On Top Spaces

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Abstract. In this paper the notion of smooth complete semidynamical systems is studied. A relation between the rank of the derivatives of a motion at a point and at the identity of that point is deduced. A method for constructing top generalized subgroups is considered. Connected component of an identity as a top generalized normal subgroup is studied. A criterion for the connectedness of an inverse image of an identity is deduced. A condition for the separability of a top space is presented. Top generalized normal subgroups and quotient space created by a top space are studied.

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1 Introduction

The notion of topological generalized groups as a generalization of topological groups considered in [2]. Let us recall its definition [2].

Definition 1.1 A topological generalized group is a non-empty set T admitting an operation

$$\begin{aligned} m_2 : T \times T &\to T \\ (g,h) &\mapsto gh \end{aligned}$$

called multiplication, subject to the set of rules given below:
(i) $(xy)z = x(yz)$ for all x, y, z in T;
(ii) For each x in T there exists a unique $e(x)$ in T such that $xe(x) = e(x)x = x$;
(iii) For each x in T there exists y in T such that $xy = yx = e(x)$;
(iv) T is a Hausdorff topological space;
(v) The mapping m_2 and the mapping

$$\begin{aligned} m_1 : T &\to T \\ g &\mapsto g^{-1} \end{aligned}$$

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are continuous maps.
Each topological generalized group T is a disjoint unions of topological groups $e^{-1}(\{e(g)\})$ where $g \in T$ [2]. Moreover if U is a neighborhood of the identity $e(g)$, then there exists a neighborhood V of $e(g)$ such that $m_1(V) = V$, and $V \subset U$ [2].

Theorem 1.1 *Let T be a topological generalized group and let the cardinality of $e(T)$ be finite. Moreover let H be a locally closed generalized subgroup of T [1]. Then H is a closed subset of T.*

Proof. We know that $\{e^{-1}(e(g)) \ : \ e(g) \in T\}$ is a partition of T by the topological generalized groups, where the topology of the topological generalized group $e^{-1}(e(g))$ is the subspace topology. If $H_{e(g)} = H \cap e^{-1}(e(g))$ is non-empty, then it is a locally closed subgroup of the topological group $e^{-1}(e(g))$, where $g \in T$. Thus it is a closed subset of $e^{-1}(e(g))$. Since $e^{-1}(e(g))$ is a closed subset of T, then $H_{e(g)}$ is a closed subset of T, for all $g \in T$. Moreover we have $H = \bigcup_{e(g)\in e(T)} H_{e(g)}$. So the finiteness of $e(T)$ implies that H is a closed subset of T.□
The following example shows that a locally closed generalized subgroup of a topological generalized group may not be closed.

Example 1.1 The set of real numbers with the binary operation $(a, b) \mapsto a$ and Euclidean norm is a topological generalized group. The open interval $(1, 2)$ is a generalized subgroup of $\mathbf{R}$ which is locally closed.
Now let us recall the definition of top spaces [3].

Definition 1.2 A topological generalized group $(T, .)$ is called a top space if:
i) The topological space T is a smooth manifold of dimension t;
ii) The mapping $m_1 : T \to T$ is defined by $m_1(u) = u^{-1}$ and the mapping $m_2 : T \times T \to T$ is defined by $m_2(u_1, u_2) = u_1u_2$ are smooth maps
iii) For all $x, y \in T$, $e(xy) = e(x)e(y)$ where e is the identity mapping.

Theorem 1.2 *If T is a top space and if the cardinality of $e(T)$ is finite, then T is a disjoint union of Lie groups.*

Proof. Since the cardinality of $e(T)$ is finite, then for all $p \in T$, $e^{-1}(e(p))$ is an open subset of T [5], and the identity of $e^{-1}(e(p))$ is $e(p)$. So $e^{-1}(e(p))$ is a Lie group as an open submanifold of T. Moreover if $e(p) \neq e(q)$ then $(e^{-1}(e(p))) \cap (e^{-1}(e(q))) = \emptyset$. So T is a disjoint union of Lie groups.□

2 Smooth complete semi-dynamical systems

In this section we assume that T is a top space and M is a smooth manifold.
The notion of complete semi-dynamical systems as a consequence of generalized vector fields has been considered in [4].
We now define a smooth complete semi-dynamical system.

Definition 2.1 A mapping $\varphi : T \times M \to M$ is called a smooth complete semi-dynamical system if
(i) φ is a C^∞ map;
(ii) $\varphi(t, \varphi(s, m)) = \varphi(ts, m)$ for all $t, s \in T$ and $m \in M$;
(iii) For all $m \in M$ there is $e(t) \in T$ such that $\varphi(e(t), m) = m$.

Example 2.1 The set of non-zero real numbers with the operation $ab = a|b|$ and usual manifold structure as an open subset of R is a top space.
$\varphi : \mathbf{R} - \{0\} \times \mathbf{R} \to \mathbf{R}$ defined by $\varphi(t, m) = t|m|$ is a smooth complete semi-dynamical system.

Remark 2.1 If T is a Lie group and $\varphi : T \times M \to M$ is a smooth complete semi-dynamical system, then φ is a smooth dynamical system.
The mapping $\mu_m : T \to M$ defined by $t \mapsto \varphi(t, m)$ is called the T-motion of the point m and the set $T^T(m) := \{\varphi(t, m) \ : \ t \in T\}$ is called the T-trajectory of a point m.
The mappings $l_t : T \to T$ and $r_t : T \to T$ defined by $l_t(s) = ts$ and $r_t(s) = st^{-1}$ are called left transformation and right transformation respectively.

Theorem 2.1 *Let $m \in M$ and let there exists $t \in T$ such that $\varphi^t : M \to M$ defined by $\varphi^t(n) = \varphi(t, n)$ has a one to one differential at $\varphi(e(t), m)$. Moreover let the cardinality of $e(T)$ be finite. Then $rank(d_{e(t)}\mu_m) = rank(d_t\mu_m)$.*

Proof. We have $\varphi^t o\mu_m = \mu_m ol_t$. Thus $d_{\varphi(e(t),m)}\varphi^t od_{e(t)}\mu_m = d_t\mu_m od_{e(t)}l_t$. Since $card(e(T)) < \infty$, then $e^{-1}(e(t))$ is an open set in T. Moreover $l_t : e^{-1}(e(t)) \to e^{-1}(e(t))$ is a diffeomorphism. So $d_{e(t)}l_t$ is an isomorphism. Since $d_m\varphi^t$ is also an isomorphism, then $rank(d_{e(t)}\mu_m) = rank(d_t\mu_m)$. □

Corollary 2.1 *If T is a top space with finite number of identities, and $\varphi : T \times T \to T$ is defined by $\varphi(t, s) = l_t(s)$. Then for all $s \in T$ and $t \in e^{-1}(e(s))$, $rank(d_{e(t)}\mu_s) = rank(d_t\mu_s)$.*

Proof. Let $s \in T$ be given. Then for all $t \in e^{-1}(e(s))$ the mappings φ^t and μ_t are local diffeomorphisms. So the corollary follows from theorem 2.1. □

Corollary 2.2 *With the assumptions of corollary 2.1 $\mu_t^{-1}(t)$ is a submanifold and a generalized subgroup of T, for all $t \in T$.*

Proof. Corollary 2.1 implies that μ_t is a subimmersion. So $\mu_t^{-1}(t)$ is a submanifold of T. If $s, u \in \mu_t^{-1}(t)$ then $ts = t$ and $tu = t$. So $t(su) = (ts)u = tu = t$. Hence $su \in \mu_t^{-1}(t)$. Moreover we have $ts^{-1} = tss^{-1} = te(s) = tse(s) = ts = t$. So $s^{-1} \in \mu_t^{-1}(t)$. Thus it is a generalized subgroup of T.□

Definition 2.2 A generalized subgroup H of T is called a top generalized subgroup if it is a submanifold of T.

Theorem 2.2 *Let $\varphi : T \times M \to M$ be a smooth complete semi-dynamical system and let $m \in M$. Moreover let $e(T)$ be a finite set and $S = \{e(n) \in M \ : \ \varphi(e(n), m) = m\}$. Then $H = (\bigcup_{s \in S} e^{-1}(s)) \cap \mu_m^{-1}(m)$ is a top generalized subgroup of T .*

Proof. If $s \in S$, then $e^{-1}(s)$ is a top generalized subgroup of T which is also a Lie group with its product. Moreover $\varphi : e^{-1}(s) \times M \to M$ is a smooth dynamical system. So $e^{-1}(s) \cap \mu_m^{-1}(m)$ is a Lie subgroup of $e^{-1}(s)$. Since $e^{-1}(s)$ is open in T, then H is submanifold of T. Other properties can deduce by the straightforward calculations. □

3 Connected components of identities as top generalized normal subgroups

In this section we assume that T is a top space and $C_{e(t)}$ is the connected component contains $e(t) \in T$.

Theorem 3.1 *If $t \in T$ then $C_{e(t)}$ is a top generalized normal subgroup of T.*

Proof. Suppose $t \in T$ be given. Then $C_{e(t)}$ is an open subset of T.
Since $r_t(t) = e(t)$, and r_t is continuous, then $r_t(C_{e(t)})$ is a connected set contains $e(t)$. So $r_t(C_{e(t)}) \subseteq C_{e(t)}$.
If $s \in C_{e(t)}$, then $e(s) \in C_{e(t)}$. Because $r_{s^{-1}}(C_{e(s)}) \subseteq C_{e(s)}$. Hence $r_{s^{-1}}(e(s)) = s \in C_{e(s)}$. Moreover we have $s \in C_{e(t)}$. Thus $C_{e(s)} \cap C_{e(t)} \neq \emptyset$. Hence $C_{e(s)} = C_{e(t)}$.
Since $s^{-1} = r_s(e(s))$, then $s^{-1} \in C_{e(t)}$.
If $s_1, s_2 \in C_{e(t)}$, then $s_1 s_2 = r_{s_2^{-1}}(s_1) \in C_{e(t)}$. So $C_{e(t)}$ is a generalized subgroup of T. Moreover it is a generalized normal subgroup. Because $C_{e(t)} = \bigcup_{a\in T}(e^{-1}(e(a)) \cap C_{e(t)})$. □

Theorem 3.2 *If there is $s \in T$ such that $s \notin \bigcup_{t\neq s} C_{e(t)}$, then $e^{-1}(e(s))$ is a connected set.*

Proof. The condition $s \notin \bigcup_{t\neq s} C_{e(t)}$ implies that $e^{-1}(e(s)) \subseteq C_{e(s)}$. Moreover $e^{-1}(e(s)) \cup (\bigcup_{t\neq s} C_{e(t)}) = T$. Thus $e^{-1}(e(s)) = C_{e(s)}$. So $e^{-1}(e(s))$ is a connected set. □

Theorem 3.3 *Let U be an open set contain $e(T)$, and $e^{-1}(e(t))$ be a connected set for all $t \in T$. Then $T = \bigcup_{n=1}^{\infty} U^n$.*

Proof. For given $t \in T$ we have: $U \cap e^{-1}(e(t))$ is an open subset of $e^{-1}(e(t))$. So $e^{-1}(e(t)) = \bigcup_{n=1}^{\infty}(U \cap e^{-1}(e(t)))^n$. Thus $T = \bigcup_{t\in T} e^{-1}(e(t)) = \bigcup_{t\in T}\bigcup_{n=1}^{\infty}(U \cap e^{-1}(e(t)))^n = \bigcup_{n=1}^{\infty}\bigcup_{t\in T}(U \cap e^{-1}(e(t)))^n = \bigcup_{n=1}^{\infty}\bigcup_{t\in T}(U^n \cap e^{-1}(e(t))) = \bigcup_{n=1}^{\infty} U^n$. □

Theorem 3.4 *If T is a union of countably many compact subsets, then T has countably many connected components.*

Proof. Let K_n are compact sets such that $T = \bigcup_{n=1}^{\infty} K_n$. Then each K_n can cover by a finite numbers of $C_{e(t)}$. So there exist $\{t_1, t_2, t_3, ...\} \subseteq T$ such that $T \subseteq \bigcup_{i=1}^{\infty} C_{e(t_i)}$.
Hence T has countably many connected components. □
The following corollary follows from the proof of theorem 3.4.

Corollary 3.1 *Let T be a union of countably many compact subsets. Then T is separable.*

4 Quotient spaces created by a top space

We begin this section with the definition of a morphism of top spaces.

Definition 4.1 If T, and S are two top spaces, then a homomorphism $f : T \to S$ is called a morphism if it is also a C^∞ map.

Definition 4.2 A top generalized subgroup N of a top space T is called a top generalized normal subgroup of T if there exists a top space S and a morphism $f : T \to S$ such that,

$$(\forall a \in T)(N_a = \emptyset \text{ or } N_a = Ker\ f_a),$$

where $N_a := N \cap e^{-1}(e(a))$ and $f_a := f|_{e^{-1}(e(a))}$.

Theorem 4.1 *Let N be a top normal generalized subgroup of T and let $e(T)$ be finite. Then $\Gamma_N = \{a \in T \mid N_a \neq \emptyset\}$ is an open top generalized subgroup of T.*

Proof. Since $e(T) < \infty$, then $e^{-1}(e(a))$ is open in T for all $a \in T$. So Γ_N is open in T. Moreover Γ_N is a generalized subgroup of T [3]. So it is an open top generalized subgroup of T. $\square$

With the assumptions of theorem 4.1 the topology of Γ_N is:

$$\{V : V \cap e^{-1}(e(a)) \text{ is open in } e^{-1}(e(a)) \text{ for all } a \in \Gamma_N\} \cup \{\Gamma_N\},$$

and we define a topology on T/N as the form

$$\{V : \pi^{-1}(V) \text{ is open in } \Gamma_N\},$$

where $\pi : \Gamma_N \to T/N$ defined by $\pi(x) := xN_x$.

Theorem 4.2 *With the above assumptions there is a unique differentiable structure on T/N such that $\pi : \Gamma_N \to T/N$ is a submersion. Moreover T/N with this differentiable structure is a top space.*

Proof. For all $a \in \Gamma_N$ there is a unique differentiable structure on $e^{-1}(e(a))/N_a$ such that $\pi|_{e^{-1}(e(a))} : e^{-1}(e(a)) \to e^{-1}(e(a))/N_a$ is a submersion. Because N_a is a Lie subgroup of the $e^{-1}(e(a))$. Since $e^{-1}(e(a))$ and $e^{-1}(e(a))/N_a$ are open in T and T/N respectively, then there is a unique differentiable structure on T/N such that $\pi : \Gamma_N \to T/N$ is a submersion. Moreover we know that T/N with the operation $(xN_x)(yN_y) = xyN_{xy}$ is a topological generalized groups. We know show that

$$\begin{array}{rcl} \hat{m}_1 : T/N & \to & T/N \\ xN_x & \mapsto & x^{-1}N_x \end{array}$$

is a smooth map. This follows from the commutativity of the following diagram:

$$\begin{array}{ccc} \Gamma_N & \xrightarrow{m_1} & \Gamma_N \\ \pi \downarrow & & \downarrow \pi \\ T/N & \xrightarrow{\hat{m}_1} & T/N \end{array} .$$

Moreover the mapping

$$\begin{array}{rcl} \hat{m}_2 : T/N \times T/N & \rightarrow & T/N \\ (xN_a, yN_b) & \mapsto & xyN_{ab} \end{array}$$

is a smooth map. Because π is a submersion and the following diagram is a commutative one.

$$\begin{array}{ccc} \Gamma_N \times \Gamma_N & \xrightarrow{m_2} & \Gamma_N \\ \pi \times \pi \downarrow & & \downarrow \pi \\ T/N \times T/N & \xrightarrow{\hat{m}_2} & T/N \end{array} . \square$$

5 Conclusion

In this paper we used of dynamical methods for constructing new top spaces. We have also deduced two conditions for separability and connectedness of top spaces. We also paid attention to the interesting properties of the identities, which can be a base for further research on this structure.

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