)$ such that $p(z) = x$ and $T_z p(Z) = X$. We define $\mathcal{F} : TM \rightarrow [0, \infty)$ by $\mathcal{F}(X) := F(Z)$. By Remark 3.4, this definition is welldefined and also left invariant.

Since $\{X_1, \dots, X_k\}$ is a basis of the Lie algebra consists of left invariant vector fields of H , so $\{\nu^* X_1, \dots, \nu^* X_k\}$ is a basis of the Lie algebra consists of right invariant vector fields of H and $\{\nu^* X_1, \dots, \nu^* X_n\}$ is a basis of the Lie algebra consists of right invariant vector fields of G . Now by using Lemma 3.2 and Theorem 3.3 and the fact that, $zH = Hz$ for any $z \in G$, we have \mathcal{F} is right invariant, therefore \mathcal{F} is bi-invariant. \square

Corollary 3.6. *Let G be any n -dimensional connected Lie group, H any connected closed Lie subgroup and $M := G/H$ the quotient manifold. Suppose that F is a left invariant (right or bi-invariant) Finsler metric on G . If \mathfrak{h} is an ideal of \mathfrak{g} then M admits a left invariant (right or bi-invariant) Finsler metric in the natural way.*

Proof: Since G and H are connected Lie groups and \mathfrak{h} is an ideal of \mathfrak{g} , by Theorem 2.13.4 of [8], H is a closed normal Lie subgroup of G . So by attention to Theorems 3.3, 3.5 and Remark 3.4 the proof will be finished. \square

Corollary 3.7. *Let G be any n -dimensional abelian Lie group and H a closed subgroup of G . If F is a left invariant (right or bi-invariant) Finsler metric on G then F induces a left invariant (right or bi-invariant) Finsler metric on $M = G/H$ in the natural way.*

Our results are true in Riemann case, and also our method for construct invariant Finsler metrics on quotient groups is compatible with method described in the first part of section 3 about invariant Riemannian metrics.

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