On "All regular Landsberg metrics are always Berwald" by Z. I. Szabó

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Abstract. In his paper [2], Z. I. Szabó claimed (Theorem 3.1) that *all sufficiently smooth Landsberg Finsler metrics are Berwald*; this claim solves the long-standing "unicorn" problem. Unfortunately, as explained below, the proof of the statement has a gap.

M.S.C. 2000: 53B40, 53C60.
Key words: Finsler metric, Landsberg space, Berwald space.

Following [2], let us consider a smooth $n-$dimensional manifold M with a proper Finsler metric $F : TM \to \mathbb{R}$. The second differential of $\frac{1}{2}F^2_{|T_xM}$ will be denoted by $g = g_{(x,y_x)}$ and should be viewed as a Riemannian metric on the punctured tangent space $T_xM - \{0\}$.

For a smooth curve $c(t)$ connecting two points $a, b \in M$, we denote by

$$\tau : T_aM \to T_bM, \quad \tau(a, \underbrace{y_a}_{\in T_aM}) = (b, \underbrace{\phi(y_a)}_{\in T_bM})$$

the Berwald parallel transport along the curve c. Following [1], Z. I. Szabó considers the following Riemannian metric $\boldsymbol{g}$ on M canonically constructed by F by the formula

$$\boldsymbol{g}_{(x)}(\xi, \eta) := \int\limits_{\substack{y_x \in T_xM \\ F(x,y_x) \le 1}} g_{(x,y_x)}(\xi, \eta) d\mu_{(x,y_x)} \tag{1.1}$$

where $\xi, \eta \in T_xM$ are two arbitrary vectors, and the volume form $d\mu$ on T_xM is given by $d\mu_{(x,y_x)} := \sqrt{\det(g_{(x,y_x)})}\, dy_x^1 \wedge \cdots \wedge dy_x^n$.

Z. I. Szabó claims that *if the Finsler metric F is Landsberg, the Berwald parallel transport preserves the Riemannian metric $\boldsymbol{g}$*. According to the definitions in Section 2 of [2], this claim means that for every $\xi, \eta, \nu \in T_aM$

$$\boldsymbol{g}_{(a)}(\xi, \eta) = \boldsymbol{g}_{(b)}(d_\nu \phi(\xi), d_\nu \phi(\eta)). \tag{1.2}$$

This claim is crucial for the proof; the remaining part of the proof is made of relatively simple standard arguments, and is correct. The claim itself is explained

Balkan Journal of Geometry and Its Applications, Vol.14, No.2, 2009, pp. 50-52.

very briefly; basically Z. I. Szabó writes that, for Landsberg metrics, the unit ball $\{y_x \in T_xM \mid F(x, y_x) \le 1\}$, the volume form $d\mu$, and the metric $g_{(x,y_x)}$ are preserved by the parallel transport, and, therefore, the metric $\boldsymbol{g}$ given by (1.1) must be preserved as well.

Indeed, for Landsberg metrics, the unit ball and the volume form $d\mu$ are preserved by the parallel transport. Unfortunately, it seems that the metric g is preserved in a slightly different way one needs to prove the claim.

More precisely, plugging (1.1) in (1.2), we obtain

$$\int_{\substack{y_a\in T_aM\\ F(a,y_a)\le 1}} g_{(a,y_a)}(\xi,\eta)d\mu_{(a,y_a)} = \int_{\substack{y_b\in T_bM\\ F(b,y_b)\le 1}} g_{(b,y_b)}(d_\nu\phi(\xi), d_\nu\phi(\eta))d\mu_{(b,y_b)}. \tag{1.3}$$

As it is explained for example in Section 2 of [2], for every Finsler metric, the parallel transport preserves the unit ball:

$$\phi(\{y_a \in T_aM \mid F(a, y_a) \le 1\}) = \{y_b \in T_bM \mid F(b, y_b) \le 1\}. \tag{1.4}$$

The condition that F is Landsberg implies $\phi_* d\mu_{(a,y_a)} = d\mu_{(b,\phi(y_a))}$. Thus, Szabó's claim is trivially true if at every $y_a \in T_aM$

$$g_{(a,y_a)}(\xi,\eta) = g_{(b,\phi(y_a))}(d_\nu\phi(\xi), d_\nu\phi(\eta)). \tag{1.5}$$

But the condition that the metric is Landsberg means that

$$g_{(a,y_a)}(\xi,\eta) = g_{(b,\phi(y_a))}(d_{y_a}\phi(\xi), d_{y_a}\phi(\eta)) \tag{1.6}$$

only, i.e., (1.5) coincides with the definition of the Landsberg metric at the only point $y_a = \nu \in T_aM$.

Since no explanation why (1.3) holds is given in the paper, I tend to suppose that Z. I. Szabó oversaw the difference between the formulas (1.5) and (1.6); anyway, at the present point, the proof of Theorem 3.1 in [2] is not complete. Unfortunately, I could not get any explanation from Z. I. Szabó by email.

The unicorn problem remains open until somebody closes the gap, or presents another proof, or proves the existence of a counterexample; at the present point I can do neither of these.

Acknowledgement: I thank Deutsche Forschungsgemeinschaft (Priority Program 1154 — Global Differential Geometry) for partial financial support.

Note added in proof: After the paper was submitted, I have known that Z. I. Szabó has written the paper [3], where he in particular accepts that his proof is incomplete.

References

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