The bounds for the squared norm of the second fundamental form of minimal submanifolds of S^{n+p}

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Abstract. The aim of this paper is to study some properties of compact minimal submanifold M of the standard Euclidean sphere S^{n+p} with flat normal connection. We will give a lower bound for the squared form S of the second fundamental form h of M in terms of the gap $n-\lambda_1$ when S is constant, where λ_1 stands for the first eigenvalue of the Laplacian of M. Moreover, we will prove that S is actually a constant if M, in addition, is non-negatively curved and give an upper bound for S as well as a lower bound. Finally, as applications of these results to the case of hypersurfaces, we will also give a lower bound for λ_1, which is better than that in [5].

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1 Introduction and main results

Let M be an n-dimensional compact minimal submanifold in the standard Euclidean sphere S^{n+p} with the second fundamental form h. We denote by S the square of the length of h. Throughout this paper, we shall make use of the convention on the ranges of indices: $1 \leq i, j, k, \cdots \leq n$; $n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p$.

There is a well-known theorem due to Simons [9] showed that if S satisfies $0 \leq S \leq \frac{n}{2-\frac{1}{p}}$, then either $S = 0$, and M is totally geodesic, or else $S = \frac{n}{2-\frac{1}{p}}$. Later, Chern, do Carmo and Kobayashi [4] further obtained that the Veronese surface in S^4 and the submanifold $S^m\left(\sqrt{\frac{m}{n}}\ \right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\ \right)$ in S^{n+1} are the only compact minimal submanifolds of dimension n in S^{n+p} satisfying $S = \frac{n}{2-\frac{1}{p}}$. According to the above results, it is plausible that the set of values for S is discrete, at least S does not arbitrarily large. If this is the case, an estimate of the value for S next to $\frac{n}{2-\frac{1}{p}}$ should be of interest. Leung [6] showed that the gap $n - \lambda_1$ is a lower bound for S provided that S is constant, where λ_1 stands for the the first eigenvalue of the Laplacian operator $\triangle$ on M. Recently, Barbosa and Barros [1] improved Leung's gap for compact minimal hypersurface $M \subset S^{n+1}$ by showing that there is a rational

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constant $k \in [\frac{n}{n-1}, n]$ depending either on h or on the first eigenfunction of $\triangle$ such that $S \geq k\frac{n-1}{n}(n-\lambda_1)$.

In the present paper, we take on two goals. First, we study the similar problems in the case of higher codimension and obtain two inequalities concerning the squared norm of second fundamental form. Following which, we obtain the lower bounds for S if it is constant.

Theorem 1.1. *Let M be an n-dimensional compact orientable minimal submanifold in the standard Euclidean sphere S^{n+p} with flat normal connection. Let f be an eigenfunction of the Laplacian of M associated to λ_1. Let $l(q)$ denotes the number of nonzero components of ∇f with respect to a principal referential $E_q = \{e_i(q)\}_{i=1}^n$ at $q \in M$. Set $l_0 = \min_{q\in M}\{l(q)|\nabla f(q) \neq 0\}$ and $k_0 = \begin{cases} \frac{n}{n-1}, & \text{if } l_0 = 1 \\ l_0, & \text{if } l_0 \geq 2 \end{cases}$. Then*

$$\int_M S|\nabla f|^2 \geq \frac{k_0(n-1)(n-\lambda_1)}{n}\int_M |\nabla f|^2.$$

In particular, if S is a constant, we have $S \geq \frac{k_0(n-1)(n-\lambda_1)}{n}$.

Theorem 1.2. *With the same assumptions on M and f as in Theorem 1.1. Let $k = \max_{n+1\leq\alpha\leq n+p}\{dim(kerA_\alpha)\}$ and set $n_0 = \begin{cases} k, & \text{if } k \leq n-2 \\ n-2, & \text{if } k = n-1 \text{ or } k = n \end{cases}$, where A_α is the shape operator in the direction e_α. Then*

$$\int_M S|\nabla f|^2 \geq \frac{(n-n_0)(n-1)(n-\lambda_1)}{n}\int_M |\nabla f|^2.$$

In particular, if S is a constant, we have $S \geq \frac{(n-n_0)(n-1)(n-\lambda_1)}{n}$.

Remark 1.1 In [14], Takahashi showed that n is an upper bound for λ_1. Therefore, either lower bound of S we obtain in Theorems 1.1 and 1.2 is nonnegative.

Remark 1.2 For codimension $p = 1$, normal connection of M in S^{n+1} is naturally flat. Therefore, Theorems 1.1 and 1.2 include that in [1] as the special cases.

Second, for submanifold M assumed in Theorem 1.1 or 1.2, applying Bochner technique, we show that S is actually a constant if M is also non-negatively curved. Furthermore, as a corollary of Theorem 1.1 or 1.2, we obtain a lower bound for S. Another method will lead to an upper bound for S. These results, applying to the case of hypersurfaces, will improve the lower bound for λ_1 in [5] (see Corollaries 4.1 and 4.2).

Theorem 1.3. *With the same assumptions on M as in Theorem 1.1. If, in addition, M is non-negatively curved, then S must be a constant. Furthermore, $\frac{k_0(n-1)(n-\lambda_1)}{n} \leq S \leq np$ or $\frac{(n-n_0)(n-1)(n-\lambda_1)}{n} \leq S \leq np$, where k_0, n_0 are given as in Theorem 1.1 and 1.2 respectively.*

In fact, most of the classification theorems for submanifolds in S^{n+p} based on the assumption of the upper bound for S (cf. [11], [12], [13], [15]). As I know, there are a few results about the estimate of upper bound of S as well as that of lower bound if we exclude the totally geodesic case. Our progress in Theorem 1.3 is to prove that S must be constant under the additional restriction—M is non-negatively curved, furthermore, to give both bounds from below and above for S. If M is an n-dimensional complete and connected minimal submanifold in the standard Euclidean sphere S^{n+p} with the parallel second fundamental form, Mo [7] obtained the same upper bound for S as that in Theorem 1.3.

2 Preliminaries

For a compact submanifold M of S^{n+p}, we choose a local field of orthonormal frames $\{e_1, \cdots, e_{n+p}\}$ in S^{n+p} such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and the remaining vectors $e_{n+1}, \cdots, e_{n+p}$ are normal to M. Then the second fundamental form h of M is given by

$$h(e_i, e_j) = \sum_{\alpha=n+1}^{n+p} h_{ij}^{\alpha} e_{\alpha},$$

where $h_{ij}^{\alpha} = \langle A_{\alpha} e_i, e_j \rangle$ and A_{α} is the shape operator in the direction e_{α}. The equations of Gauss, Codazzi and Ricci are respectively

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_{\alpha=n+1}^{n+p} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}), \tag{2.1}$$

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}, \tag{2.2}$$

$$R_{\alpha\beta ij}^{\perp} = \langle [A_{\alpha}, A_{\beta}](e_i), e_j \rangle, \tag{2.3}$$

where $R, R^{\perp}$ are the curvature tensors corresponding to the connection ∇ on M and the normal connection $\nabla^{\perp}$ respectively. For $X, Y, Z, W \in \mathcal{X}(M)$, $\mathcal{X}(M)$ is the Lie algebra of smooth vector fields on M, the first and second covariant derivatives of h are given by

$$(\nabla h)(X, Y, Z) = \nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

$$\begin{aligned}(\nabla^2 h)(X, Y, Z, W) = {} & \nabla_X^{\perp}((\nabla h)(Y, Z, W)) - (\nabla h)(\nabla_X Y, Z, W) \\ & - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W).\end{aligned}$$

Also, the Ricci identity reads as

$$\begin{aligned}& (\nabla^2 h)(X, Y, Z, W) - (\nabla^2 h)(Y, X, Z, W) \\ & = R^{\perp}(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W).\end{aligned} \tag{2.4}$$

We recall now the Bochner formula (cf. [2] or [10]), which states that for a differentiable function $f : M \to \mathbb{R}$,

$$\frac{1}{2}\triangle(|\nabla f|^2) = \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla(\triangle f)\rangle + |\mathrm{Hess} f|^2, \tag{2.5}$$

where Ric denote the Ricci tensor of M, and for $X, Y \in \mathcal{X}(M)$,

$$\langle \nabla f, X\rangle = X(f), \quad \mathrm{Hess}\, f(X,Y) = \langle \nabla_X(\nabla f), Y\rangle, \quad \triangle\, f = \mathrm{tr}(\mathrm{Hess}\, f).$$

For a bilinear form A, the norm of A considered here is the Euclidean, which is given by $|A|^2 = \mathrm{tr}(AA^t)$.

Let I denotes the identity operator on the tangent bundle TM of M, for any $t \in \mathbb{R}$, we have

$$|\mathrm{Hess} f - tfI|^2 = |\mathrm{Hess} f|^2 - 2tf\triangle f + nt^2 f^2.$$

Therefore, if $\triangle f + \lambda_1 f = 0$, then

$$\int_M |\mathrm{Hess} f - tfI|^2 = \int_M |\mathrm{Hess} f|^2 + \Big(2t + \frac{n}{\lambda_1}t^2\Big)\int_M |\nabla f|^2. \tag{2.6}$$

In particular, putting $t = -\frac{\lambda_1}{n}$ into (2.6), we get

$$\begin{aligned} \int_M |\mathrm{Hess} f|^2 &= \int_M |\mathrm{Hess} f + \frac{\lambda_1}{n} fI|^2 + \frac{\lambda_1}{n}\int_M |\nabla f|^2 \\ &\geq \frac{\lambda_1}{n}\int_M |\nabla f|^2. \end{aligned} \tag{2.7}$$

Moreover, the equality holds if and only if M is isometric to the sphere $S^n(\sqrt{\lambda_1/n}\,)$ (see Obata [8, Theorem A]).

Also, we need the following lemma in the rest sections.

Lemma 2.1.([1]) *Let V be an inner product space of finite dimension n and $T : V \to V$ be a nontrivial traceless symmetric linear operator. Let $\{e_1, \cdots, e_n\}$ be an orthonormal referential such that $Te_i = u_i e_i, i = 1, \cdots, n$. Then given a nonzero vector $v = \sum_{i=1}^{n} v_i e_i$, we have*

$$\frac{1}{n-k}|T|^2|v|^2 \geq \sum_{i=1}^{n} u_i^2 v_i^2, \tag{2.8}$$

or

$$\frac{1}{k_0}|T|^2|v|^2 \geq \sum_{i=1}^{n} u_i^2 v_i^2, \tag{2.9}$$

where $k = \dim(\ker T), k_0 = \begin{cases} \frac{n}{n-1}, & \text{if } l_0 = 1 \\ l_0, & \text{if } l_0 \geq 2 \end{cases}$ and l_0 be the number of nonzero components v_i of v.

3 Proof of Theorems

Proof of Theorem 1.1. Since the normal connection is flat, for every point $q \in M$, all the shape operators A_α can be diagonalized simultaneously with respect to the same local orthonormal frame $\{e_1, \cdots, e_n\}$(cf. [3], p.127). Choose a local orthonormal frame $\{e_{n+1}, \cdots, e_{n+p}\}$ of normals such that $A_\alpha e_i = \lambda_i^\alpha e_i$, where λ_i^α are the smooth functions. Since M is minimal, we have from (2.1) that

$$\mathrm{Ric}(e_i, e_i) = (n-1) - \sum_{\alpha=n+1}^{n+p} (\lambda_i^\alpha)^2.$$

Now for a differentiable function f defined on M, writing $\nabla f = \sum_{i=1}^{n} f_i e_i$ at $q \in M$, we get

$$\mathrm{Ric}(\nabla f, \nabla f) = (n-1)|\nabla f|^2 - \sum_{\alpha=n+1}^{n+p} \Big(\sum_{i=1}^{n} (\lambda_i^\alpha)^2 f_i^2\Big).$$

We may apply (2.9) at each point of M to obtain the inequality

$$\frac{1}{k_0}\Big(\sum_{i=1}^{n} (\lambda_i^\alpha)^2\Big)|\nabla f|^2 \geq \sum_{i=1}^{n} (\lambda_i^\alpha)^2 f_i^2,$$

where k_0 is given as in Theorem 1.1. Consequently, we derive

$$\mathrm{Ric}(\nabla f, \nabla f) \geq (n-1)|\nabla f|^2 - \frac{1}{k_0} S|\nabla f|^2, \tag{3.1}$$

in addition $\triangle f = -\lambda_1 f$, then the Bochner formula (2.5) leads to

$$\frac{1}{2}\triangle(|\nabla f|^2) = \mathrm{Ric}(\nabla f, \nabla f) + |\mathrm{Hess} f|^2 - \lambda_1 |\nabla f|^2. \tag{3.2}$$

Integrating (3.2) on M and using (2.7) and (3.1), we get

$$0 \geq \frac{\lambda_1}{n}\int_M |\nabla f|^2 + (n-1)\int_M |\nabla f|^2 - \frac{1}{k_0}\int_M S|\nabla f|^2 - \lambda_1 \int_M |\nabla f|^2.$$

Therefore,

$$\int_M S|\nabla f|^2 \geq \frac{k_0(n-1)(n-\lambda_1)}{n}\int_M |\nabla f|^2,$$

which completes the proof of the Theorem 1.1. □

Proof of Theorem 1.2. With the same symbols as in the proof of Theorem 1.1. Let f be an eigenfunction associated to the first eigenvalue λ_1 of Laplacian operator on M, and write $\nabla f = \sum_{i=1}^{n} f_i e_i$. Then,

(1) in the case of $\dim(\ker A_\alpha) \leq n-2$, it follows from (2.8) that

$$\frac{1}{n - n_0^\alpha}\Big(\sum_{i=1}^{n} (\lambda_i^\alpha)^2\Big)|\nabla f|^2 \geq \sum_{i=1}^{n} (\lambda_i^\alpha)^2 f_i^2, \tag{3.3}$$

where $n_0^\alpha = \dim(\ker A_\alpha)$.

(2) in the case of $\dim(\ker A_\alpha) \geq n-1$, i.e. $\dim(\ker A_\alpha) = n-1$ or n, we note $A_\alpha \equiv 0$, because M is minimal. In this case, setting $n_0^\alpha = n-2$, so (3.3) also holds.

Setting $n_0 = \max_{n+1\leq\alpha\leq n+p}\{n_0^\alpha\}$, we have from either of the cases (1) or (2) that

$$\frac{1}{n-n_0}S|\nabla f|^2 \geq \sum_{\alpha=n+1}^{n+p}\Big(\sum_{i=1}^{n}(\lambda_i^\alpha)^2 f_i^2\Big).$$

Therefore,

$$\mathrm{Ric}(\nabla f, \nabla f) \geq (n-1)|\nabla f|^2 - \frac{1}{n-n_0}S|\nabla f|^2. \tag{3.4}$$

and the rest proof follows as in the proof of Theorem 1.1 after integrating (3.2) and using (3.4). □

Remark 3.1 In fact, in the course of the above proof, if M is totally geodesic, i.e. $A_\alpha \equiv 0$, we have $\lambda_1 = n$.

Proof of Theorem 1.3. With the same symbols as in the proof of Theorem 1.1. Then we have

$$\begin{aligned}
&\sum_{i,j=1}^{n}\mathrm{Ric}(e_j, A_{h(e_i,e_j)}e_j) - \sum_{i,j,k=1}^{n}\langle R(e_k,e_i)e_j, A_{h(e_i,e_j)}e_k\rangle\\
=&\sum_{\alpha=n+1}^{n+p}\Big[\sum_{i,j=1}^{n}\langle A_\alpha e_i, e_j\rangle \mathrm{Ric}(e_i, A_\alpha e_j) - \sum_{i,j,k=1}^{n}\langle A_\alpha e_i, e_j\rangle\langle R(e_k,e_i)e_j, A_\alpha e_k\rangle\Big]\\
=&\frac{1}{2}\sum_{\alpha=n+1}^{n+p}\sum_{j,k=1}^{n}(\lambda_j^\alpha - \lambda_k^\alpha)^2\langle R(e_k,e_j)e_j, e_k\rangle\\
\geq& 0 \text{ (since } M \text{ is non-negatively curved).}
\end{aligned} \tag{3.5}$$

Define $F : M \to \mathbb{R}$ by $F = \frac{1}{2}S$, then the Laplacian of F is given by

$$\begin{aligned}
\triangle F =& \sum_{k=1}^{n}[e_k e_k(F) - (\nabla_{e_k}e_k)(F)]\\
=& \sum_{i,j,k=1}^{n}\langle(\nabla^2 h)(e_k,e_k,e_i,e_j), h(e_i,e_j)\rangle + \sum_{i,j,k=1}^{n}\|(\nabla h)(e_i,e_j,e_k)\|^2.
\end{aligned} \tag{3.6}$$

When M is minimal, for $X, Y \in \mathcal{X}(M)$, we have

$$\sum_{i=1}^{n}h(e_i,e_i) = 0, \quad \sum_{i=1}^{n}(\nabla h)(X,e_i,e_i) = 0, \quad \sum_{i=1}^{n}(\nabla^2 h)(X,Y,e_i,e_i) = 0. \tag{3.7}$$

Substituting (2.2), (2.4) and (3.7) into (3.6) and noticing that $R^\perp = 0$, we get

$$\triangle F = \sum_{i,j=1}^{n} \mathrm{Ric}(e_j, A_{h(e_i,e_j)}e_j) - \sum_{i,j,k=1}^{n} \langle R(e_k, e_i)e_j, A_{h(e_i,e_j)}e_k\rangle + \|\nabla h\|^2. \tag{3.8}$$

Integrating (3.8) on M, we get

$$\int_M \Big[\|\nabla h\|^2 + \sum_{i,j=1}^{n} \mathrm{Ric}(e_j, A_{h(e_i,e_j)}e_j) - \sum_{i,j,k=1}^{n} \langle R(e_k, e_i)e_j, A_{h(e_i,e_j)}e_k\rangle\Big] = 0.$$

Using (3.5), we have

$$\sum_{i,j=1}^{n} \mathrm{Ric}(e_j, A_{h(e_i,e_j)}e_j) = \sum_{i,j,k=1}^{n} \langle R(e_k, e_i)e_j, A_{h(e_i,e_j)}e_k\rangle, \tag{3.9}$$

and

$$\|\nabla h\|^2 = 0. \tag{3.10}$$

Together with (3.8), we have $\triangle F = 0$, then $F = \frac{1}{2}S = \text{const.}$, according to Theorem 1.1, we get $S \geq \frac{k_0(n-1)(n-\lambda_1)}{n}$. Similarly, it follows from Theorem 1.2 that $S \geq \frac{(n-n_0)(n-1)(n-\lambda_1)}{n}$.

Now, we turn to estimate the upper bound of S. Equation (2.1) implies

$$A_{h(e_j,e_k)}e_i = R(e_i, e_k)e_j - \delta_{kj}e_i + \delta_{ij}e_k + A_{h(e_i,e_j)}e_k. \tag{3.11}$$

Taking inner product in (3.11) with $A_{h(e_i,e_j)}e_k$, we get

$$\sum_{i,j,k=1}^{n} \langle A_{h(e_i,e_j)}e_k, A_{h(e_j,e_k)}e_i\rangle = \|A_h\|^2 - \sum_{i,j,k=1}^{n} \langle R(e_k, e_i)e_j, A_{h(e_i,e_j)}e_k\rangle - S, \tag{3.12}$$

where $\|A_h\|^2 = \sum\limits_{i,j,k=1}^{n} \|A_{h(e_i,e_j)}e_k\|^2$. Similarly, we have from (2.1)

$$\sum_{i,j,k=1}^{n} \langle A_{h(e_j,e_k)}e_k, A_{h(e_i,e_j)}e_i\rangle = (n-1)S - \sum_{i,j=1}^{n} \mathrm{Ric}(e_j, A_{h(e_i,e_j)}e_i). \tag{3.13}$$

Since $R^{\perp} = 0$, substituting (3.12) and (3.13) into (2.3), we arrive at

$$\|A_h\|^2 - \sum_{i,j,k=1}^{n} \langle R(e_k, e_i)e_j, A_{h(e_i,e_j)}e_k\rangle = nS - \sum_{i,j=1}^{n} \mathrm{Ric}(e_j, A_{h(e_i,e_j)}e_j). \tag{3.14}$$

Combining (3.9) and (3.14), we have

$$\|A_h\|^2 = nS. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\|A_h\|^2 &= \sum_{i,j,k=1}^{n} \|A_{h(e_i,e_j)}e_k\|^2\\ &= \sum_{\alpha=n+1}^{n+p}\sum_{i,j,k=1}^{n} \langle A_\alpha e_i, e_j\rangle^2 \|A_\alpha e_k\|^2\\ &= \sum_{\alpha=n+1}^{n+p} \|A_\alpha\|^4.\end{aligned}$$

Also from (3.15), we have $\sum_{\alpha=n+1}^{n+p}(\|A_\alpha\|^4 - n\|A_\alpha\|^2) = 0$ or equivalently,

$$\sum_{\alpha=n+1}^{n+p}\Big(\|A_\alpha\|^2 - \frac{n}{2}\Big)^2 = \frac{n^2 p}{4}. \tag{3.16}$$

Now using Schwarz inequality, we get

$$\sum_{\alpha=n+1}^{n+p}\Big(\|A_\alpha\|^2 - \frac{n}{2}\Big)^2 \geq \frac{1}{p}\Big[\sum_{\alpha=n+1}^{n+p}(\|A_\alpha\|^2 - \frac{n}{2})\Big]^2 = \frac{1}{p}(S - \frac{np}{2})^2. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$S(S - np) \leq 0.$$

which leads to $S = 0$ or $S \leq np$. If $S \leq np$, Theorem 1.3 holds. If $S = 0$, according to remark 3.1, we have $\lambda_1 = n$. Therefore, $S = \frac{k_0(n-1)(n-\lambda_1)}{n} = \frac{(n-n_0)(n-1)(n-\lambda_1)}{n}$, and Theorem 1.3 also holds. In this way, we complete the proof of the Theorem 1.3. □

4 Applications

If $\varphi : M \to S^{n+p}$ is a minimal immersion, it was proved that n is an upper bound for λ_1 by Takahashi [14]. So it was conjectured by Yau [16] that for any embedded compact minimal hypersurface $M \subset S^{n+1}$, the first eigenvalue λ_1 of the Laplacian of M satisfies $\lambda_1 = n$. Later, Choi and Wang [5] proved that $\lambda_1 \geq \frac{n}{2}$. Now, as applications of Theorem 1.3 to the case of hypersurfaces, we have:

Corollary 4.1. *Let $M^n(n \geq 2)$ be a compact orientable non-negatively curved minimal hypersurface of the standard Euclidean sphere S^{n+1}. Then $\lambda_1 \geq n - \frac{n^2}{k_0(n-1)}$, where k_0 is given as in Theorem 1.1.*

Similarly, we have:

Corollary 4.2. *Let $M^n(n \geq 2)$ be a compact orientable non-negatively curved minimal hypersurface of the standard Euclidean sphere S^{n+1}. Then $\lambda_1 \geq n - \frac{n^2}{(n-n_0)(n-1)}$, where n_0 is given as in Theorem 1.2.*

Remark 4.1 For $n \geq 3$, when $k_0 \in [3, n]$, then $n - \frac{n^2}{k_0(n-1)} \geq \frac{n}{2}$. In this case, Corollary 4.1 provides a better lower bound than that in [5]. For Corollary 4.2, we can similarly discuss.

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