Hopf hypersurfaces in nearly Kaehler 6-sphere

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Abstract. We obtain a characterization for a compact Hopf hypersurface in the nearly Kaehler sphere S^6 using a pinching on the scalar curvature of the hypersurface. It has been also observed that the totally geodesic sphere S^5 in S^6 has induced Sasakian structure as a hypersurface of the nearly Kaehler sphere S^6.

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1 Introduction

It is known that the 6-dimensional unit sphere S^6 has a nearly Kaehler structure $(J, \overline{g})$, where J is an almost complex structure defined on S^6 using the vector cross product of purely imaginary Cayley numbers R^7 and $\overline{g}$ is the induced metric on S^6 as a hypersurface of R^7. Also S^6 can be expressed as $S^6 = G_2/SU(3)$ a homogeneous almost Hermitian manifold, where G_2 is the compact Lie group of all automorphisms of the Cayley division algebra R^8. Regarding the submanifolds of the nearly Kaehler S^6, Gray [13] proved that it does not have any complex hypersurface. However, there are 4-dimensional CR-submanifolds in S^6 and have been studied by Sekigawa and otheres [15], [17]. Moreover, 2- and 3-dimensional totally real submanifolds of S^6 also have been extensively studied [9], [11], [12]. Recently, Berndt et. al [1] have shown that the geometry of almost complex curves (2-dimensional almost complex submanifolds) in S^6 is related to Hopf hypersurfaces (Real hypersurfaces with the 1-dimensional foliation induced by the distribution which is obtained by applying almost complex structure J to the normal bundle of the hypersurface is totally geodesic) of S^6. This relationship between the almost complex curves and Hopf hypersurfaces in S^6 makes the study of Hopf hypersurfaces in S^6 more interesting. In [1], the authors proved that a connected Hopf hypersurface of the nearly Kaehler S^6 is an open part of either a geodesic hypersphere of S^6 or a tube around an almost complex curve in S^6. It is therefore interesting question to obtain characterizations of the Hopf hypersurface which is totally geodesic hypersphere in S^6 and the one which is tube around almost complex curve. In this paper we obtain a characterization for the Hopf hypersurface which is totally geodesic hypersphere.

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Let M be an orientable real hypersurface of S^6 with unit normal vector field N and $\xi = -JN$ be the characteristic vector field on M. If A is the shape operator of the hypersurface we define $f : M \to R$ by $f = g(A\xi, \xi)$, where g is the induced metric on M. Let $\alpha = \frac{1}{5} trA$ be the mean curvature and S be the scalar curvature of the hypersurface M. In this paper our main result is the following

Theorem 1.1. *Let M be an orientable compact and connected real hypersurface of the nearly Kaehler S^6. If the scalar curvature S of M satisfies*

$$S \geq 20 + 5\alpha(5\alpha - f),$$

then M is a Hopf hypersurface, the totally geodesic hypersphere S^5.

We also show that this Hopf hypersurface (totally geodesic hypersphere) has naturally induced Sasakian structure (cf. Theorem 3.1). It will be an interesting question as to know whether the other class of Hopf hypersurface (tubes around almost complex curves) in S^6 too carries an induced Sasakian structure.

2 Preliminaries

Let S^6 be the nearly Kaehler 6-sphere with nearly Kaehler structure $(J, \overline{g})$, where J is the almost complex structure and $\overline{g}$ is the induced metric on S^6. Then we have

$$\left(\overline{\nabla}_X J\right)(X) = 0, \quad \overline{g}\,(JX, JY) = \overline{g}\,(X, Y)\,, \quad X, Y \in \mathfrak{X}(S^6), \tag{2.1}$$

where $\overline{\nabla}$ is the Riemannian connection with respect to the almost Hermitian metric $\overline{g}$ and $\mathfrak{X}(S^6)$ is the Lie algebra of smooth vector fields on S^6. The tensor field G of type $(2, 1)$ defined by $G(X, Y) = \left(\overline{\nabla}_X J\right)(Y)$, $X, Y \in \mathfrak{X}(S^6)$ has the properties as described in the following

Lemma 2.1. *([12]) (a) $G(X, JY) = -JG(X, Y)$, (b) $G(X, Y) = -G(Y, X)$, (c) $\left(\overline{\nabla}_X G\right)(Y, Z) = \overline{g}(Y, JZ)X + \overline{g}(X, Z)JY - \overline{g}(X, Y)JZ$, $X, Y, Z \in \mathfrak{X}(S^6)$.*

Let M be an orientable real hypersurface of S^6, ∇ be the Riemannian connection with respect to the induced metric g on M and N be the unit normal vector field. Then we have

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M), \tag{2.2}$$

where A is the shape operator of the hypersurface M. The Guass and Codazzi equations for the hypersurface are

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY \tag{2.3}$$

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \tag{2.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$, where $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$. The Ricci tensor Ric and the scalar curvature S of the hypersurface are given by

(2.5) $$Ric(X,Y) = 4g(X,Y) + 5\alpha g(AX,Y) - g(AX,AY)$$

(2.6) $$S = 20 + 25\alpha^2 - \|A\|^2,$$

where $\alpha = \frac{1}{5}trA$ is the mean curvature and $\|A\|^2 = trA^2$ is the square of the length of the shape operator of the hypersurface.

Using the almost complex structure J of S^6, we define a unit vector field $\xi \in \mathfrak{X}(M)$ by $\xi = -JN$, with dual 1-form $\eta(X) = g(X,\xi)$. For a $X \in \mathfrak{X}(M)$, we set $JX = \phi(X) + \eta(X)N$, where $\phi(X)$ is the tangential component of JX. Then it follows that ϕ is a $(1,1)$ tensor field on M. Using $J^2 = -I$, it is easy to see that (ϕ,ξ,η,g) defines an almost contact metric structure on M, that is

(2.7) $$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi(\xi) = 0,$$

and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, $X, Y \in \mathfrak{X}(M)$. Using the fact $G(X,X) = 0$, $X \in \mathfrak{X}(M)$, we immediately obtain the following

Lemma 2.2. *Let M be an orientable real hypersurface of S^6. Then the structure (ϕ,ξ,η,g) on M satisfies*

(i) $(\nabla_X \phi)(X) = \eta(X)AX - g(AX,X)\xi$,

(ii) $g(\nabla_X \xi, X) = g(\phi AX, X)$.

Note that ϕ is skewsymmetric and for a unit vector field e_1 orthogonal to ξ, $\{e_1, \phi e_1\}$ is an orthonormal set of vector fields and that if e_2 is a unit vector field orthogonal to $e_1, \phi e_1$ and ξ, then $\{e_1, \phi e_1, e_2, \phi e_2, \xi\}$ forms a local orthonormal frame on the hypersurface M, called an adapted frame. Using an adapted frame together with Lemma 2.2 one immediately concludes the following

Corollary 2.1. *Let M be an orientable real hypersurface of S^6. Then $div\xi = 0$.*

Corollary 2.2. *Let M be an orientable real hypersurface of S^6 with almost contact structure (ϕ,ξ,η,g). Then $\|\pounds_\xi g\| = \|\phi A - A\phi\|^2$.*

Proof. We have using on Lemma 2.2 for $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned}
(\pounds_\xi g)(X,Y) &= g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\
&= g(\nabla_{X+Y}\xi, X+Y) - g(\nabla_X \xi, X) - g(\nabla_Y \xi, Y) \\
&= g(\phi A(X+Y), X+Y) - g(\phi AX, X) - g(\phi AY, Y) \\
&= g((\phi A - A\phi)(X), Y).
\end{aligned}$$

Note that $\phi A - A\phi$ is a symmetric operator and consequently we get $\|\pounds_\xi g\| = \|\phi A - A\phi\|^2$. □

Lemma 2.3. *Let M be an orientable real hypersurface of S^6 with almost contact structure (ϕ,ξ,η,g). Then*

$$\nabla_X \xi = \phi AX - G(X,N), \quad X \in \mathfrak{X}(M).$$

Proof. We have

$$\begin{aligned} G(X,\xi) &= \overline{\nabla}_X N - J\left(\nabla_X \xi + g(AX,\xi)N\right) \\ &= -AX - J\nabla_X \xi + g(AX,\xi)\xi. \end{aligned}$$

Operating J on this equation and using Lemma 2.1 we get the result. □

On an orientable hypersurface M of S^6 we let $D = Ker\eta = \{X \in \mathfrak{X}(M) : \eta(X) = 0\}$. Then D is a 4-dimensional smooth distribution on M, and that for each $X \in D$, $JX \in D$, that is D is invariant under the almost complex structure J.

Lemma 2.4. *Let M be an orientable real hypersurface of S^6 with almost contact structure (ϕ, ξ, η, g). Then for a unit vector field $X \in D$*

$$G\left(X, G(X,\xi)\right) = -\xi.$$

Proof. Using the properties of the tensor G in Lemma 2.1, it is easy to see that for $X \in D$, $G(X,\xi) \in D$ as $G(X,\xi) \perp \xi$ and N. We have

$$\begin{aligned} G\left(X, G(X,\xi)\right) &= \left(\overline{\nabla}_X G\right) G(X,\xi) = -\overline{\nabla}_X G(X,N) - J\overline{\nabla}_X G(X,\xi) \\ &= -\left[\left(\overline{\nabla}_X G\right)(X,N) + G(\overline{\nabla}_X X, N) - G(X,AX)\right] \\ &- J\left[\left(\overline{\nabla}_X G\right)(X,\xi) + G(\overline{\nabla}_X X,\xi) + G(X,\overline{\nabla}_X \xi)\right]. \end{aligned}$$

Using Lemma 2.1 and $JN = -\xi$ we arrive at $G\left(X, G(X,\xi)\right) = -2\xi - G\left(X, G(X,\xi)\right)$ and this proves the Lemma. □

Lemma 2.5. *Let M be an orientable real hypersurface of S^6 and $e \in D$ be the unit vector field. Then*

$$\|G(e,\xi)\|^2 = 1.$$

Proof. It is an easy consequence of the definition of G to check that $\overline{g}(G(X,Y),Z) = -\overline{g}\left(Y, G(X,Z)\right)$, $X,Y,Z \in \mathfrak{X}(M)$. Consequently the Lemma 2.4 yields $\overline{g}(G(e,\xi), G(e,\xi)) = -\overline{g}\left(\xi, G(e, G(e,\xi))\right) = 1$. □

Lemma 2.6. *Let M be an orientable real hypersurface of S^6. Then for each unit vector field $X \in D$, the set $\{X, JX, G(X,\xi), JG(X,\xi), \xi\}$ is a local orthonormal frame on M, and if it is denoted by $\{e_1, e_2, e_3, e_4, \xi\}$, it satisfies*

$$G(e_1,e_2) = 0,\ G(e_1,e_3) = -\xi,\ G(e_1,e_4) = N,\ G(e_1,\ \xi) = e_3,\ G(e_1,N) = -e_4$$

$$G(e_2,e_3) = -N,\ G(e_2,e_4) = \xi,\ G(e_2,\xi) = -e_4,\ G(e_2,N) = -e_3,\ G(e_3,e_4) = 0$$

$$G(e_3,\xi) = -e_1,\ G(e_3,N) = e_2,\ G(e_4,\xi) = e_2,\ G(e_4,N) = e_1.$$

Proof. The proof directly follows from Lemma 2.1 and Lemma 2.4. □

Lemma 2.7. *Let M be an orientable compact real hypersurface of S^6. Then*

$$\int_M \left\{ Ric(\xi,\xi) - 4 + tr(\phi A)^2 \right\} dv = 0.$$

Proof. Define $u \in \mathfrak{X}(M)$ by $u = \nabla_\xi \xi$. Then using Lemma 2.3 we compute

$$\nabla_X u = R(X,\xi)\xi + \nabla_\xi (\phi AX - G(X,N)) + \phi A [X,\xi] - G([X,\xi],N).$$

Consequently,

$$\begin{aligned} divu &= Ric(\xi,\xi) + \sum_{i=1}^{5} [\xi g(\phi Ae_i, e_i) - \xi g(G(e_i,N),e_i) \\ &- g(A[e_i,\xi],\phi e_i) - g(G([e_i,\xi],N),e_i)], \end{aligned} \tag{2.8}$$

where we have used a pointwise constant local orthonormal frame $\{e_1, ..., e_5\}$ on M. Note that

$$g(G(e_i,N),e_i) = -g(N, G(e_i,e_i)) = 0, \tag{2.9}$$

and for an adapted frame

$$\sum_{i=1}^{5} g(\phi Ae_i, e_i) = 0, \tag{2.10}$$

and that $\sum_{i=1}^{5} g(\phi Ae_i, e_i)$ remains same for any local orthonormal frame. Thus equation (2.8) reduces to

$$divu = Ric(\xi,\xi) - \sum_{i=1}^{5} \left[g(A\nabla_{e_i}\xi, \phi e_i) + g(G(\nabla_{e_i}\xi, N), e_i) \right]. \tag{2.11}$$

Also we have

$$\begin{aligned} g(A(\nabla_{e_i}\xi),\phi e_i) &= g(\phi Ae_i - G(e_i,N), A\phi e_i) \\ &= -g\left((\phi A)^2(e_i), e_i\right) - g(G(e_i,N),\phi Ae_i), \end{aligned}$$

which gives

$$\sum_{i=1}^{5} g((A(\nabla_{e_i}\xi),\phi e_i) = -tr(\phi A)^2 - \sum_{i=1}^{5} g(G(e_i,N),\phi Ae_i). \tag{2.12}$$

Now we use the local orthonormal frame in Lemma 2.6 to show

$$\sum_{i=1}^{5} g(G(e_i,N),\phi Ae_i) = 0, \tag{2.13}$$

as the trace of the linear forms is invariant with respect to the local orthonormal frames. Finally we observe that $\sum_{i=1}^{5} g(G(\nabla_{e_i}\xi,N),e_i) = \sum_{i=1}^{5} g(\nabla_{e_i}\xi, G(N,e_i))$. Using Lemmas 2.3, 2.5 and equation (2.13) we conclude

$$\sum_{i=1}^{5} g\left(G(\nabla_{e_i}\xi, N), e_i\right) = 4. \tag{2.14}$$

Thus using equations (2.12), (2.13) and (2.14) in equation (2.11), we arrive at $divu = Ric(\xi,\xi) - 4 + tr\,(\phi A)^2$, which on integration yields the Lemma. □

3 Hopf Hypersurfaces

A real hypersurface of S^6 is said to be a Hopf hypersurface if the integral curves of the characteristic vector field $\xi = -JN$ are geodesics. The geometry of Hopf hypersurfaces in S^6 is related to the geometry of 2-dimensional almost complex submanifolds of S^6 (cf. [1]). A Hopf hypersurface in S^6 is known to be orientable and that the characteristic vector field is an eigen vector of the shape operator A that is $A\xi = \lambda\xi$ and it is also known that λ is a constant ([1]). Now we proceed to prove our main Theorem

Proof of the Main Theorem. Let M be an orientable compact hypersurface of S^6 satisfying the hypothesis of the Theorem. Define a smooth function $f : M \to R$ by $f = g(A\xi,\xi)$. Then equation (2.6) gives

$$Ric(\xi,\xi) = 4 + 5\alpha f - \|A\xi\|^2. \tag{3.1}$$

Since $tr(\phi A)^2 = tr(A\phi)^2$ and that $\phi A - A\phi$ is a symmetric operator, using an adapted local orthonormal frame we arrive at

$$\frac{1}{2}\|\phi A - A\phi\|^2 = tr(\phi A)^2 + \|A\|^2 - \|A\xi\|^2. \tag{3.2}$$

Thus using equations (3.1), (3.2) and (2.6) in Lemma 2.7, we arrive at

$$\int_M \left\{\frac{1}{2}\|\phi A - A\phi\|^2 + S - 20 - 5\alpha(5\alpha - f)\right\} dv = 0. \tag{3.3}$$

This together with the condition in the hypothesis of the Theorem yields $\phi A = A\phi$ which gives $\phi A\xi = 0$ and consequently $A\xi = f\xi$. Thus by Lemma 2.3 we get that M is a Hopf hypersurface and that f is a constant. Now, we proceed to show that this Hopf hypersurface is the totally geodesic hypersphere.

From the equation (3.3) we conclude that

$$S = 20 + 25\alpha^2 - 5\alpha f, \quad \phi A = A\phi. \tag{3.4}$$

Note that the second equation in (3.4) implies that if $AX = \lambda X$, then $A\phi X = \lambda\phi X$, $X \in \mathfrak{X}(M)$, that is, the shape operator A is diagonalized in an adapted local orthonormal frame $\{e_1, ..., e_4, \xi\} = \{e_1, \phi e_1, e_2, \phi e_2, \xi\}$, with $Ae_i = \lambda_1 e_i$, $i = 1, 2$ and $Ae_j = \lambda_2 e_j$, $j = 3, 4$, $A\xi = f\xi$, where λ_1, λ_2 are smooth functions and f is a constant. Also from equations (2.6) and (3.4), we have $\|A\|^2 = 5\alpha f$, which gives

$$\lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)f. \tag{3.5}$$

For the connected Hopf hypersurface M, we discuss the two cases $\lambda_1 + \lambda_2 = 0$ and $\lambda_1 + \lambda_2 \neq 0$.

Case (i). If $\lambda_1 + \lambda_2 = 0$, then by equation (3.5) we get $\lambda_1 = \lambda_2 = 0$. Thus we conclude that $A(X) = 0$, $X \in D$. Using Codazzi equation (2.4) for $X, Y \in D$, we get $A([X, Y]) = 0$, which on taking inner product with ξ yields $fg([X, Y], \xi) = 0$. Thus either the constant $f = 0$ or else $[X, Y] \in D$. However $[X, Y] \in D$ implies that D is integrable and as $JD = D$, the leaves of the distribution are 4-dimensional almost complex submanifolds of S^6 which do not exist (cf. [13]). Hence $f = 0$ which implies, as f is a constant, that M is totally geodesic sphere S^5.

Case (ii). If $\lambda_1 + \lambda_2 \neq 0$, then on the open subset where it happens, we choose an adapted local orthonormal frame that diagonalizes A and use Lemmas 2.3 and 2.6 to write the following structure equations

$$\begin{aligned}
\nabla_{e_1}\xi &= \lambda_1 e_2 + e_4, \quad \nabla_{e_2}\xi = -\lambda_1 e_2 + e_3, \quad \nabla_{e_3}\xi = \lambda_2 e_4 - e_2,\\
\nabla_{e_4}\xi &= -\lambda_3 e_2 - e_1, \nabla_\xi e_1 = ae_2 + be_3 + \mu e_4,\\
\nabla_\xi e_2 &= -ae_1 + xe_3 + ye_4, \quad \nabla_\xi e_3 = -be_1 - xe_2 + ze_4,\\
\nabla_\xi e_4 &= -\mu e_1 - ye_2 - ze_3,
\end{aligned} \tag{3.6}$$

where a, b, μ, x, y, z are smooth functions. Using the definition of the curvature tensor field

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \mathfrak{X}(M),$$

and equations in (3.6), we compute

$$\begin{aligned}
R(e_1, \xi)\xi &= (\lambda_1^2 + 1)e_1 + (-\xi(\lambda_1) + y - b)e_2\\
&+ (-\lambda_1 x - \lambda_2\mu + z - \lambda_1 + \lambda_2 + a)e_3 + (-\lambda_1 y + b\lambda_2)e_4
\end{aligned} \tag{3.7}$$

However, by Gauss equation (2.3), we have $R(e_1, \xi)\xi = (1 + \lambda_1 f)e_1$, consequently equation (3.7) yields

$$\begin{aligned}
\lambda_1^2 &= \lambda_1 f, \quad \xi(\lambda_1) = y - b, \quad \lambda_1 y = b\lambda_2\\
\lambda_1 x + \lambda_2\mu &= z - \lambda_1 + \lambda_2 + a.
\end{aligned} \tag{3.8}$$

Repeating the same proceedure for $R(e_3, \xi)\xi$, we arrive at

$$\begin{aligned}
\lambda_2^2 &= \lambda_2 f, \quad \xi(\lambda_2) = y - b, \quad \lambda_2 y = b\lambda_1\\
\lambda_1 x + \lambda_2\mu &= z + \lambda_1 - \lambda_2 + a.
\end{aligned} \tag{3.9}$$

Using last equations in (3.8) and (3.9), we get $\lambda_1 = \lambda_2$, and consequently together with the assumpsion $\lambda_1 + \lambda_2 \neq 0$, we get both $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and this together with equation (3.5) gives $\lambda_1 = \lambda_2 = f$, that is, in this case M is totally umbilical.

Since M is connected, we have either M is totally geodesic hypersphere S^5 or else M is totally umbilical hypersurface. Using Theorem 3 in [1], we get that M is not a tube around almost complex curve. This completes the proof of our Theorem (cf. Theorem 2 in [1]). □

As an application of integral formula (3.3) we have following :

Corollary 3.1. *Let M be an orientable compact minimal hypersurface of the nearly Kaehler sphere S^6 with commuting operators A and ϕ, then M is totally geodesic.*

Proof. Since for a minimal hypersurface $S - 20 = -\|A\|^2$, the integral formula (3.3) gives $\|A\|^2 = 0$, that is M is totally geodesic. □

It is well known that S^5, as totally umbilic hypersurface of complex space C^3, is a Sasakian space form ([3]). However, it also has an induced Sasakian structure as a totally geodesic hypersurface of the nearly Kaehler S^6 as seen in the following

Theorem 3.1. *The totally geodesic sphere S^5 has induced Sasakian structure as a hypersurface of the nearly Kaehler S^6 and it is a Sasakian space form.*

Proof. For the hypersurface S^5 the Lemma 2.3 gives

$$\nabla_X \xi = -G(X, N), \quad X \in \mathfrak{X}(S^5). \tag{3.10}$$

Define $\psi : \mathfrak{X}(S^5) \to \mathfrak{X}(S^5)$ by $\psi(X) = G(X, N)$, then it follows that $\psi(\xi) = 0$ and that $\psi(X) \in D$ for $X \in \mathfrak{X}(S^5)$. Also as S^5 is totally geodesic, by Corollary 2.2, $\pounds_\xi g = 0$, that is ξ is a Killing vector field. Using Lemma 2.1, equation (3.10) and the fact that $\overline{\nabla}_X Y = \nabla_X Y, \quad X, Y \in \mathfrak{X}(S^5)$, we compute

$$\begin{aligned} G\left(G\left(X,N\right),N\right) &= -\overline{\nabla}_N JG(X,N) + J\overline{\nabla}_N G(X,N) \\ &= -\left(\overline{\nabla}_N G\right)(X,\xi) - G\left(\overline{\nabla}_N X,\xi\right) - G\left(X,\overline{\nabla}_N \xi\right) \\ &+ J\left[\left(\overline{\nabla}_N G\right)(X,N) + G\left(\overline{\nabla}_N X,N\right) + G\left(X,\overline{\nabla}_N N\right)\right] \\ &= \eta(X)\xi - X. \end{aligned}$$

Consequently, $\psi^2(X) = -X + \eta(X)\xi$ holds. Also we have

$$\begin{aligned} g(\psi(X),\psi(Y)) &= g\left(G(X,N),G(Y,N)\right) = g\left(X,G((N,G(Y,N)))\right) \\ &= g\left(X,Y-\eta(Y)\xi\right) = g(X,Y) - \eta(X)\eta(Y). \end{aligned}$$

This implies that (ψ, ξ, η, g) is an almost contact metric structure on S^5. Moreover using equation (3.10), we have

$$\begin{aligned} \left(\nabla_X \psi\right)(Y) &= \overline{\nabla}_X \psi(Y) - \psi(\overline{\nabla}_X Y) = \overline{\nabla}_X G(Y,N) - G(\overline{\nabla}_X Y,N) \\ &= \left(\overline{\nabla}_X G\right)(Y,N) = g(X,Y)\xi - \eta(Y)X, \end{aligned}$$

where we used Lemma 2.1 and the fact that $\overline{\nabla}_X N = 0$. The last equation proves that (ψ, ξ, η, g) is a Sasakian structure on S^5 (cf. [3]). Also for each unit $X \in D$, the holomorphic sectional curvature $H(X) = 1$ and hence S^5 is a Sasakian space form with respect to this induced Sasakian structure. □

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