A new proof of the Riemann-Poincaré uniformization theorem

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Abstract. Within the framework of the prescribing curvature problem for Riemannian 2-manifolds, we give a new proof of the Riemann-Poincaré Uniformization Theorem. The approach is variational and the method is based on a lemma of Brézis [2]. As a significant feature, we have been able to reveal an intimate bond between the Differential Geometry of a Riemannian surface and a space of functions defined on it.

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1 Introduction

Let (S, g) be a compact connected 2-dimensional Riemannian C^∞-manifold with Gauss curvature k and Euler-Poincaré characteristic χ. After the Gauss-Bonnet Theorem, the total curvature of (S, g) and χ are related by

$$\int_S k dS = 2\pi\chi.$$

The pointwise conformal class of (S, g) consists of the Riemannian 2-manifolds $(S, \tilde{g})$ such that $\tilde{g} = ge^{2u}$, for some $u \in F = C^\infty(S; \mathbb{R})$. The purpose of this paper is to give a new proof of the following version of the Riemann-Poincaré Uniformization Theorem, cf. [6], [5].

Theorem 1.1. *If $\chi = 0$ (respectively $\chi < 0$, $\chi > 0$), then there is a member $(S, \tilde{g})$ in the pointwise conformal class of (S, g) with Gauss curvature $K \equiv 0$ (respectively $K \equiv -1$, $K \equiv 1$).*

The space F is one of the most evident functional spaces to study the geometry of (S, g). The family of seminorms $p_n(u) = \max\{|D^\alpha u(p)| : p \in S, \alpha \leq n\}$ furnishes F with a structure of nonnormable locally convex Fréchet (and so, Hausdorff) space with the Heine-Borel property (so, it is locally compact). One possible choice for a metric $d : F \times F \to \mathbb{R}$ is given by

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$$d(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(u-v)}{1+p_n(u-v)}.$$

In order to allow the use of standard compactness arguments and simplify the proofs, the one-point compactification $E = F \cup \{\infty\}$ of F will be used instead of F itself. This procedure should be regarded as the construction of a (relative, finer) topology for F suitable to handle the geometrical problem. Furthermore, to avoid the persistent reference to the subspace C of constant functions on S, we will sometimes make use of E/C, the quotient space obtained from E by the equivalence relation $u \sim v \Leftrightarrow u - v \in C$.

With the previous notions, it is possible to characterize the problem under consideration in terms of a nonlinear partial differential equation or a related variational problem.

Proposition 1.2. *The following assertions are equivalent :*

(i) $c \in C$ *is the Gauss curvature of a member in the conformal class of* (S, g).

(ii) $\Delta u - k + ce^{2u} = 0$ *in* (S, g) *for some* $u \in E$.

(iii) The function $f : E \to \mathbb{R}$ *defined by*

$$f(u) = \frac{1}{2} \int_S \langle \nabla u, \nabla u \rangle + 2ku - ce^{2u} dS$$

has a critical point.

Proof. (Sketch) (i) $\Leftrightarrow$ (ii) is a straightforward geometric computation, cf. [4], pp. 15-16 and also [7], for an elementary explanation. As f is C^{∞}-differentiable in E, (ii) $\Leftrightarrow$ (iii) comes after calculating the first derivative

$$(dfu, v) = \int_S \langle \nabla u, \nabla v \rangle + (k - ce^{2u}) v dS = \int_S (-\Delta u + k - ce^{2u}) v dS$$

together with the "fundamental lemma" of the Calculus of Variations, cf. [9]. □

By focusing on the variational approach, we will appeal to certain monotonicity-like properties of *df* which guarantee the existence of critical points $u \in E$ of f. Concretely, we will use the general notion of m-map, cf. [2], pp. 123-124.

Definition 1.3. *df* is an m-map if it verifies the following two properties :

(i) For each sequence u_i in a compact subset of E such that $u_i \longrightarrow u$, $dfu_i \longrightarrow z$ and $\limsup (dfu_i, u_i) \leq (z, u)$, we have $dfu = z$.

(ii) The restrictions of *df* to the finite-dimensional subspaces of E are continuous.

Continuous *df* maps are m-maps. Prominently, any monotone hemicontinuous map *df* is an m-map and f is convex if and only if *df* is monotone hemicontinuous. The analytical instrument to prove the existence of critical points will be the following lemma, cf. [2], pp. 124–126.

Lemma 1.4 (Brézis). *Assume $A \subset E$ is a convex compact subset of E containing zero and df is an m-map such that $(dfv, v) \neq 0$ for all $v \in E - A$. Then, the set of critical points $\{u \in A : dfu = 0\}$ is nonempty and compact.*

Section 2 is devoted to the proof of Theorem 1.1. At the end, some conclusions regarding the strong bond between the analytical methods employed and the geometry of the underlying surface will be drawn out.

2 Pointwise uniformization

Proof of Theorem 1.1. Let $k \in E$ and $\int_S kdS = 0$. For the sake of explicitness, we look first at Laplace equation $\Delta u = 0$ in (S, g), $u \in E/C$. The function

$$f(u) = \frac{1}{2}\int_S \langle \nabla u, \nabla u \rangle dS$$

(Dirichlet integral) is convex. This means df is monotone hemicontinuous and henceforth, an m-map. $A = \{0\} \subset E/C$ is trivially convex, closed and compact. Since $(dfv, v) = \int_S \langle \nabla v, \nabla v \rangle dS > 0$, for all $v \in E/C - A$, Lemma 1.4 yields $0 \in E/C$ is the unique harmonic function on (S, g), cf. [1], p. 142. So, we move on straight into Poisson equation. Then, k changes sign in S and there is an open set $\varnothing \subsetneq T \subsetneq S$ in which k is negative (respectively positive). The subset

$$A = \left\{ v \in E : (dfv, v) = \int_S \langle \nabla v, \nabla v \rangle + kv \; dS \leq 0 \right\}$$

contains zero and is closed because it is the inverse image of the interval $(-\infty, 0]$ under the continuous convex form $(df\cdot, \cdot) : E \to \mathbb{R}$. Hence, A is compact and convex. We notice $E - A \neq \varnothing$, as any nonpositive (respectively nonnegative) v with support in $\bar{T}$ belongs to it. Also, the convexity of the function

$$f(u) = \frac{1}{2}\int_S \langle \nabla u, \nabla u \rangle + 2ku \; dS$$

implies df is monotone hemicontinuous. Consequently, f has a critical point in A. We conclude $\Delta u = k \in E$ is solvable in (S, g) if and only if k has zero mean in (S, g). The solution u is unique in E/C.

If $\int_S kdS = 2\pi\chi < 0$, k is negative in a nonempty open subset of S. Without loss of generality, we suppose k is nonconstant and $k < -1$ somewhere. Let $G \subset E$ denote the subspace of functions with support in $T = \{p \in S : k(p) \leq -\frac{1}{e}\}$. Hence,

$$A = \left\{ v \in G : g(v) = (dfv, v) = \int_S \langle \nabla v, \nabla v \rangle + (k + e^{2v})v \; dS \leq 0 \right\}$$

contains zero, is closed and so, compact as above. Besides $A \subset B$, where

$$B = \left\{ v \in G : \exists\, p \in T, \log\sqrt{-k(p)} \leq v(p) \leq 0 \text{ or } 0 \leq v(p) \leq \log\sqrt{-k(p)} \right\}.$$

Because of our choice of T, any $v \in A$ has a $p \in T$ for which $v(p) \geq -\frac{1}{2}$. Next, we profit the nearness of A to $C = \{v \in G : \forall\, p \in T, v(p) \geq -\frac{1}{2}\}$. Let $[u, u'] = \{(1-t)u + tu' :$

$t \in [0,1]\}$ denote the segment of linear convex combinations of u and u'. For $v, v' \in A$, let $w, w' \in C$ and $h : [v, v'] \longrightarrow [w, w']$ be such that $h((1-t)v + tv') = (1-t)w + tw'$ and $g((1-t)v + tv') = g((1-t)w + tw')$, for all $t \in [0,1]$. Since xe^{2x} is convex on the real interval $[-\frac{1}{2}, \infty) \ni x$, $g((1-t)v + tv') = g((1-t)w + tw') \leq (1-t)g(w) + tg(w') = (1-t)g(v) + tg(v') \leq 0$. This proves A is convex. Furthermore, the function $f : G \longrightarrow \mathbb{R}$,

$$f(u) = \frac{1}{2}\int_S \langle \nabla u, \nabla u\rangle + 2ku + e^{2u}\ dS,$$

is convex and so, *df* is an m-map. We conclude f has a critical point $u \in A$. This point is unique for, if u, u' satisfy $\Delta u - k - e^{2u}, \Delta u' - k - e^{2u'} = 0$ then, $\Delta(u - u') = e^{2u'} - e^{2u}$. Multiplying by $u - u'$, integrating on S and using Green identities, we find $\int_S \langle \nabla(u-u'), \nabla(u-u')\rangle\ dS \leq 0$. Therefore, $u = u'$.

When $\int_S kdS = 2\pi\chi > 0$, k is positive in a nonempty subset of S, which can be the whole surface S. We may assume k is nonconstant and $k > 1$ somewhere. Define $T = \{p \in S : k(p) \geq \frac{1}{e}\}$ and let $G \subset E$ be the space of functions with support in T. The subset

$$G - A = \Big\{v \in G : g(v) < 0\Big\},\ g(v) = (dfv, v) = \int_S \langle \nabla v, \nabla v\rangle + (k - e^{2v})v\ dS,$$

does not contain zero and is open. Also $G - A \subset G - C$, where

$$G - C = \Big\{v \in G : \exists\ p \in T, v(p) < \min\{0, \log\sqrt{k(p)}\} \text{ or } v(p) > \max\{0, \log\sqrt{k(p)}\Big\}.$$

In this way, A compact and contains the set $C = \{v \in G : \forall\ p \in T, 0 \leq v(p) \leq \log\sqrt{k(p)}$ or $\log\sqrt{k(p)} \leq v(p) \leq 0\}$. Note that C is convex and $0 \in C$. For $v, v' \in A$, let $w, w' \in C$ and h as above. Then, A is convex. As *df* is continuous,

$$f(u) = \frac{1}{2}\int_S \langle \nabla u, \nabla u\rangle + 2ku - e^{2u}\ dS$$

has a critical point in A. However, this time f has infinite critical points depending on a convenient choice of T and so, of G, A and C. For example, if $T = \{p \in S : 0 \leq k(p) \leq \frac{1}{e}\}$ we can repeat, *verbatim mutatis mutandis*, the existence proof with the convex set $C = \{v \in G : \forall\ p \in T, \log\sqrt{k(p)} \leq v(p) \leq 0\}$. □

3 Concluding remarks

The technique succeeds as a result of a mixture of crucial ingredients. First, the compactification of F endows E with a topology suitable to handle the geometry of the underlying surfaces. On the other hand, Brézis Lemma (1.4) provides an existence result which is valid in a wide class of situations, including the general case of the conformal deformation functional. By the way, the hemicontinuous monotone case could have been solved by usual Convex Analysis, cf. [3] §25 (and even by more classical techniques such as Legendre's or Jacobi's criteria, cf. [8]). Last but not least, our proof relies primarily on the rich geometry of the space E. In this regard, it differs from the standard proof of the Theorem, cf. e.g., [10].

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