Affine classification of n-curves

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Abstract. The classification of curves up to affine transformations in a finite dimensional space was studied by some different methods. In this paper, we obtain the exact formulas of affine invariants via the equivalence problem in view of Cartan's theorem and then, we state a necessary and sufficient condition for the classification of n–Curves.

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1 Introduction

This paper devoted to the study of curve invariants in an arbitrary finite dimensional space, under the action of special affine transformations. This work was done before in some different methods. Furthermore, these invariants were just pointed out by Spivak [6], in the method of Cartan's theorem, but they were not explicitly determined. Now, we will exactly determine these invariants in view of Cartan's theorem and of the equivalence problem.

An *affine transformation* in an $n-$dimensional space, is generated by the action of the general linear group $\mathrm{GL}(n, \mathbf{R})$ and then, of the translation group $\mathbf{R}^n$. If we restrict $\mathrm{GL}(n, \mathbf{R})$ to the special linear group $\mathrm{SL}(n, \mathbf{R})$ of matrices with determinant equal to 1, we have a *special* affine transformation. The group of special affine transformations has $n^2 + n - 1$ parameters. This number coincides with the dimension of the Lie algebra of the Lie group of special affine transformations. The natural condition of differentiability is $\mathcal{C}^{n+2}$.

In the next section, we state some preliminaries about the Maurer–Cartan forms, Cartan's theorem for the equivalence problem, and a theorem about the number of invariants in a space. In section three, we obtain the invariants and then, by them, we classify the $n-$curves of the space.

2 Preliminaries

Let $G \subset \mathrm{GL}(n, \mathbf{R})$ be a matrix Lie group with Lie algebra $\mathfrak{g}$ and let $P : G \to Mat(n \times n)$ be a matrix-valued function which embeds G into $Mat(n \times n)$, the vector space of

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$n \times n$ matrices with real entries. Its differential is $dP_B : T_B G \to T_{P(B)} Mat(n \times n) \simeq Mat(n \times n)$.

Definition 2.1 The following form of G is called *Maurer-Cartan form*:

$$\omega_B = \{P(B)\}^{-1} \, . \, dP_B$$

that it is often written $\omega_B = P^{-1} \, . \, dP$. The Maurer-Cartan form is the key to classifying maps into homogeneous spaces of G, and this process needs the following result (for the proof we refer to [2]):

Theorem 2.2 (Cartan) *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω. Let M be a manifold on which there exists a $\mathfrak{g}-$valued 1-form ϕ satisfying $d\,\phi = -\phi \wedge \phi$. Then for any point $x \in M$ there exists a neighborhood U of x and a map $f : U \to G$ such that $f^*\,\omega = \phi$. Moreover, any two such maps f_1, f_2 must satisfy $f_1 = L_B \circ f_2$ for some fixed $B \in G$ (L_B is the left action of B on G).*

Corollary 2.3 *Given the maps $f_1, f_2 : M \to G$, then $f_1^*\,\omega = f_2^*\,\omega$, that is, this pull-back is invariant, if and only if $f_1 = L_B \circ f_2$ for some fixed $B \in G$.*

The next section is devoted to the study of some properties of $n-$curves invariants, under the special affine transformations group. The number of essential parameters (the dimension of the Lie algebra) is $n^2 + n - 1$. The natural assumption of differentiability is $\mathcal{C}^{n+2}$.

We obtain all the invariants of an $n-$curve with respect to special affine transformations, and by theorem 2.2, two $n-$curves in $\mathbf{R}^n$ will be equivalent under special affine transformations, if they differ by a left action introduced by an element of $\mathrm{SL}(n, \mathbf{R})$ and then by a translation.

3 Classification of $n-$curves

Let $C : [a, b] \to \mathbf{R}^n$ be a curve of class $\mathcal{C}^{n+2}$ in the finite dimensional space $\mathbf{R}^n$, *$n-$space*, which satisfies the condition

$$\det(C', C'', \cdots, C^{(n)}) \neq 0; \tag{3.1}$$

we call this curve as *$n-$curve*. The condition (3.1) guarantees that C', C'', $\cdots$, and $C^{(n)}$ are independent, and therefore, the curve does not turn into the lower dimensional cases. Also, we may assume that

$$\det(C', C'', \cdots, C^{(n)}) > 0, \tag{3.2}$$

for avoiding the absolute value for its computation.

For the $n-$curve C, we define a new curve $\alpha_C(t) : [a, b] \to \mathrm{SL}(n, \mathbf{R})$ of the following form

$$\alpha_C(t) := \frac{(C', C'', \cdots, C^{(n)})}{\sqrt[n]{\det(C', C'', \cdots, C^{(n)})}} \, . \tag{3.3}$$

Obviously, this is well-defined on $[a, b]$. We can study this new curve with respect to special affine transformations, that is the action of affine transformations on first, second, ..., and n^{th} differentiation of C. For A, the special affine transformation, there is a unique representation $A = \tau \circ B$, where B is an element of $\mathrm{SL}(n, \mathbf{R})$ and τ is a translation in $\mathbf{R}^n$. If two $n-$curves C and $\bar{C}$ coincide mod some special affine transformation, that is, $\bar{C} = A \circ C$, then from [4], we have

$$\bar{C}' = B \circ C', \quad \bar{C}'' = B \circ C'', \quad \ldots, \quad \bar{C}^{(n)} = B \circ C^{(n)}. \tag{3.4}$$

We can relate the determinants of these curves as follows

$$\begin{aligned} \det(\bar{C}', \bar{C}'', \cdots, \bar{C}^{(n)}) &= \det(B \circ C', B \circ C'', \cdots, B \circ \bar{C}^{(n)}) \\ &= \det(B \circ (C', C'', \cdots, C^{(n)})) \\ &= \det(C', C'', \cdots, C^{(n)}). \end{aligned} \tag{3.5}$$

Hence we conclude that $\alpha_{\bar{C}}(t) = B \circ \alpha_C(t)$ and thus $\alpha_{\bar{C}} = L_B \circ \alpha_C$ that L_B is a left translation by $B \in \mathrm{SL}(n, \mathbf{R})$.

This condition is also necessary because when C and $\bar{C}$ are two curves in $\mathbf{R}^n$ such that for an element $B \in \mathrm{SL}(n, \mathbf{R})$, we have $\alpha_{\bar{C}} = L_B \circ \alpha_C$, thus we can write

$$\begin{aligned} \alpha_{\bar{C}}(t) &= \det(\bar{C}', \bar{C}'', \cdots, \bar{C}^{(n)})^{-1/n} (\bar{C}', \bar{C}'', \cdots, \bar{C}^{(n)}) \\ &= \det(B \circ (C', C'', \cdots, C^{(n)}))^{-1/n} B \circ (C', C'', \cdots, C^{(n)}) \\ &= \det(C', C'', \cdots, C^{(n)})^{-1/n} B \circ (C', C'', \cdots, C^{(n)}). \end{aligned} \tag{3.6}$$

Therefore, we have $\bar{C}' = B \circ C'$, and hence there exists a translation τ such that $A = \tau \circ B$, and so, we have $\bar{C} = A \circ C$ where A is an $n-$dimensional affine transformation. Therefore, we have

Theorem 3.1 *Two $n-$curves C and $\bar{C}$ in $\mathbf{R}^n$ coincide mod some special affine transformations that is, $\bar{C} = A \circ C$, with $A = \tau \circ B$ for a translation τ in $\mathbf{R}^n$ and $B \in \mathrm{SL}(n, \mathbf{R})$, if and only if, $\alpha_{\bar{C}} = L_B \circ \alpha_C$, where L_B is left translation by B.*

From Cartan's theorem, a necessary and sufficient condition for $\alpha_{\bar{C}} = L_B \circ \alpha_C$ by $B \in \mathrm{SL}(n, \mathbf{R})$, is that for any left invariant 1-form ω^i on $\mathrm{SL}(n, \mathbf{R})$ we have $\alpha^*_{\bar{C}}(\omega^i) = \alpha^*_C(\omega^i)$, that is equivalent with $\alpha^*_{\bar{C}}(\omega) = \alpha^*_C(\omega)$, for natural $\mathfrak{sl}(n, \mathbf{R})$-valued 1-form $\omega = P^{-1} \,.\, dP$, where P is the Maurer–Cartan form.

Thereby, we must compute $\alpha^*_C(P^{-1}.dP)$, which is invariant under special affine transformations, that is, its entries are invariant functions of the $n-$curve. This $n \times n$ matrix form consists of arrays that are coefficients of dt.Since $\alpha^*_C(P^{-1} \,.\, dP) = \alpha_C^{-1} \,.\, d\alpha_C$, for finding the invariants, it is sufficient to calculate the matrix $\alpha_C(t)^{-1} . d\alpha_C(t)$. Thus, we compute $\alpha^*_C(P^{-1} \,.\, dP)$. We have

$$\alpha_C^{-1} = \sqrt[n]{\det(C', C'', \cdots, C^{(n)})} \,.\, (C', C'', \cdots, C^{(n)})^{-1}. \tag{3.7}$$

We assume that C is in the form $(C_1 \;\; C_2 \;\; \cdots \;\; C_n)^T$. By differentiating of determinant, we have

$$
\begin{aligned}
[\det(C',C'',\cdots,C^{(n)})]' &= \det(C'',C'',\cdots,C^{(n)}) \\
&\quad +\det(C',C''',\cdots,C^{(n)}) \\
&\quad \vdots \\
&\quad +\det(C',C'',\cdots,C^{(n-1)},C^{(n+1)}) \\
&= \det(C',C'',\cdots,C^{(n-1)},C^{(n+1)})\,.
\end{aligned}
\tag{3.8}
$$

Thus, we conclude that

$$
\begin{aligned}
\alpha'_C &= \{\det(C',C'',\cdots,C^{(n)})\}^{-1/n}\cdot \begin{pmatrix} C''_1 & C'''_1 & \cdots & C^{(n)}_1 \\ C''_2 & C'''_2 & \cdots & C^{(n)}_2 \\ \vdots & \vdots & & \vdots \\ C''_3 & C'''_3 & \cdots & C^{(n)}_n) \end{pmatrix} \\
&\quad -\frac{1}{n}\det(C',C'',C''')\}^{-(n+1)/n}\cdot \begin{pmatrix} C'_1 & C''_1 & \cdots & C^{(n)}_1 \\ C'_2 & C''_2 & \cdots & C^{(n)}_2 \\ \vdots & \vdots & & \vdots \\ C'_3 & C''_3 & \cdots & C^{(n)}_3 \end{pmatrix}.
\end{aligned}
\tag{3.9}
$$

Therefore, we have $\alpha_C^{-1}\cdot d\alpha_C$ as the following matrix multiplied with dt:

$$
\left(\begin{array}{ccccc|c} a & 0 & \cdots & 0 & 0 & \\ 1 & a & \cdots & 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & M\,.\,C^{(n+1)}+\begin{pmatrix} 0 \\ 0 \\ \vdots \\ a \end{pmatrix} \\ 0 & 0 & \cdots & 1 & a & \\ 0 & 0 & \cdots & 0 & 1 & \end{array}\right)
\tag{3.10}
$$

where the latest column, $M\,.\,C^{(n+1)}+(0,0,\cdots,a)^T$, is multiple of M by $C^{(n+1)}$ added by the transpose of $(0,0,\cdots,a)$, where M is the inverse of the matrix $(C',C'',\cdots,C^{(n)})$ and also, we assumed that

$$
a=-\frac{\det(C',C'',\cdots,C^{(n-1)},C^{(n+1)})}{n\ \det(C',C'',\cdots,C^{(n)})}.
\tag{3.11}
$$

Using Crammer's law, we compute $M\,.\,C^{(n+1)}$. If $M\,.\,C^{(n+1)}=X=(X_1,X_2,\cdots,X_n)^T$, then $M^{-1}.X=C^{(n+1)}$. Therefore, for each $i=1,2,\cdots,n$ we conclude that

$$
X_i=\frac{\det(C',C'',\cdots,C^{(i-1)},C^{(n+1)},C^{(i+1)},\cdots,C^{(n)})}{\det(C',C'',\cdots,C^{(n)})}
\tag{3.12}
$$

Finally, $\alpha_C^{-1}\cdot d\alpha_C$ is the following multiple of dt:

$$
\left(\begin{array}{cccccc}
a & 0 & \cdots & 0 & 0 & (-1)^{n-1}\frac{\det(C'',\cdots,C^{(n+1)})}{\det(C',C'',\cdots,C^{(n)})}\\
1 & a & \cdots & 0 & 0 & (-1)^{n-2}\frac{\det(C',C''',\cdots,C^{(n)})}{\det(C',C'',\cdots,C^{(n)})}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
o & 0 & \cdots & 1 & a & -\frac{\det(C',\cdots,C^{(n-2)},C^{(n)},C^{(n+1)})}{\det(C',C'',\cdots,C^{(n)})}\\
0 & 0 & \cdots & 0 & 1 & \frac{n-1}{n}\frac{\det(C',\cdots,C^{(n-1)},C^{(n+1)})}{\det(C',C'',\cdots,C^{(n)})}
\end{array}\right), \tag{3.13}
$$

where the coefficient $(-1)^{i-1}$ for the i^{th} entry of the last column is provided by the translation of $C^{(n+1)}$ to the n^{th} column of the matrix

$$
(C', C'', \cdots, C^{(i-1)}, C^{(n+1)}, C^{(i+1)}, \cdots, C^{(n)}). \tag{3.14}
$$

Clearly, the trace of the matrix (3.13) is zero. The entries of $\alpha_C^*(P^{-1} . dP)$, and hence the arrays of the matrix (3.13) are invariants of the group action.

Two $n-$curves $C, \bar{C} : [a, b] \to \mathbf{R}^n$ coincide mod a special affine transformations, if we have

$$
\begin{aligned}
\frac{\det(C''(t), \cdots, C^{(n+1)}(t))}{\det(C'(t), C''(t), \cdots, C^{(n)}(t))} &= \frac{\det(\bar{C}''(t), \cdots, \bar{C}^{(n+1)}(t))}{\det(\bar{C}'(t), \bar{C}''(t), \cdots, \bar{C}^{(n)}(t))}\\
\frac{\det(C'(t), C'''(t), \cdots, C^{(n+1)}(t))}{\det(C'(t), C''(t), \cdots, C^{(n)}(t))} &= \frac{\det(\bar{C}'(t), \bar{C}'''(t), \cdots, \bar{C}^{(n+1)}(t))}{\det(\bar{C}'(t), \bar{C}''(t), \cdots, \bar{C}^{(n)}(t))}\\
&\vdots\\
\frac{\det(C'(t), \cdots, C^{(n-1)}(t), C^{(n+1)})}{\det(C'(t), C''(t), \cdots, C^{(n)}(t))} &= \frac{\det(\bar{C}'(t), \cdots, \bar{C}^{(n-1)}(t), \bar{C}^{(n+1)})}{\det(\bar{C}'(t), \bar{C}''(t), \cdots, \bar{C}^{(n)}(t))}.
\end{aligned} \tag{3.15}
$$

We may use of a proper parametrization $\gamma : [a, b] \to [0, l]$, such that the parameterized curve, $\gamma = C \circ \sigma^{-1}$, satisfies in condition

$$
\det(\gamma'(s), \gamma''(s), \cdots, \gamma^{(n-1)}(s), \gamma^{(n+1)}(s)) = 0, \tag{3.16}
$$

then, the arrays on the main diagonal of $\alpha_\gamma^*(dP . P^{-1})$ will be zero. But the last determinant is given by the differentiation of $\det(\gamma'(s), \gamma''(s), \cdots, \gamma^{(n)}(s))$, and thus it is sufficient to assume that

$$
\det(\gamma'(s), \gamma''(s), \cdots, \gamma^{(n)}(s)) = 1. \tag{3.17}
$$

On the other hand, we have

$$
\begin{aligned}
C' &= (\gamma \circ \sigma)' = \sigma'.(\gamma' \circ \sigma)\\
C'' &= (\sigma')^2.(\gamma'' \circ \sigma) + \sigma''.(\gamma' \circ \sigma)\\
&\vdots\\
C^{(n)} &= (\sigma')^{(n)}.(\gamma^{(n)} \circ \sigma) + n\, \sigma^{(n-1)}\sigma'.(\gamma^{(n-1)} \circ \sigma)\\
&\quad + \frac{n(n-1)}{2}\sigma^{(n-2)}\sigma''.(\gamma^{(n-2)} \circ \sigma) + \cdots + \sigma^{(n)}.(\gamma' \circ \sigma)
\end{aligned} \tag{3.18}
$$

Therefore, the $C^{(i)}$s for $1 \leq i \leq n$, are some expressions in terms of $\gamma^{(j)} \circ \sigma$, $1 \leq j \leq n$. We conclude that

$$\begin{aligned}
\det(C', C'', \cdots, C^{(n)}) &= \det(\sigma'.(\gamma' \circ \sigma)\,,\, (\sigma')^2.(\gamma'' \circ \sigma) + \sigma''.(\gamma' \circ \sigma)\,, \\
&\quad \cdots, (\sigma')^{(n)}.(\gamma^{(n)} \circ \sigma) + n\,\sigma^{(n-1)}\sigma'.(\gamma^{(n-1)} \circ \sigma) \\
&\quad + \frac{n(n-1)}{2}\,\sigma^{(n-2)}\sigma''.(\gamma^{(n-2)} \circ \sigma) + \cdots + \sigma^{(n)}.(\gamma' \circ \sigma)) \qquad (3.19)\\
&= \det(\sigma'.(\gamma' \circ \sigma)\,,\, (\sigma')^2.(\gamma'' \circ \sigma)\,,\, \cdots\,,\, (\sigma')^n.(\gamma^{(n)} \circ \sigma)) \\
&= \sigma'^{\frac{n(n-1)}{2}}.\det(\gamma' \circ \sigma\,,\, \gamma'' \circ \sigma\,,\, \cdots\,,\, \gamma^{(n)} \circ \sigma) \\
&= \sigma'^{\frac{n(n-1)}{2}}\,,
\end{aligned}$$

The last expression signifies σ. Therefore, we define the *special affine arc length* as follows

$$\sigma(t) := \int_a^t \left\{\det(C'(u), C''(u), \cdots, C^{(n)}(u))\right\}^{\frac{2}{n(n-1)}} du. \qquad (3.20)$$

So, σ is the natural parameter for $n-$curves under the action of special affine transformations, that is, when C is parameterized with σ, then for each special affine transformation A, $A \circ C$ will also be parameterized with the same σ. Furthermore, every $n-$curve parameterized with σ with respect to special affine transformations, will lead to the following invariants

$$\begin{aligned}
\chi_1 &= (-1)^{n-1}\det(C'', \cdots, C^{(n+1)}) \\
\chi_2 &= (-1)^{n-2}\det(C', C''', \cdots, C^{(n)}) \\
&\;\;\vdots \qquad\qquad (3.21)\\
\chi_{n-1} &= \det(C', \cdots, C^{(n-2)}, C^{(n)}, C^{(n+1)}).
\end{aligned}$$

We call $\chi_1, \chi_2, \cdots$, and χ_{n-1} as (respectively) the first, second, ..., and $n-1^{th}$ *special affine curvatures.* In fact, we proved the following

Theorem 3.2 *A curve of class $\mathcal{C}^{n+2}$ in $\mathbf{R}^n$ which satisfies the condition (3.1), up to special affine transformations has $n-1$ invariants χ_1, χ_2, ..., and χ_{n-1}, the first, second, ..., and $n-1^{th}$ affine curvatures that are defined in formulas (3.3).*

Theorem 3.3 *Two $n-$curves $C, \bar{C} : [a, b] \rightarrow \mathbf{R}^n$ of class $\mathcal{C}^{n+2}$, that satisfy in the condition (3.1), are special affine equivalent, if and only if, $\chi_1^C = \chi_1^{\bar{C}}$, $\cdots$, and $\chi_{n-1}^C = \chi_{n-1}^{\bar{C}}$.*

Proof: The proof is completely similar to the three dimensional case [5]. The first part of the theorem was proved above. For the other part, we assume that C and $\bar{C}$ are $n-$curves of class $\mathcal{C}^{n+2}$ satisfying the conditions (resp.):

$$\det(C', C'', \cdots, C^{(n)}) > 0, \qquad \det(\bar{C}', \bar{C}'', \cdots, \bar{C^{(n)}}) > 0, \qquad (3.22)$$

this meaning that they are not $(n-1)$–curves. Also, we suppose that they have the same same χ_1, $\cdots$, and χ_{n-1}.

By changing the parameter to the natural parameter (σ) discussed above, we obtain two new curves γ and $\bar{\gamma}$ resp. that the determinants (3.22) will be equal to 1. We prove that γ and $\bar{\gamma}$ are special affine equivalent, so there exists a special affine transformation A, such that $\bar{\gamma} = A \circ \gamma$, and then we have $\bar{C} = A \circ C$, and the proof will be complete.

First, we replace the curve γ with $\delta := \tau(\gamma)$ properly, in which case δ intersects $\bar{\gamma}$, and τ is a translation defined by translating one point of γ to one point of $\bar{\gamma}$. We correspond $t_0 \in [a, b]$, to the intersection point of δ and $\bar{\gamma}$; thus, $\delta(t_0) = \bar{\gamma}(t_0)$. One can find a unique element B of the general linear group $\mathrm{GL}(n, \mathbf{R})$, such that this maps the basis $\{\delta'(t_0), \delta''(t_0), \cdots, \delta^{(n)}(t_0)\}$ of the tangent space $T_{\delta(t_0)}\mathbf{R}^3$ to its basis $\{\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \cdots, \bar{\gamma}^{(n)}(t_0)\}$. So, we have $B \circ \delta'(t_0) = \bar{\gamma}'(t_0)$, $B \circ \delta''(t_0) = \bar{\gamma}''(t_0)$, $\cdots$, and $B \circ \delta^{(n)}(t_0) = \bar{\gamma}^{(n)}(t_0)$. B also is an element of the special linear group, $\mathrm{SL}(n, \mathbf{R})$, since we have

$$\begin{aligned} &\det(\gamma'(t_0), \gamma''(t_0), \cdots, \gamma^{(n)}(t_0)) = \\ &\qquad = \det(\delta'(t_0), \delta''(t_0), \cdots, \delta^{(n)}(t_0)), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &\det(\delta'(t_0), \delta''(t_0), \cdots, \delta^{(n)}(t_0)) = \\ &\qquad \det\Big(B \circ (\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \cdots, \bar{\gamma}^{(n)}(t_0))\Big), \end{aligned} \tag{3.24}$$

so, $\det(B) = 1$. If we denote $\eta := B \circ \delta$ as equal to $\bar{\gamma}$ on $[a, b]$, then by choosing $A = \tau \circ B$, the claim follows.

For the curves η and $\bar{\gamma}$ we have (resp.)

$$\begin{aligned} &(\eta', \eta'', \cdots, \eta^{(n)})' = \\ &\qquad = (\eta', \eta'', \cdots, \eta^{(n)}).\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \chi_1^\eta \\ 1 & 0 & \cdots & 0 & 0 & -\chi_2^\eta \\ 0 & 1 & \cdots & 0 & 0 & \chi_3^\eta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & (-1)^{(n-2)}\chi_{n-1}^\eta \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &(\bar{\gamma}', \bar{\gamma}'', \cdots, \bar{\gamma}^{(n)})' = \\ &\qquad = (\bar{\gamma}', \bar{\gamma}'', \cdots, \bar{\gamma}^{(n)}).\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \chi_1^{\bar{\gamma}} \\ 1 & 0 & \cdots & 0 & 0 & -\chi_2^{\bar{\gamma}} \\ 0 & 1 & \cdots & 0 & 0 & \chi_3^{\bar{\gamma}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & (-1)^{(n-2)}\chi_{n-1}^{\bar{\gamma}} \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \end{aligned} \tag{3.26}$$

Since, χ_1, $\cdots$, and χ_{n-1}, are invariant under special affine transformations so, we have

$$\chi_i^\eta = \chi_i^\gamma = \chi_i^{\bar{\gamma}}, \qquad (i = 1, \cdots, n-1). \tag{3.27}$$

Therefore, we conclude that η and $\bar{\gamma}$ are solutions of the ordinary differential equation of degree $n+1$:

$$Y^{n+1} + (-1)^{n-1}\chi_{n-1}Y^{(n)} + \cdots + \chi_2\, Y'' - \chi_1\, Y' = 0,$$

where, Y depends on the parameter t. Due to having identical initial conditions

$$\eta^{(i)}(t_0) = B \circ \delta(t_0) = \bar{\gamma}^{(i)}(t_0), \tag{3.28}$$

for $i = 0, \cdots, n$, and to the generalization of the existence and uniqueness theorem of solutions, we have $\eta = \bar{\gamma}$ in a neighborhood of t_0, that can be extended to all $[a, b]$. □

Corollary 3.4 *The number of invariants of the special affine transformations group acting on* $\mathbf{R}^n$ *is* $n-1$.

This coincides with the results provided by other methods (e.g., see [1]).

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