A model of Gauge Theory with invariant variables

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Abstract. A model for $SU(2)$ gauge theory with invariant variables is presented. The strength tensor of the $SU(2)$ gauge potentials and its Hodge dual are obtained for a spherical symmetric model. Then, a metric tensor associated to these tensors is constructed. The components of this metric tensor are interpreted as gauge invariant variables for $SU(2)$ theory. The property of self-duality of the gauge model with respect to the metric tensor is studied. The self-duality equations are written and their solution is obtained. A comparison with the Yang-Mills field equations is also given.

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1 Introduction

Usually, the gauge theories are formulated in terms of non-gauge invariant variables, like potentials $A_\mu^a(x)$ [2]. But, the physical observables are gauge invariant, and this rises many difficulties both at classical and quantum level. Some models of gauge theories on Euclidean and Minkowski 3-dimensional spaces have been developed [5,6] in terms of gauge invariant variables. The fundamental quantity used in these theories is the gauge invariant metric tensor $g_{ij} = -\frac{1}{2}Tr\left({}^*F_i\,{}^*F_j\right)$, where ${}^*F_i = \frac{1}{2}\varepsilon_{ijk}F^{jk}$ is the dual of the gauge field tensor F^{ij} $(i,j=1,2,3)$. It has been shown that this metric tensor satisfies the Einstein equations with the right-hand side of a simple form [5]. This theory was generalized to the case of a curved space-time [11]. Namely, a $SU(2)$ gauge theory on the 3-dimensional sphere S^3 has been formulated. The manifold S^3 is a space with constant curvature, and the generalization of the theory to this case is not trivial. The corresponding model has the advantage that the dimensions of the $SU(2)$ group and of the sphere S^3 are the same.

In this paper, we develope a model of $SU(2)$ theory in terms of local gauge invariant variables defined on a 4-dimensional space-time. In Section 2 we determine the components (gauge potentials) of the 2-form F and its dual ${}^*F$ and give the equations of structure for the gauge group $SU(2)$. We define a metric tensor $\overline{g}_{\mu\nu}$, $\mu,\nu=0,1,2,3$, in the Section 3 starting with the components of the curvature 2-form F and its Hodge dual ${}^*F$. The components $\overline{g}_{\mu\nu}$ are interpreted as new local gauge

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variables and they are calculated for a particular gauge field defined over a Minkowski space-time. In order to assure the property of self-duality of the 2-form F a convenient scaling factor Δ is introduced into the expression of the metric $\overline{g}_{\mu\nu}$. In Section 4 we obtain the self-duality equations and determine the independent gauge variables $\overline{g}_{\mu\nu}$ and $W_{\mu\nu\rho\sigma}$ which are exactly the freedom degrees that are left after eliminating the gauge degrees. The Section 5 is devoted to the study of compatibility between self-duality and Yang-Mills equations. In fact, we will write the Einstein-Yang-Mills equations and analyze only the Yang-Mills sector. The Einstein equations can not be obtained of course from self-duality. They should be obtained if we would consider a gauge theory having $P \times SU(2)$ as symmetry group, where P is the Poincaré group. More generally, a gauge theory of N-extended supersymmetry can be developed by imposing the self-duality condition.

2 Gauge potentials

We develop a $SU(2)$ Yang-Mills gauge theory over a Minkowski space-time M_4 endowed with the spherically symmetric metric:

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2.1}$$

where the coordinates are chosen such that $(x^\mu) = (t, r, \theta, \varphi)$, $\mu = 0, 1, 2, 3$. The components of the metric tensor are:

$$g_{00} = 1, \quad g_{11} = -1, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta, \tag{2.2}$$

and its determinant is

$$g = \det(g_{\mu\nu}) = -r^4\sin^2\theta, \quad \sqrt{-g} = r^2\sin\theta. \tag{2.3}$$

Let $P(M_4, SU(2), \pi)$ be the principal fibre bundle with M_4 as base manifold and $SU(2)$ as structural group. The mapping $\pi : P \longrightarrow M_4$ is the natural projection of P onto M_4. The Lie algebra of $SU(2)$ group is characterized by the following equations of structure:

$$[T_a, T_b] = \varepsilon_{abc}T_c, \quad a, b, c = 1, 2, 3, \tag{2.4}$$

where ε_{abc} is the Levi-Civita symbol of rank 3 with $\varepsilon_{123} = +1$. The gauge potentials $A_\mu = A_\mu^a T_a$, with values in the Lie algebra of the group $SU(2)$, determine a connection on the principal fibre bundle $P(M_4, SU(2), \pi)$ [8]. The Lie algebra-valued 1-form of connection on P is $A = A_\mu^a T_a dx^\mu$. Its 2-form of curvature F is defined by the formula:

$$F = dA + \frac{1}{2}[A, A]. \tag{2.5}$$

If we write F in the form

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu}^a T_a dx^\mu \wedge dx^\nu, \tag{2.6}$$

then we obtain the following expression for its components $F_{\mu\nu}^a$ (strength tensor of the gauge fields):

$$F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon_{abc} A^b_\mu A^c_\nu. \tag{2.7}$$

The spherically symmetric $SU(2)$ gauge potentials A^a_μ will be parametrized as (Witten ansatz) [7]:

$$A = -W \sin\theta d\varphi T_1 + W d\theta T_2 + (U dt + \cos\theta d\varphi) T_3, \tag{2.8}$$

where U and W are functions depending only on the variable r. Using (2.8), we obtain the following non-null components of the strength tensor:

$$F^1_{02} = -UW, \quad F^1_{13} = -W' \sin\theta, \tag{2.9a}$$

$$F^2_{03} = -UW \sin\theta, \quad F^2_{12} = W', \tag{2.9b}$$

$$F^3_{01} = -U', \; F^3_{23} = \left(W^2 - 1\right) \sin\theta \tag{2.9c}$$

with $U' = \frac{dU}{dr}$ and $W' = \frac{dW}{dr}$.

Now, we introduce the dual 2-form $^*F$ (the symbol "$*$" denoting the Hodge dual map) whose components are defined by

$$^*F^a_{\mu\nu} = \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma}, \tag{2.10}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol of rank 4 with $\varepsilon_{0123} = +1$, and $F^{a\rho\sigma} = g^{\rho\lambda} g^{\sigma\tau} F^a_{\lambda\tau}$. The non-null components of $^*F$ are:

$$^*F^1_{02} = W', \quad ^*F^1_{13} = -UW \sin\theta, \tag{2.11a}$$

$$^*F^2_{03} = W' \sin\theta, \quad ^*F^2_{12} = UW, \tag{2.11b}$$

$$^*F^3_{01} = \frac{W^2 - 1}{r^2}, \; F^3_{23} = r^2 U' \sin\theta. \tag{2.11c}$$

In the next section we will determine a metric tensor $\overline{g}_{\mu\nu}$ starting with the components $F^a_{\mu\nu}$ and $^*F^a_{\mu\nu}$. The components of this tensor will be interpreted as local gauge-invariant variables for the $SU(2)$ Yang-Mills gauge theory.

3 Local gauge variables

The gauge potentials A^a_μ are not invariant under the gauge transformations. But, we can introduce new local gauge-invariant variables $\overline{g}_{\mu\nu}$, given by [4, 10]:

$$\overline{g}_{\mu\nu} = \frac{1}{3\Delta^{1/3}} \varepsilon_{abc} F^a_{\mu\alpha} \, ^*F^{b\alpha\beta} F^c_{\beta\nu}, \tag{3.1}$$

and

(3.2) $$\overline{g}^{\mu\nu} = \frac{2}{3\Delta^{2/3}}\varepsilon_{abc}\ {}^*F^{a\mu\alpha}F^b_{\alpha\beta}\ {}^*F^{c\beta\nu}.$$

Here, Δ is a scale factor which will be chosen of a convenient form in what follows. The contravariant components of the dual 2-form ${}^*F$ are defined as usually

(3.3) $${}^*F^{a\mu\nu} = g^{\mu\rho}g^{\nu\sigma*}F^a_{\rho\sigma}.$$

The non-null components in our model are:

(3.4a) $${}^*F^{102} = -\frac{W'}{r^2},\quad {}^*F^{113} = -\frac{UW}{r^2\sin\theta},$$

(3.4b) $${}^*F^{203} = -\frac{W'}{r^2\sin\theta},\quad {}^*F^{212} = \frac{UW}{r^2},$$

(3.4c) $${}^*F^{301} = \frac{1-W^2}{r^2},\quad {}^*F^{323} = \frac{U'}{r^2\sin\theta},$$

Introducing the expressions (2.9) and (3.4) into the definition (3.1), we obtain the following non-null components of $\overline{g}_{\mu\nu}$:

(3.5a) $$\overline{g}_{00} = \frac{2}{r^2\Delta^{1/3}}W^2U^2U',$$

(3.5b) $$\overline{g}_{11} = \frac{2}{r^2\Delta^{1/3}}W'^2U',$$

(3.5c) $$\overline{g}_{22} = \frac{2}{r^2\Delta^{1/3}}(W^2-1)UWW',$$

(3.5c) $$\overline{g}_{33} = \frac{2}{r^2\Delta^{1/3}}(W^2-1)UWW'\sin^2\theta.$$

Having these quantities determined, we introduce a new metric manifold, whose line element written in the spherically variables (t,r,θ,φ) is

(3.6) $$d\sigma^2 = \overline{g}_{00}dt^2 + \overline{g}_{11}dr^2 + \overline{g}_{22}d\theta^2 + \overline{g}_{33}d\varphi^2,$$

or

(3.7) $$d\sigma^2 = \frac{2W^2U^2U'}{r^2}\left[dt^2 + \frac{W'^2}{W^2U^2}dr^2 + \frac{W'\left(W^2-1\right)}{WUU'}\left(d\theta^2+\sin^2\theta d\varphi^2\right)\right]$$

If we chose now the scale factor Δ in the form

(3.8) $$\Delta^{1/3} = \frac{2W^2U^2U'}{r^2},$$

then (3.7) reduces to (2.1) if we impose the following supplementary conditions:

$$W' = \frac{dW}{dr} = -iUW, \quad U' = \frac{dU}{dr} = i\frac{W^2 - 1}{r^2}. \tag{3.9}$$

But, these conditions are nothing else than the Yang-Mills field equations for the potentials $A_{\mu}^{a}(x)$ [13]. In fact, the Yang-Mills equations are differential equations of second order; in the case of ansatz (2.8), they have the following form:

$$\frac{d}{dr}\left(\frac{dW}{dr^2}\right) = -WU^2 + \frac{W\left(W^2 - 1\right)}{r^2}, \tag{3.10}$$

$$\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = 2UW^2. \tag{3.11}$$

It is easy to verify that the equations (3.10) and (3.11) result from the first-order equations given in (3.9).

Therefore, we conclude that the scale factor Δ chosen in (3.8), together with the field equations (3.9), reduce the new metric $\overline{g}_{\mu\nu}$ to that of the Minkowski space-time M_4.

4 Self-duality equations

A self-dual (or anti-self-dual) form T over a differential manifold M can be constructed only if M is of even dimension and the following equation is satisfied [3]:

$$**T = \lambda T; \ rankT = \frac{1}{2}\dim M. \tag{4.1}$$

But, the dual map (or the Hodge-duality) has the property:

$$\begin{aligned} **T &= (-1)^{k(n-k)}T \ (for\ Euclidean\ metric), \\ **T &= -(-1)^{k(n-k)}T \ (for\ Minkowski\ metric), \end{aligned} \tag{4.2}$$

where k is the rank of T and n is the dimension of M. This means that the quantity λ in (4.1) is constrained to very special values:

$$\pm T = **T = *(\lambda T) = \lambda^2 T;$$

that is

$$\begin{aligned} \lambda &= \pm 1, \ if \ **T = T\ , \ (Euclidean\ metric), \\ \lambda &= \pm i, \ if \ **T = -T, \ (Minkowski\ metric). \end{aligned} \tag{4.3}$$

In our model, the rank of F is $k = 2$ and the dimension of the space-time M_4 is $n = 4$. Then, the self-duality condition is [3]:

$$*F = iF \tag{4.4}$$

Now, if we introduce the components (2.9) and (2.11) in (4.4), we obtain the self-duality equations that coincide with the supplementary conditions (3.9). Therefore, the $SU(2)$ gauge theory with the Witten ansatz (2.8) is a self-dual with respect to the Minkowski metric (2.1). It is also self-dual with respect to the new metric $\overline{g}_{\mu\nu}$ defined by the equations (3.1) and (3.2). Indeed, the condition of self-duality with respect to the metric $\overline{g}_{\mu\nu}$ is [4]:

$$\frac{1}{2}\sqrt{-\overline{g}}\varepsilon_{\mu\nu\rho\sigma}\overline{F}^{a\rho\sigma} = iF^a_{\mu\nu}, \tag{4.5}$$

where $\overline{g} = \det\left(\overline{g}_{\mu\nu}\right)$ and $\overline{F}^{a\rho\sigma} = \overline{g}^{\rho\lambda}\overline{g}^{\sigma\tau}F^a_{\lambda\tau}$. The metric tensor $\overline{g}_{\mu\nu}$ is symmetric and has 10 independent component. We have

$$\overline{g} \equiv \det\left(\overline{g}_{\mu\nu}\right) = \frac{1}{4}\Delta^{2/3}. \tag{4.6}$$

The Δ scalings in (3.1) and (3.2) have been chosen so that $\overline{g}_{\mu\nu}$ will be a covariant tensor. Actually, the distinction between self-dual and anti-self-dual properties here is just what sign we take in $\sqrt{-\overline{g}} = \pm i\Delta^{1/3}$.

We define now, the tensor

$$W_{\mu\nu\rho\sigma} = F^a_{\mu\nu}F^a_{\rho\sigma} - \frac{1}{24\sqrt{-\overline{g}}}\varepsilon^{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta}\left(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} - \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho} + \sqrt{-\overline{g}}\varepsilon_{\mu\nu\rho\sigma}\right). \tag{4.7}$$

It is traceless and $\overline{g}_{\mu\nu}$-self-dual tensor and has only five independent components. Therefore, the 10 fields of the metric $\overline{g}_{\mu\nu}$ together with the 5 independent fields corresponding to $W_{\mu\nu\rho\sigma}$ form 15 independent variables, which is exactly the number of degrees of freedom that are left after eliminating the gauge degrees.

5 Einstein-Yang-Mills equations

We will impose the condition that the metric $\overline{g}_{\mu\nu}$ determines a spherically symmetric line element of the form [7,10]:

$$ds^2 = Ndt^2 - \frac{1}{N}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{5.1}$$

where N is a function of the variable r only. For $N = 1 - \frac{2m}{r}$ we obtain the Schwarzschild metric, while for $N = 1 - \frac{2m}{r} + \frac{Q^2+1}{r^2}$ we have the Reissner-Nordström (RS) metric. In this case, the supplementary conditions (3.9) have to be changed by:

$$NW' = -iUW, \quad U' = i\frac{W^2-1}{r^2}. \tag{5.2}$$

In addition, the scaling factor Δ is supposed to be defined as:

$$\Delta^{1/3} = \frac{2W^2U^2U'}{r^2N}. \tag{5.3}$$

It is easy to verify that the conditions (5.2) express the property of self duality for the strength tensor $F^a_{\mu\nu}$ with respect to the new metric considered in (5.1).

The integral action of our model is [7]:

$$S_{EYM} = \int \left(-\frac{1}{16\pi G} R - \frac{1}{4K\alpha_s^2} Tr F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-\overline{g}}\ d^4x, \tag{5.4}$$

where α_s is the $SU(2)$ gauge coupling (strong) constant, R is the scalar curvature associated to $\overline{g}_{\mu\nu}$ and $Tr(T_a T_b) = K\delta_{ab}$. For $G = SU(2)$ we choose $T_a = \frac{1}{2}\tau_a$ (τ_a being the Pauli matrices) and then $K = \frac{1}{2}$. The gravitational constant G is the only dimensionful quantity in the action (the units $\hbar = c = 1$ are understood).

Taking $\delta S_{EYM} = 0$ with respect to A_μ^a and $\overline{g}_{\mu\nu}$ fields, we obtain the following general form of the EYM equations [9]:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-\overline{g}} F^{a\mu\nu}) + f_{bc}^a A_\mu^b F^{c\mu\nu} = 0, (Yang - Mills\ equations), \tag{5.5}$$

where $f_{bc}^a = -f_{cb}^a$ are the structure constants of the gauge group, and respectively

$$R_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\ R = 8\pi G T_{\mu\nu},\ (\text{E}instein\ equations), \tag{5.6}$$

with the gauge-invariant stress-energy tensor

$$T_{\mu\nu} = \frac{1}{K\alpha_s^2} Tr \left(-F_{\mu\rho} F_\nu{}^{\ \rho} + \frac{1}{4} F_{\rho\lambda} F^{\rho\lambda} \overline{g}_{\mu\nu} \right), \tag{5.7}$$

For the $SU(2)$ gauge group the structure constants f_{bc}^a are given by the Levi-Civita symbol ε_{abc} of rank 3, with $\varepsilon_{123} = +1$. Then, introducing the metric components $\overline{g}_{\mu\nu}$ in (5.5) and (5.6), we obtain the Einstein-Yang-Mills (EYM) equations of our model:

$$(NW')' = \frac{W\left(W^2 - 1\right)}{r^2} - \frac{U^2 W}{N}, \tag{5.8}$$

$$\left(r^2 U'\right)' = \frac{2UW^2}{N}, \tag{5.9}$$

$$W'^2 + \frac{W^2 U^2}{rN^2} = 0, \tag{5.10}$$

$$\frac{1}{2}\left(N'r + N - 1\right) + \frac{r^2 U'^2}{2} + \frac{U^2 W^2}{N} + NW'^2 + \frac{\left(W^2 - 1\right)^2}{2r^2} = 0, \tag{5.11}$$

where we used $K = \frac{1}{2}$ and $\frac{4\pi G}{\alpha_s^2} = 1$ units. These equations admit the particular solution [6,8]:

$$U = 0,\ \ W = \pm 1,\ \ N = 1 - \frac{2m}{r}, \tag{5.12}$$

which describes the Schwarzschild metric and a pure gauge Yang-Mills field. Therefore, the $SU(2)$ gauge model (2.8) has the property of self-duality on a Schwarzschild space-time.

The EYM field equations (5.8)-(5.11) admit also the solution with a non-trivial gauge field describing colored black holes [1]:

$$U = U_0 + \frac{Q}{r},\ W = 0,\ N = 1 - \frac{2m}{r} + \frac{Q^2+1}{r^2}. \tag{5.13}$$

where U_0 is a constant. It corresponds to the RN metric with the electric charge Q and the unit magnetic charge. However, it is not a solution of the self-duality equations (5.2), so that the model (2.8) can not be self-dual on a RN space-time.

Many others solutions (particle-like, sphaleron type, with Λ-term, stringy type, axially symmetric etc.) for the $SU(2)$ gauge theory are given by Volkov and Gal'tsov [7]. Local solutions of the static, spherically symmetric, EYM equations with $SU(2)$ gauge group are studied by Zotov [14] on the basis of dynamical system methods. In this case it is proven the existence of solutions with oscillating metric as well as the existence of local solutions for all known formal series expansions. Exact solutions for $SU(2)$ gauge theory with axial symmetry are given in Ref. [10]. However, these solutions are not self-dual.

Let us compare now the self-duality equations (5.2) with the first two EYM equations (5.8) and (5.9). If we take the derivatives with respect to r of the equations (5.2), then we obtain:

$$\begin{aligned} &(NW')' = -i\left(U'W + UW'\right), \\ &\left(r^2U'\right)' = 2iWW'. \end{aligned} \tag{5.14}$$

Now, if we replace iW' and iU' deduced from (5.2) into the right-hand sides of (5.14), then we obtain the EYM equations (5.8) and (5.9). Of course, the other two EYM equations (5.10) and (5.11) can not be obtained from the self-duality equations of the gauge fields. This may be possible if we develope a gauge theory with the gauge group $P \times SU(2)$, where P is the Poincaré group [10].

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