Measured foliations and mapping class groups of surfaces

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Abstract. We prove a rigidity result on the action of the extended mapping class group $\Gamma^*(S)$ of a closed surface S of genus ≥ 2 on Thurston's space of measured foliations $\mathcal{MF}(S)$. More precisely, we describe a natural homomorphism from $\Gamma^*(S)$ to the automorphism group of the train track PL structure of $\mathcal{MF}(S)$ and we prove that if the genus of S is ≥ 3, then this natural homomorphism is an isomorphism, and if the genus of S is 2, then the homomorphism is surjective and its kernel is $\mathbb{Z}_2$ generated by the hyperelliptic involution.

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1 Introduction

A beautiful and recurrent theme in low-dimensional topology says that when the extended mapping class group of a surface acts on some space X of geometric structures on that surface, then this group coincides with the automorphism group of X. Famous instances of this rigidity phenomenon include the results of Royden stating that the extended mapping class group is (except for some special surfaces) the group of isometries of Teichmüller space of the surface, equipped with its Teichmüller metric, and that it is also the group of biholomorphic maps of the same Teichmüller space equipped with its complex structure (see [7]). Another well-known result of the same sort is Ivanov's result saying that the extended mapping class group is (again, with a few exceptions) the group of simplicial automorphisms of the complex of curves of the surface (see [2]). Feng Luo obtained a rigidity result concerning the action of the extended mapping class group on Thurston's space of measured foliations of the surface, in terms of homeomorphisms of this space preserving the intersection functions (see [3]). Recently, I obtained a rigidity result of a different character, concerning the action of the mapping class group on the space of unmeasured foliations of the surface (see [5]). In this paper, I present another rigidity result that concerns the action of

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the mapping class group on Thurston's measured foliations space equipped with its piecewise-linear structure.

The plan of the paper is the following. I start by recalling some necessary backgound material on measured foliations and on train tracks, and then I prove the following (Theorem 4.9): Let S be an oriented connected closed surface of genus $g \geq 2$ and let $\mathrm{Aut}(\mathcal{MF}, \mathcal{P})$ be the group of homeomorphisms of the space $\mathcal{MF}$ that preserve the train track PL structure (the precise definition is given below). Then, there is a natural homomorphism

$$\Gamma^*(S) \to \mathrm{Aut}(\mathcal{MF}, \mathcal{P}).$$

This homomorphism is an isomorphism if $g \geq 3$. If $g = 2$, the homomorphism is surjective and its kernel is $\mathbb{Z}_2$ generated by the hyperelliptic involution.

2 Measured foliations

In what follows, $S = S_g$ is a connected closed oriented surface of genus $g \geq 2$. An isotopy of S is a homeomorphism which is isotopic to the identity. The *extended mapping class group*, $\Gamma^* = \Gamma^*(S)$, is the group of isotopy classes of homeomorphisms of S. The *mapping class group*, $\Gamma = \Gamma(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of S. Elements of the (extended) mapping class group are called *(extended) mapping classes.* Mapping class groups of surfaces are certainly the most important groups in low-dimensional topology. These groups have been studied from several points of view, and these studies use actions of these groups on various spaces. In this paper, I present a new result on the action of Γ^* on the space of measured foliations on the surface.

I start by recalling the notion of measured foliation, which was introduced by Thurston (see [9] and [1]).

We consider foliations on S with singular points, where the local model of a singular point is of the type represented in Figure 1, that is, the point is a "saddle with k separatrices" with all values of $k \geq 3$ allowed.

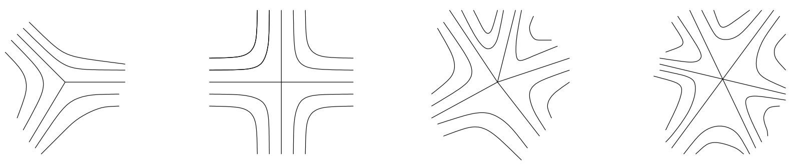

Figure 1: Singular points of measured foliations with k-separatrices, for $k = 3, 4, 5, 6$ respectively.

By an Euler-Poincaré count, and since the genus of S is ≥ 2, any foliation on S has at least one singular point.

We shall call a *transverse measure* for a foliation a measure on each transverse arc that is equivalent to the Lebesgue measure of an interval of $\mathbb{R}$, such that these measures on arcs are invariant by the local holonomy maps, that is, by isotopies of arcs that keep each point on the same leaf.

A *measured foliation* on S is a foliation equipped with a transverse measure.

A *Whitehead move* on a measured foliation is an operation on the surface S that modifies a measured foliation by isotopy and by contracting to a point a compact leaf that joins two singular points, or by the inverse operation. An example of a Whitehead move is represented in Figure 2. The equivalence relation between measured foliations that is generated by isotopy and Whitehead moves is called *Whitehead equivalence*. The space of Whitehead equivalence classes of measured foliations is called *measured foliations space*, and it is denoted by $\mathcal{MF}$ or $\mathcal{MF}(S)$.

Given a measured foliations F on S, we shall use the notation $[F]$ for its equivalence class in $\mathcal{MF}$.

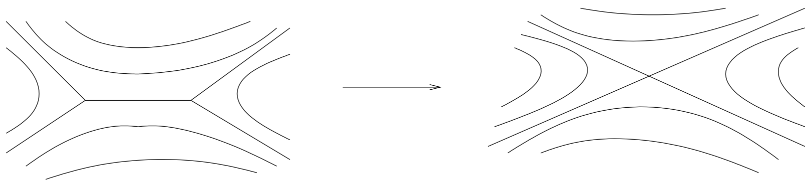

Figure 2: Whitehead move: collapsing an arc joining two singular points.

A *partial measured foliation* on S is a measured foliations whose support is a nonempty subsurface with boundary of S. The support is not necessarily connected. We shall denote by $\mathrm{Supp}(F)$ the support of a partial measured foliation F. From a partial measured foliation F, one can obtain a genuine measured foliation F_0 by collapsing each connected component of $S \setminus \mathrm{Supp}(F)$ onto a one-dimensional complex (a *spine* of $S \setminus \mathrm{Supp}(F)$). The equivalence class of F_0 does not depend on the chosen spine, and the collapse operation therefore gives a well defined element of $\mathcal{MF}$.

A simple closed curve on S is a homeomorphic image of a circle. Such a curve is said to be *essential* if it does not bound a disk on S.

We let $\mathcal{S}$ be the set of isotopy classes of essential simple closed curves on S. If c is a simple closed curve on S, we denote its isotopy class by $[c]$.

We denote by $\mathbb{R}_+^*$ the set of positive real numbers. There is a natural map from $\mathbb{R}_+^* \times \mathcal{S}$ into $\mathcal{MF}$ defined by assigning to each pair $(r, [c])$ an annulus N in the interior of S which is isotopic to a regular neighborhood of c, foliated by closed leaves in the homotopy class $[c]$ and equipped with a transverse measure such that the total transverse measure of a segment transverse to the foliation and joining the two boundary components of the annulus is equal to r. The resulting partial measured foliation on S gives a well defined element of $\mathcal{MF}$. This defines a map $\mathbb{R}_+^* \times \mathcal{S} \to \mathcal{MF}$, and it is known that this map is injective.

Any measured foliation on S can be decomposed into a union of finitely many components, where each component is a partial measured foliation whose support is contained in the support of S and which is of one of the following two sorts:

- a partial measured foliation in which every leaf is dense;
- a partial measured foliation all whose leaves are closed and homotopic to each other, and whose support is maximal with respect to inclusion.

The components of a measured foliation give a well-defined set of elements of $\mathcal{MF}$.

For any measured foliation F and for any element γ of $\mathcal{S}$, *the geometric intersection number* $i(F,\gamma)$ is the infimum of the total measure of c, where c varies over the set of closed curves c in the homotopy class γ which are made up of arcs transverse to F and arcs contained in the leaves of F. The geometric intersection number $i(F,\gamma)$ does not depend on the choice of the Whitehead equivalence class. Thus, we obtain a map

$$i : \mathcal{MF} \times \mathcal{S} \to \mathbb{R}_+$$

called the *geometric intersection function*.

Using this function, we can map the space $\mathcal{MF}$ into the space $\mathbb{R}_+^{\mathcal{S}}$ of nonnegative functions on $\mathcal{S}$. It is known that this map is injective. From this injection, the set $\mathcal{MF}$ inherits a topology which makes it homeomorphic to $\mathbb{R}^{6g-6}\setminus\{0\}$. It is sometimes useful to consider the "empty foliation" as an element of $\mathcal{MF}$. When we add to $\mathcal{MF}$ this empty foliation, $\mathcal{MF}$ becomes homeomorphic to $\mathbb{R}^{6g-6}$.

There is a natural action of the group of positive reals $\mathbb{R}_+^*$ on $\mathcal{MF}$, induced from the action of this group on measured foliations defined by multiplying the transverse measures by a constant. The quotient of $\mathcal{MF}$ by this action is denoted by $\mathcal{PMF}$. The embedding of $\mathcal{MF}$ into the space $\mathbb{R}_+^{\mathcal{S}}$ induces an embedding of $\mathcal{PMF}$ in the projectived space $P\mathbb{R}_+^{\mathcal{S}}$.

3 A review of train tracks

A *train track* τ on S is a (non-necessarily connected) graph on S with trivalent vertices such that the three half-edges that abut on any vertex have a well defined tangent at that point. The local structure at a vertex is represented in Figure 3. Train tracks were introduced by Thurston in [8]. We refer to the monograph [6] by Penner and Harer for a detailed study of train tracks.

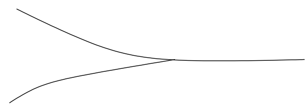

Figure 3: The local model at a switch.

We also allow a train track to have no vertices, that is, τ can consist of a collection of disjoint simple closed curves. An *edge* of such a train track will denote a connected component, that is, a simple closed curve. A vertex of a train track is also called a *switch*. At each switch, there is a well-dfined notion of two half-edges abutting from one side and one half-edge abutting from the other side. A *corner* of the surface S equipped with a train track τ is a region in a neighborhood of a switch of τ contained between the two half-edges that abut from the same side. All the train tracks τ that we consider satisfy the following property: any connected component of $S \setminus \tau$ is not a disk with 0, 1 or 2 corners, or an annulus with no corner (cf. Figure 4). A train track τ on S is said to be *maximal* if every component of $S \setminus \tau$ is a disk with three corners on its boundary.

A train track τ has a *regular neighborhood* $N(\tau)$ foliated by segments that are called the *ties*, whose endpoints lie on the boundary of $N(\tau)$. The local picture of the foliation by the ties near a switch is represented in Figure 5. The ties are the fibres of a natural projection $N(\tau) \searrow \tau$. The regular neighborhood $N(\tau)$, equipped with its foliation by the ties, is well defined up to an isotopy of the surface.

Let τ be a train track and let $e_1, \ldots, e_N$ be its edges. We consider the vector space $\mathbb{R}^N$ of real weights $(x_1, \ldots, x_N)$ on the edges of τ, and we let $V_\tau \subset \mathbb{R}^N$ denote the closed convex cone defined by the system

$$\begin{cases} x_i \geq 0 & \text{for every } i = 1, \ldots, N \\ x_i = x_j + x_k & \text{for every switch of } \tau \end{cases}$$

where, in the equations, x_j and x_k denote the weights of the two edges that abut from the same side at the given switch, and x_i is the weight on the edge that abuts from the other side.

A train track τ is said to be *recurrent* if there exists an element $(x_1, \ldots, x_N)$ of V_τ satisfying $x_i > 0$ for all $i = 1, \ldots, N$.

Let $N(\tau)$ be a regular neighborhood of τ equipped with a projection $N(\tau) \searrow \tau$. Let $(x_1, \ldots, x_N)$ be a nonzero element of V_τ. For each nonzero coordinate x_i, consider the inverse image of the edge e_i by the projection $N(\tau) \searrow \tau$. This inverse image has a natural structure of a rectangle equipped with a foliation induced by the ties, which we call the "vertical" foliation. We equip this rectangle with another foliation, which we call the "horizontal" foliation, whose leaves are segments transverse to the ties and joining the two edges of the rectangle that consist of ties. Furthermore, we equip this horizontal foliation with a transverse measure whose total mass is equal to x_i. Gluing by measure-preserving homeomorphisms all such horizontally foliated

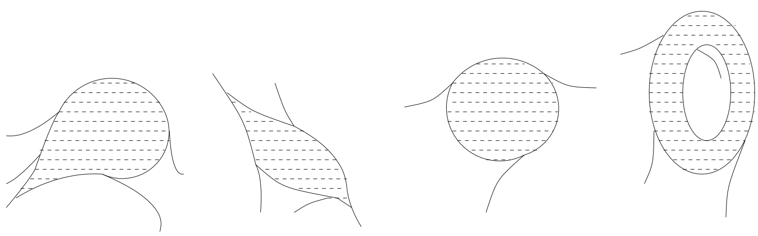

Figure 4: The shaded regions represent the four types of excluded components of the complement of a train track.

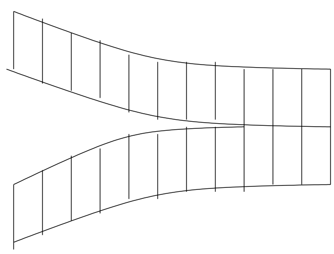

Figure 5: The regular neighborhood and the local structure of the ties near a switch

rectangles along their vertical sides, we obtain a well defined element of $\mathcal{MF}$. The zero element of V_τ is sent to the empty foliation. This defines a map $\varphi_\tau : V_\tau \to \mathcal{MF}$ which is a homeomorphism onto its image. In the case where τ is maximal and recurrent, the image $\varphi_\tau(V_\tau) = U_\tau$ has nonempty interior in $\mathcal{MF}$ (see [4] p. 20).

A measured foliation F (or its equivalence class $[F] \in \mathcal{MF}$) is said to be *carried by* a train track τ if $[F]$ is in the image U_τ of V_τ by the map φ_τ.

Given two train tracks τ and σ on S, we say that τ is *carried* by σ, and we write this relation as $\tau \prec \sigma$, if τ is isotopic to a train track τ' which is contained in a regular neighborhood $N(\sigma)$ of σ and which is transverse to the ties (see Figure 6). When $\tau \prec \sigma$, there is a natural linear map (independent of the choice of the representative τ' isotopic to τ) from the closed convex cone V_τ to the closed convex cone V_σ, which induces the inclusion map at the level of the two subspaces $\varphi_\tau(V_\tau)$ and $\varphi_\sigma(V_\sigma)$ of $\mathcal{MF}$. The linear map $V_\tau \to V_\sigma$ is obtained by using the natural identification between the edges of τ and the edges of a train track τ' isotopic to τ, contained in $N(\sigma)$ and transverse to the ties. A set of weights on τ naturally induces a set of weights on τ', which in turn induces a set of weights on σ by assigning to each edge e of σ the total transverse measure induced by the weighted train track τ' on a tie that lies above that edge e with respect to the projection $N(\sigma) \searrow \sigma$. The system of weights thus obtained on the edges of σ does not depend on the choice of the train track τ' isotopic to τ, contained in $N(\sigma)$ and transverse to the ties.

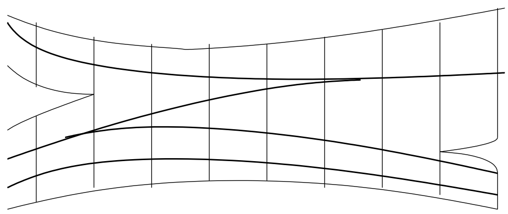

Figure 6: The train track in bold lines is carried by the train track whose regular neighborhood is represented.

We shall use some standard operations on train tracks, called *shift* and *split* operations. These operations are represented in Figure 7. A shift operation on a train track σ produces a train track σ' satisfying $\sigma' \prec \sigma$ and such that the inclusion $\mathcal{MF}(\sigma') \subset \mathcal{MF}(\sigma)$ induces the identity map (that is, it is also onto). There are two splitting operations, a right splitting and a left splitting. Each of these operations produces a train track σ' (respectively σ'') out of σ satisfying $\sigma' \prec \sigma$ (respectively $\sigma'' \prec \sigma$). In general, the induced inclusion map from the set of measured foliations carried by σ to the set of measured foliations carried by σ' is not onto. Rather, the interiors in $\mathcal{MF}$ of the two sets $\mathcal{MF}(\sigma')$ and $\mathcal{MF}(\sigma'')$ are disjoint, and the union of these sets is equal to $\mathcal{MF}(\sigma)$.

4 Automorphisms of the train track PL structure

Thurston showed that the space $\mathcal{MF}$ is equipped with a natural piecewise-linear structure defined by an atlas whose charts use the train track coordinates.

More precisely, for each maximal recurrent train track τ, we let $\psi_\tau : U_\tau \to V_\tau$ denote the inverse of the homeomorphism $\varphi_\tau : V_\tau \to U_\tau$ defined above, and we consider the set

$$\mathcal{A} = \{(U_\tau, \psi_\tau) \mid \tau \text{ is a maximal recurrent train track }\}.$$

Thurston proved in [8] the following

Theorem 4.1. *The set $\mathcal{A}$ is an atlas of a PL structure on $\mathcal{MF}$.*

In what follows, we prove a rigidity result concerning the automorphisms of this structure. This result is based on a precise analysis of the singular set of the coordinate changes of this atlas. To understand this, we need to introduce a few notions that concern the singular set of a PL map.

Let $M \geq 1$ be an integer. A subset V of $\mathbb{R}^M$ will be called a *linear polytope* (or a *polytope*, for brevity) if V is the intersection of a finite number of closed linear half-spaces. Note that the set V is closed and convex, and that it is noncompact unless it is empty. The *dimension* of V is the smallest dimension of a vector subspace of $\mathbb{R}^M$ that contains it.

Let M and N be two positive integers. Let V be a finite union of polytopes $V_1, \ldots, V_n$ in $\mathbb{R}^M$, all having the same dimension. A function $f : V \to \mathbb{R}^N$ is said to be piecewise-linear (PL for brevity) if f is continuous and if the restriction of f to the relative interior of each of the polytopes $V_1, \ldots, V_n$ is the restriction of a linear function from $\mathbb{R}^M$ to $\mathbb{R}^N$.

Given a PL function $f : V \to \mathbb{R}^N$, its *singular set*, denoted by $\mathrm{Sing}(f)$, is the set of points $x \in V_1 \cup \ldots \cup V_n$ such that f is not linear in any neighborhood of x.

We observe the following:

1) The set $\mathrm{Sing}(f)$ is a union of codimension one faces, each of which is the intersection of two sets in the collection of polytopes $\{V_1, \ldots, V_n\}$.

2) If the polytopes V have dimension D, then the set $\mathrm{Sing}(f)$ has a natural structure of a union of linear polytopes of dimension $D-1$ in $\mathbb{R}^M$, and the restriction of f to $\mathrm{Sing}(f)$ is PL.

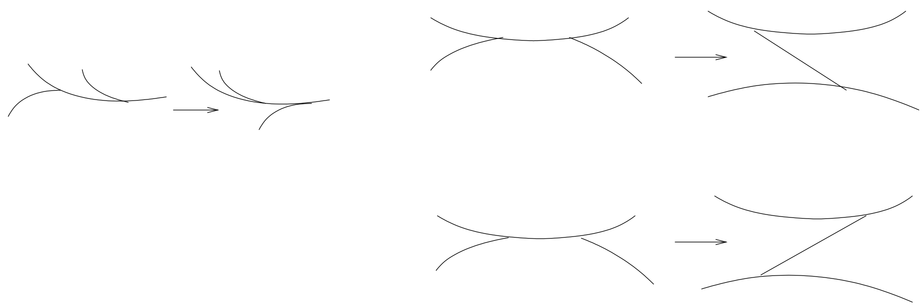

Figure 7: In the situation to the left, the train track is obtained by a shift operation, and in the two situations to the right, the train tracks are obtained by splitting operations.

Using this fact, we associate inductively to such a PL function $f : V \to \mathbb{R}^N$ a nested sequence of subsets of V

$$V \supset \mathrm{Sing}_0(f) \supset \mathrm{Sing}_1(f) \supset \dots \mathrm{Sing}_k(f),$$

characterized by the following conditions:

1. $\mathrm{Sing}_0(f) = V \setminus \mathrm{Sing}(f)$;
2. $\mathrm{Sing}_1(f) = \mathrm{Sing}(f)$;
3. for each integer i satisfying $2 \leq i \leq k$, $\mathrm{Sing}_i(f)$ is the singular set of the restriction of f to $\mathrm{Sing}_{i-1}(f)$;
4. the restriction of f to $\mathrm{Sing}_k(f)$ is linear.

We note that for each $2 \leq i \leq k$, $\mathrm{Sing}_i(f)$ is a codimension-1 subset of $\mathrm{Sing}_{i-1}(f)$ (see Observation 2 above) and that $k \leq D$.

The sequence $\mathrm{Sing}_0(f) \supset \mathrm{Sing}_1(f) \supset \dots \mathrm{Sing}_k(f)$ associated to the PL function f defines a stratification of V, each stratum being characterized by its codimension in V.

We shall call this stratification the *flag* of f, and we shall denote it by $\mathrm{Fl}(f)$.

We need to give a precise description of the coordinate changes $\psi_{\tau\sigma} = \psi_\sigma \circ \psi_\tau^{-1}$ of the atlas $\mathcal{A}$. We start by introducing the following notion.

Definition 4.2 (Adapted family of train tracks). *Let τ be a maximal recurrent train track on S and let $T = \{\tau_1, \dots, \tau_n\}$ be a family of maximal recurrent train tracks. We say that the family T is adapted to τ if the following properties are satisfied:*

1. *for each $i = 1, \dots, n$, $\tau_i \prec \tau$;*
2. *for each i and j satisfying $1 \leq i < j \leq n$, the interiors of U_{τ_i} and U_{τ_j} are disjoint.*

A typical example of a family of train tracks adapted to a train track σ is a collection $\{\sigma', \sigma''\}$ of two train tracks obtained from σ by a left and a right shift operation on some edge of σ.

Proposition 4.3. *Let (U_τ, ψ_τ) and (U_σ, ψ_σ) be two charts in $\mathcal{A}$ and let $\psi_{\tau\sigma}$ be the corresponding coordinate change, defined on the subset $\psi_\tau(U_\tau \cap U_\sigma)$ of V_τ. Then, for every point F in the interior of $\psi_\tau(U_\tau \cap U_\sigma)$, we can find a family $T = \{\tau_1, \dots, \tau_n\}$ of train tracks which is adapted to τ and which furthermore satisfies the following properties:*

1. *$\tau_i \prec \sigma$ for all $i = 1, \dots, n$;*
2. *the union $\bigcup_{i=1}^n U_{\tau_i}$ is a neighborhood of F in $\mathcal{MF}$;*
3. *F belongs to each set U_{τ_i}, for all $i = 1, \dots, n$;*
4. *the PL map $\psi_{\tau\sigma}$ is a union of linear maps that are induced by the relations $\tau_i \prec \tau$ and $\tau_i \prec \sigma$, and inverses of such maps.*

Proof. The family $T = \{\tau_1, \ldots, \tau_n\}$ of train tracks is obtained by a sequence of successive splittings and shifts that start from the train track τ, and we shall now describe how to obtain it. We note right away that each shift operation that we use produces a new train track, but as for splitting operations, some of them produce one train track, and some of them produce two.

A *singular leaf* of a partial foliation is a leaf that starts at a singular point.

The sequence of splittings and shifts that we use is obtained step by step, by "following the singular leaves" of a partial measured foliation representing the element F of $\mathcal{MF}$, whose support is equal to a regular neighborhood of a train track obtained at the preceding step, and whose leaves are transverse to the foliation of that neighborhood by the ties. The process of "following a singular leaf" is illustrated in Figure 8. At each step, opening up the regular neighborhood by following the singular leaf produces a new train track out of the old one, either by a shift or by a splitting, except that in the case where the singular leaf abuts on another singular point (as illustrated in Figure 9), the operation produces two train tracks. Indeed, in this case, it is necessary to take the two train tracks σ' and σ'' obtained from the given train track σ by splitting, in order to insure that the union $\mathcal{MF}(\sigma') \cup \mathcal{MF}(\sigma'')$ is a neighborhood of F in $\mathcal{MF}$. The fact that F is also carried by σ insures that by performing a finite number of appropriate shift and splitting operations, we obtain a sequence of train tracks satisfying the required properties. Note that there is no natural choice of the singular leaf that one uses at each step, but it is important to choose the singular leaves in such a way that the maximal transverse measure of the rectangles that appear in the construction becomes less than any amount that is fixed in advance. (Equivalently, the maximal weight that is induced by the foliation F on the edges of the train tracks resulting from the construction can be made smaller than an amount that is fixed in advance.) To insure this, it may be necessary not to use the same singular leaf at each step, and one possible way to proceed is to use, in any given order, all the singular leaves one after the other. The details of this construction are contained in Chapter 1 of [4]. □

Let us note that it follows from the proof of Proposition 4.3 that if the foliation representing the class F does not have any compact leaf joining singular points, then the process described produces a family T consisting of a single train track, and the coordinate change in the neighborhood of F in $\mathcal{MF}$ is linear, and not only PL.

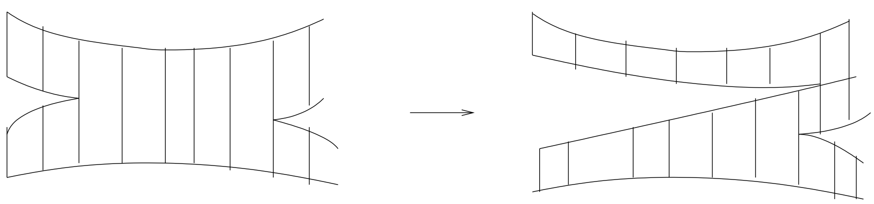

Figure 8: The operation of following a leaf, used in the proof of Proposition 4.3. In the case represented, the operation corresponds, at the level of train tracks, to a splitting.

We note that Property (4) of Proposition 4.3 implies that the restriction of the coordinate change map $\psi_{\tau\sigma}$ to the neighborhood $N(F)$ of F is linear on each subset

U_{τ_i} of $N(F)$. Proposition 4.3 is the basic technical result which implies that the coordinate changes in the atlas $\mathcal{A}$ are piecewise-linear.

The study of the automorphisms of the train track PL structure will be based on considerations on the singular set of a PL function.

Definition 4.4 (Train track PL function). *Let N be a positive integer. A function $f : \mathcal{MF} \to \mathbb{R}^N$ is said to be a train track* PL *function if for every x in $\mathcal{MF}$, there exists a chart (U_τ, ψ_τ) belonging to the atlas $\mathcal{A}$ such that the set U_τ contains x in its interior and the function $f \circ \psi_\tau^{-1}$ defined on $V_\tau = \psi_\tau(U_\tau)$ is PL. Furthermore, we require that there exists a coordinate change map $\psi_{\tau\sigma}$ belonging to the atlas $\mathcal{A}$, having $\psi(x)$ in the interior of its domain and such that the singular sets of the restrictions of the maps $f \circ \psi_\tau^{-1}$ and $\psi_{\tau\sigma}$ to the set V_τ coincide in a neighborhood of $\psi_\tau(x)$ in V_τ.*

The last condition is the important part of the definition. It says that a train track PL function is not allowed to have singularities that do not already appear in a coordinate change function.

Let $\mathcal{P}$ be the set of train track PL functions on $\mathcal{MF}$. In some sense, $\mathcal{P}$ is the set of smoothest possible PL functions on $\mathcal{MF}$ relatively to the atlas $\mathcal{A}$.

Definition 4.5 (Automorphism of the train track PL structure). *We shall say that a homeomorphism $h : \mathcal{MF} \to \mathcal{MF}$ is an automorphism of the train track PL structure of $\mathcal{MF}$ if for every element f of $\mathcal{P}$, $f \circ h$ is also in $\mathcal{P}$.*

In other words, an automorphism of the PL structure is a homeomorphism of $\mathcal{MF}$ that preserves the set $\mathcal{P}$ of PL functions.

We denote by $\mathrm{Aut}(\mathcal{MF}, \mathcal{P})$ the automorphism group of the train track PL structure.

Proposition 4.6. *Any homeomorphism of $\mathcal{MF}$ that is induced by an element of the extended mapping class group is an automorphism of the train track PL structure of $\mathcal{MF}$.*

Proof. Let $h : \mathcal{MF} \to \mathcal{MF}$ be a homeomorphism induced by an extended mapping class, let N be a positive integer and let $f : \mathcal{MF} \to \mathbb{R}^N$ be a function in $\mathcal{P}$. For each x in $\mathcal{MF}$, let (U_τ, ψ_τ) and $\psi_{\tau\sigma}$ be respectively a chart in $\mathcal{A}$ and a coordinate change map in $\mathcal{A}$ satisfying the properties required in Definition 4.4. Then, $\tau' = h(\tau)$

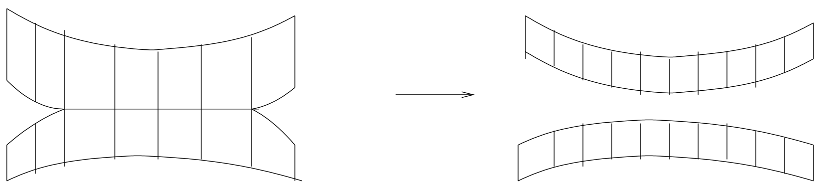

Figure 9: In the case considered (where the singular leaf abuts on another singular point), the operation of following a leaf, used in the proof of Proposition 4.3, produces two train tracks, each one obtained by a splitting operation (one right splitting and one left splitting).

and $\sigma' = h(\sigma)$ are maximal recurrent train tracks on S, $(h(U_\tau), \psi_{\tau'})$ is a chart in $\mathcal{A}$, and $\psi_{\tau'\sigma'}$ is a coordinate change in $\mathcal{A}$ which also satisfies the properties required in Definition 4.4, with respect to the point $h(x)$ instead of the point x. From this the proof follows. $\square$

From Proposition 4.6, we obtain a homomorphism

$$\Gamma^*(S) \to \mathrm{Aut}(\mathcal{MF}, \mathcal{P}).$$

Theorem 4.9 below says that this homomorphism is an isomorphism, except in genus two, where the homomorphism is surjective with kernel $\mathbb{Z}_2$. Before proving this theorem, we need to establish a few more notations.

A *system of curves* on S is the isotopy class of a collection of disjoint and pairwise non-isotopic essential curves on S. Note that for $g \geq 2$, the number of elements in such a collection is bounded above by $3g - 3$.

Let $\mathcal{S}'$ be the set of systems of curves on S.

For each integer k satisfying $1 \leq k \leq 3g - 3$, we denote by $\mathcal{S}_k$ the subset of $\mathcal{S}'$ consisting of isotopy classes of curves of cardinality k. In particular $\mathcal{S}_1 = \mathcal{S}$.

For each k satisfying $1 \leq k \leq 3g - 3$, we let $\mathcal{MF}_k \subset \mathcal{MF}$ be the set of measured foliation classes x having the following properties:

(C1) For each i satisfying $0 \leq i \leq k - 1$, there is no chart (U_τ, ψ_τ) in $\mathcal{A}$ having x in the interior of its domain U_τ with a coordinate change $\psi_{\tau\sigma}$ having $\psi_\tau(x)$ in the interior of its domain, and with $\psi_\tau(x)$ on a stratum of dimension i of the flag $\mathrm{Fl}(\psi_{\tau\sigma})$.

(C2) There exists a coordinate chart (U_τ, ψ_τ) in $\mathcal{A}$ having x in the interior of its domain and a coordinate change $\psi_{\tau\sigma}$ having $\psi_\tau(x)$ in the interior of its domain such that $\psi(x)$ is on a stratum of dimension k of the flag $\mathrm{Fl}(\psi_{\tau\sigma})$. Furthermore, we require that $\psi_\tau(x)$ is a convex combination of k elements in the 1-stratum of $\mathrm{Fl}(\psi_{\tau\sigma})$, with respect to the linear structure of V_τ being the one induced from its inclusion in $\mathbb{R}^N$.

Note that $\mathcal{MF}_1 \subset \mathcal{MF}$ is simply the set of measured foliation classes x such that there exists a coordinate chart (U_τ, ψ_τ) in $\mathcal{A}$ having x in the interior of its domain and a coordinate change $\psi_{\tau\psi}$ having $\psi(x)$ in the interior of its domain, and such that x is on a stratum of dimension 1 of the flag defined by the singular set $\mathrm{Fl}(\psi_{\tau\sigma})$.

Proposition 4.7. *For any $k \geq 0$, any element of* $\mathrm{Aut}(\mathcal{MF}, \mathcal{P})$ *preserves the set* $\mathcal{MF}_k$.

Proof. An automorphism of $(\mathcal{MF}, \mathcal{P})$ acts on the set of flags of the coordinate changes $\psi_{\tau\sigma}$ of $\mathcal{A}$, that is, it carries the flag of any coordinate change in $\mathcal{A}$ to a flag of some coordinate change in $\mathcal{A}$, and it preserves the properties defining the elements of $\mathcal{MF}_k$, for each $k \geq 1$. $\square$

For each integer k satisfying $1 \leq k \leq 3g - 3$, there is a natural inclusion $j_k : (\mathbb{R}_+^*)^k \times \mathcal{S}_k \hookrightarrow \mathcal{MF}$, defined by associating to each vector v in $(\mathbb{R}_+^*)^k$ and to each element C in $\mathcal{S}_k$ the equivalence class of a partial measured foliation F satisfying the following properties:

1. the support of F is the union of disjoint annuli $A_1, \dots, A_k$ foliated by closed leaves;

2. for each $1 \leq i \leq k$, the annulus A_i is a regular neighborhood of a closed curve c_i, where $c_1, \dots, c_k$ are the components of a system of curves representing the isotopy class C;

3. for each $1 \leq i \leq k$, the total transverse measure of the annulus A_i is equal to the i-th coordinate of v.

We shall call a foliation on S representing an element of $\mathcal{MF}$ which is the image of some element of $\mathcal{S}'$ by one of the maps j_k an *annular foliation*.

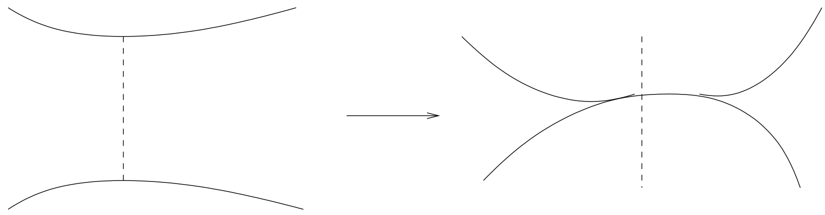

Figure 10: The pinching operation that is used in the proof of Proposition 4.8.

Proposition 4.8. *For every $k \geq 1$, the image of $(\mathbb{R}^*)^k \times \mathcal{S}_k$ in $\mathcal{MF}$ by the map j_k is equal to $\mathcal{MF}_k$.*

Proof. Let $F \in \mathcal{MF}$ be a measured foliation class which is in the image of an element of $(\mathbb{R}^*)^k \times \mathcal{S}_k$ by j_k. We must show that it satisfies Properties (C1) and (C2) above. We start by representing F by a system of weights on a train track consisting of a union of k disjoint simple closed curves representing the given element of $(\mathbb{R}^*)^k \times \mathcal{S}_k$. We can pinch this system of curves along a system of disjoint arcs having their endpoints on these curves so as to obtain a maximal recurrent train track τ such that F is in the interior of the linear polytope V_τ associated to τ, as represented in Figure 10. By choosing a different system of arcs, we can obtain a maximal recurrent train track σ such that F is in the interior of the associated linear polytope V_σ, and we can choose this new system of arcs so that F is in the codimension-k skeleton of the flag $\mathrm{Fl}(\psi_{\tau\sigma})$. From this it follows that F is in $\mathcal{MF}_k$. (This uses the description of the coordinate changes that is contained in Proposition 4.3 above.) It is easy to see that conversely, if a measured foliation satisfies Properties (C1) and (C2), then it is in the image of $(\mathbb{R}^*)^k \times \mathcal{S}_k$.

□

Theorem 4.9. *If the genus of S is ≥ 3, then the homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(\mathcal{MF}, \mathcal{P})$$

is an isomorphism. If the genus is 2, this homomorphism is surjective, and its kernel is $\mathbb{Z}_2$ generated by the hyperelliptic involution.

Proof. Consider an element f of $\mathrm{Aut}(\mathcal{MF},\mathcal{P})$. We use the action of f on the curve complex $C(S)$ of S, that is, the flag complex whose k-simplices, for each $k \geq 0$, are the isotopy classes of disjoint $k+1$ disjoint and pairwise non-homotopic essential simple closed curves on S. We refer to [2] for a study of this complex. By Proposition 4.7, f preserves the subset $\mathcal{MF}_1$ of $\mathcal{MF}$. By Proposition 4.8, the set $\mathcal{MF}_1$ is the natural image of $\mathbb{R}_+^* \times \mathcal{S}$ in $\mathcal{MF}$, that is, it is the set of measured foliation classes that are representable by foliations all whose nonsingular leaves are closed curves homotopic to a simple closed curve. Thus, $\mathcal{MF}_1$ is in natural one-to-one correspondence with the set $\mathbb{R}_+^* \times \mathcal{S}$ of isotopy classes of weighted essential curves on S. In particular, f acts on the set of isotopy classes of weighted essential simple closed curves on S. Since f acts linearly on rays, it acts on the set $\mathcal{S}$ of isotopy classes of essential curves, which is the set of vertices of the curve complex $C(S)$. Therefore, f naturally defines a self-map of the vertex set of $C(S)$, and it follows from the fact that f is a homeomorphism that this map is a bijection.

Similarly, by Proposition 4.7, for each $k = 2, \ldots, 3g-3$, f preserves the set $\mathcal{MF}_k$ of $\mathcal{MF}$ which, again using Proposition 4.8, can be naturally identified with the set $\mathcal{S}_k$ of isotopy classes of weighted systems of curves which have k components. This implies that f also induces a self-map of the set of $(k-1)$-simplices of $C(S)$. Thus, the bijection induced by f on the vertex set of $C(S)$ extends naturally to a simplicial automorphism of $C(S)$. Now by a theorem of Ivanov [2], the action of f on $C(S)$ is induced by an element γ of the extended mapping class group of S. It is clear from the definitions of these actions that the restriction of f and of the extended mapping class γ on the image of $\{1\} \times \mathcal{S}$ (and, even, of $\{1\} \times \mathcal{S}'$) in $\mathcal{MF}$ coincide. Since f and γ are linear on each ray in $\mathcal{MF}$, these actions coincide on the subset $\mathbb{R}_+^* \times \mathcal{S}$ of $\mathcal{MF}$. Since the image of $\mathbb{R}_+^* \times \mathcal{S}$ in $\mathcal{MF}$ is dense and since the actions of f and γ on $\mathcal{MF}$ are continuous, the two actions coincide on the space $\mathcal{MF}$. Thus, each automorphism of $(\mathcal{MF},\mathcal{P})$ is induced by an extended mapping class. This proves the surjectivity of the homomorphism $\Gamma^*(S) \to \mathrm{Aut}(\mathcal{MF},\mathcal{P})$. The results about the injectivity in genus ≥ 3 and the kernel in genus 2 follow from the fact that the homomorphism from the extended mapping class group to the automorphism group of the curve complex is injective except in genus two, in which case the kernel is $\mathbb{Z}_2$. □

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