L_p-Minkowski and Aleksandrov-Fenchel type inequalities

Chang-jian Zhao and Mihály Bencze

Abstract. In this paper we establish the L_p-Minkowski inequality and L_p-Aleksandrov-Fenchel type inequality for L_p-dual mixed volumes of star duality of mixed intersection bodies, respectively. As applications, we get some related results. The paper new contributions that illustrate this duality of projection and intersection bodies will be presented.

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1 Introduction

The intersection operator and the class of intersection bodies were defined by Lutwak [23]. The closure of the class of intersection bodies was studied by Goody, Lutwak, and Weil [12]. The intersection operator and the class of intersection bodies played a critical role in Gardner [8] and Zhang [32] solution of the famous Busemann-Petty problem in three dimensions and four dimensions, respectively. (See also Gardner, Koldobsky, Schlumprecht [11].)

Just as the period from the mid 60's to the mid 80's was a time of great advances in the understanding of the projection operator and the class of projection bodies, during the past 20 years significant advances have been made in our understanding of the intersection operator and the class of intersection bodies by Koldobsky, Campi, Goodey, Gardner, Lutwak, Grinberg, Fallert, Weil, Zhang, Ludwig and others (see, e.g.,[1]-[7], [9]-[10], [12]-[23], [29], [32]-[34].)

As Lutwak [23] shows (and as is further elaborated in Gardner's book [9]), there is a duality between projection and intersection bodies (that at present is not yet understood). Consider the following illustrative example: It is well known that the projections (onto lower dimensional subspaces) of projection bodies are themselves projection bodies. Lutwak conjectured the "dual": When intersection bodies are intersected with lower dimensional subspaces, the results are intersection bodies (within the lower dimensional subspaces). This was proven by Fallert, Goodey and Weil [4]. In this paper new contributions that illustrate this mysterious duality will be presented.

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In [26] (see also [24]-[25]), Lutwak introduced mixed projection bodies and derived their fundamental inequalities. Following Lutwak, Zhao and Leng [34] established polar forms of Lutwak's mixed projection bodies inequalities. In this work we shall derive, for star duality of intersection bodies, the analogous inequalities for polar mixed projection bodies inequalities.

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures(compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of convex body K. The support function of $K \in \mathcal{K}^n$, $h(K, \cdot)$, defined on $\mathbb{R}^n$ by $h(K, \cdot) = Max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K, u) = Max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let φ^n denote the set of star bodies with 0 in $\mathbb{R}^n$.

2 Background material and main results

2.1 Star duality and polar

In [29], Moszyńska introduced the notion of star duality of star body(See also Moszyńska [30]) as follows.

For the star bodies with 0 in the kernel and positive continuous radial function, such a duality $\circ$ was introduced; it is called the *star duality.*

Let $i : \mathbb{R}^n \setminus 0 \longrightarrow \mathbb{R}^n \setminus 0$ be inversion with respect to S^{n-1}: $i(x) := \frac{x}{\|x\|^2}$.

Definition 2.1 *For every* $K \in \varphi^n$, $K^\circ := \mathrm{cl}(\mathbb{R}^n \setminus i(K))$.

Definition 2.2 *For every* $K \in \varphi^n$,

$$\rho(K^\circ, u) = \frac{1}{\rho(K, u)}. \tag{2.1}$$

If K is a convex body that contains the origin in its interior, the polar body of K, K^*, was defined by $K^* := \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}$.

2.2 L_p-dual mixed volumes

The classical dual mixed volume of star bodies $K_1, \ldots, K_n$ is written as[27] $\tilde{V}(K_1, \ldots, K_n)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volumes is written as $\tilde{V}_i(K, L)$. The dual mixed volumes $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$. If $K_i \in \varphi^n (i = 1, 2, \ldots, n-1)$, then the dual mixed volume of $K_i \cap E_u (i = 1, 2, \ldots, n-1)$ will be denoted by $\tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$. If $K_1 = \ldots = K_{n-1-i} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then $\tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$ is written $\tilde{v}_i(K \cap E_u, L \cap E_u)$. If $L = B$, then $\tilde{v}_i(K \cap E_u, B \cap E_u)$ is written $\tilde{w}_i(K \cap E_u)$.

Let $K, L \in \varphi^n$ and $p \neq 0$, define a star body $K\tilde{+}_pL$ by

$$\rho(K\tilde{+}_pL, u)^p = \rho(K, u)^p + \rho(L, u)^p. \tag{2.2}$$

The operation $\tilde{+}_p$ is called L_p-radial addition. The radial addition $\tilde{+}$ is the special case of the L_p-radial addition.

The L_p-dual volume was defined:

$$\tilde{V}_p(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^p dS(u), \quad -\infty < p < +\infty. \tag{2.3}$$

2.3 Intersection bodies and its star duality

For $K \in \varphi^n$, there is a unique star body IK whose radial function satisfies for $u \in S^{n-1}$,

$$\rho(IK, u) = v(K \cap E_u), \tag{2.4}$$

It is called the *intersection bodies* of K. From a result of Busemann, it follows that IK is a convex if K is convex and centrally symmetric with respect to the origin. Clearly any intersection body is centered. Volume of the intersection bodies is given by $V(IK) = \frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^n dS(u)$.

The mixed intersection bodies of $K_1, \ldots, K_{n-1} \in \varphi^n$, $I(K_1, \ldots, K_{n-1})$, whose radial function is defined by

$$\rho(I(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u), \tag{2.5}$$

where $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume.

If $K \in \varphi^n$ with $\rho(K, u) \in C(S^{n-1})$, and $i \in \mathbb{R}$ is positive, the *intersection body of order i* of K is the centered star body I_iK such that $\rho(I_iK) = \frac{1}{n-1}\int_{S^{n-1}} \rho(K, u)^{n-i-1} dS(u)$, for $u \in S^{n-1}$, where $I_iK = I(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $I(K_1, \ldots, K_{n-1})$ is written as $I_i(K, L)$. If $L = B$, then $I_i(K, L)$ is written as I_iK is called the ith intersection body of K. For I_0K simply write IK.

The star duality of the mixed intersection bodies of $K_1, \ldots, K_{n-1} \in \varphi^n$ will be written as $I^\circ(K_1, \ldots, K_{n-1})$. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $I^\circ(K_1, \ldots, K_{n-1})$ is written as $I_i^\circ(K, L)$. If $L = B$, then $I_i^\circ(K, L)$ is written as $I_i^\circ K$ is called the star duality ith intersection body of K. For $I_0^\circ K$ simply write $I^\circ K$.

The following property will be used later: If $K, L, M, K_1, \ldots, K_{n-1} \in \varphi^n$, and $\lambda, \mu, \lambda_1, \ldots, \lambda_{n-1} > 0$, then

$$I(\lambda K\tilde{+}\mu L, M) = \lambda I(K, M)\tilde{+}\mu I(L, M), \ M = (K_1, \ldots, K_{n-2}). \tag{2.6}$$

In this paper we establish Minkowski-type inequality and Aleksabdrov-Fenchel-type inequality for star duality L_p-dual mixed volumes of mixed intersection bodies, respectively.

3 The Minkowski inequality for L_p-dual mixed volumes of star duality of mixed intersection bodies

If $K, D \in \varphi^n$, then the dual Quermassintegral sum function of star bodies K and D, $S_{\tilde{w}_i}(K, D)$, is defined by[36]

$$S_{\tilde{w}_i}(K, D) = \tilde{W}_i(K) + \tilde{W}_i(D), \quad (0 \leq i \leq n-1).$$

Similarly, L_p-dual volume sum function of star bodies K and D, $S_{\tilde{v}_p}(K, D)$, is denoted as

$$S_{\tilde{v}_p}(K, D) = \tilde{V}_p(K) + \tilde{V}_p(D), \quad (-\infty < p < \infty).$$

Theorem 3.1 *Let $K, L, D, D' \in \varphi^n$. Let D' be a dilates copy of D, and $-\infty < p < 0$.*

(i) *If $0 < j < n-1$, then*

$$S_{\tilde{v}_p}(I_j^\circ(K, L), I_j^\circ(D, D'))^{n-1} \leq S_{\tilde{v}_p}(I^\circ K, I^\circ D)^{n-j-1} S_{\tilde{v}_p}(I^\circ L, I^\circ D')^j, \tag{3.1}$$

with equality if and only if K and L are dilates.

(ii) *If $j > n-1$, then*

$$S_{\tilde{v}_p}(I_j^\circ(K, L), I_j^\circ(D, D'))^{n-1} \geq S_{\tilde{v}_p}(I^\circ K, I^\circ D)^{n-j-1} S_{\tilde{v}_p}(I^\circ L, I^\circ D')^j, \tag{3.2}$$

with equality if and only if K and L are dilates.

The following Lemmas will be required to prove Theorem 3.1.

Lemma 3.1 *If $K, L \in \varphi^n$, $-\infty < p < \infty$, then*

$$\tilde{V}_p(I^\circ K) = \frac{1}{n} \int_{S^{n-1}} v(K \cap u)^{-p} dS(u), \tag{3.3}$$

$$\tilde{V}_p(I_j^\circ K) = \frac{1}{n} \int_{S^{n-1}} \tilde{w}_j(K \cap u)^{-p} dS(u), \tag{3.4}$$

$$\tilde{V}_p(I_j^\circ(K, L)) = \frac{1}{n} \int_{S^{n-1}} \tilde{v}_j(K \cap u, L \cap u)^{-p} dS(u). \tag{3.5}$$

Proof From (2.1), (2.3) and (2.5), we obtain that

$$\begin{aligned} \tilde{V}_p(I_j^\circ(K, L)) &= \frac{1}{n} \int_{S^{n-1}} \rho(I_j^\circ(K, L), u)^p dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \rho(I_j(K, L), u)^{-p} dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \tilde{v}_j(K \cap u, L \cap u)^{-p} dS(u). \end{aligned}$$

The proof of the identity (3.5) is complete.

Taking for $K = L$ in (3.5), (3.5) changes to (3.3). Taking for $L = B$ in (3.5), (3.5) changes to (3.4).

Lutwak, Yang and Zhang [28] introduced a elementary inequality as follows.

Lemma 3.2 *If* $a, b \geq 0$ *and* $c, d > 0$, *then for* $p > 1$

$$(a+b)^p(c+d)^{1-p} \leq a^p c^{1-p} + b^p d^{1-p}, \tag{3.6}$$

with equality if and only if $ad = bc$.

In fact, the following reverse of inequality (3.6) easy follows:

Lemma 3.3 *If* $a, b, c, d > 0$, *then for* $0 < p < 1$

$$(a+b)^p(c+d)^{1-p} \geq a^p c^{1-p} + b^p d^{1-p}, \tag{3.7}$$

with equality if and only if $ad = bc$.

Lemma 3.4 ([37]). *If If* $K, L \in \varphi^n$ *and* $i < n-1$, *then*

$$\tilde{W}_i(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i-1}\tilde{W}_i(L), \tag{3.8}$$

with equality if and only if K *is a dilation of* L.

The inequality is reverse for $i > n$ *or* $(n-1) < i < n$.

Proof of Theorem 3.1 We first give the proof of the case $j > n-1$ as follows.

In view of the reverse of inequality (3.8), we obtain that

$$\tilde{v}_j(K \cap E_u, L \cap E_u)^{-p} \geq v(K \cap E_u)^{\frac{-p(n-j-1)}{n-1}} v(L \cap E_u)^{\frac{-jp}{n-1}}. \tag{3.9}$$

with equality if and only if $K \cap E_u$ and $L \cap E_u$ are dilates, it follows if and only if K and L are dilates.

From Lemma 3.1, (3.9) and in view of reverse Hölder inequality for integral, we have

$$\begin{aligned}
\tilde{V}_p(I_j^\circ(K, L)) &= \frac{1}{n}\int_{S^{n-1}} \tilde{v}_j\left(K \cap E_u, L \cap E_u)\right)^{-p} dS(u) \\
&\geq \frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^{\frac{-p(n-j-1)}{n-1}} v(L \cap E_u)^{\frac{-jp}{n-1}} dS(u) \\
&\geq \left(\frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^{-p} dS(u)\right)^{\frac{(n-j-1)}{n-1}} \left(\frac{1}{n}\int_{S^{n-1}} v(L \cap E_u)^{-p} dS(u)\right)^{\frac{j}{n-1}} \\
&= \tilde{V}_p(I^\circ K)^{\frac{(n-j-1)}{n-1}} \tilde{V}_p(I^\circ L)^{\frac{j}{n-1}}. \qquad (3.10)
\end{aligned}$$

In view of the equality conditions of (3.10) and Hölder inequality for integral, it follows that the equality holds if and only if K and L are dilates.

Therefore, from the inequality (3.10) and in view of D' is a dilates copy of D , we obtain that

$$\tilde{V}_p(I_j^\circ(K, L))^{n-1} \geq \tilde{V}_p(I^\circ K)^{n-j-1}\tilde{V}_p(I^\circ L)^j,$$

and

$$\tilde{V}_p(I_j^\circ(D, D'))^{n-1} = \tilde{V}_p(I^\circ D)^{n-j-1}\tilde{V}_p(I^\circ D')^j.$$

Hence, from Lemma 3.2, we have

$$S_{\tilde{v}_p}(I_j^\circ(K, L), I_j^\circ(D, D'))$$

$$\geq \tilde{V}_p(I^\circ K)^{(n-j-1)/(n-1)}\tilde{V}_p(I^\circ L)^{j/(n-1)}+\tilde{V}_p(I^\circ D)^{(n-j-1)/(n-1)}\tilde{V}_p(I^\circ D')^{j/(n-1)}$$

$$\geq S_{\tilde{v}_p}(I^\circ K, I^\circ D)^{\frac{n-j-1}{n-1}} S_{\tilde{v}_p}(I^\circ L, I^\circ D')^{\frac{j}{n-1}},$$

with equality if and only if K is a dilation of L.

Similarly, from Lemma 3.1, inequalities (3.7), (3.8) and in view of Hölder inequality for integral, the proof of the case $0 < j < (n-1)$ can be completed by the same steps as in the proof of the case $j > (n-1)$ with suitable changes. Here, we omit the details.

The proof of Theorem 3.1 is complete.

Remark 3.1 In Theorem 3.1, let D, D' are single points, Theorem 3.1 changes to L_p-Minkowski-type inequality for star dual of mixed intersection bodies.

Let D, D' are single points and taking for $p = -(n-i)$(where, $0 \leq i < n$) in Theorem 3.1., and in view of

$$\tilde{V}_{-(n-i)}(I_j^\circ(K,L)) = \tilde{W}_i(I_j(K,L)),\ 0 \leq i < n,\ K, L \in \varphi^n,$$

then Theorem 3.1 changes to the following result.

Corollary 3.1 *Let $K, L \in \varphi^n$ and $0 \leq i < n$.*

(i) *If $0 < j < n-1$, then*

$$\tilde{W}_i(I_j(K,L))^{n-1} \leq \tilde{W}_i(IK)^{n-j-1}\tilde{W}_i(IL)^j, \tag{3.11}$$

with equality if and only if K and L are dilates.

(ii) *If $j > n-1$, then*

$$\tilde{W}_i(I_j(K,L))^{n-1} \geq \tilde{W}_i(IK)^{n-j-1}\tilde{W}_i(IL)^j, \tag{3.12}$$

with equality if and only if K and L are dilates.

Remark 3.2 The inequality (3.11) is just *the Minkowski inequality for mixed intersection bodies* which was given in [36].

A somewhat surprising consequence of Theorem 3.1 is the following version.

Corollary 3.2 *If $K, L \in \eta \subset \varphi^n$, and $-\infty < p < 0$, while $j \in \mathbb{R}$, and if either*

$$\tilde{V}_p(I_j^\circ(K,M)) = \tilde{V}_p(I_j^\circ(L,M)),\ for\ M \in \eta \tag{3.13}$$

or

$$\tilde{V}_p(I_j^\circ(M,K)) = \tilde{V}_p(I_j^\circ(M,L)),\ for\ M \in \eta \tag{3.14}$$

hold, then it follows that $K = L$,up to translation.

This is just is the dual form of the following result which was given in [35].

If $K, L \in \gamma \subset \mathcal{K}^n$, and $0 \leq i < n$, while $0 < j < n-1$ and if either

$$W_i(\Pi_j^*(K,M)) = W_i(\Pi_j^*(L,M)),\ for\ M \in \gamma$$

or

$$W_i(\Pi_j^*(M,K)) = W_i(\Pi_j^*(M,L)),\ for\ M \in \gamma$$

hold, then it follows that $K = L$,up to translation.

Proof Suppose that (3.13) holds. Take K for M in (3.13) and use Theorem 3.1, we obtain that

$$\tilde{V}_p(I^\circ K) = \tilde{V}_p(I_j^\circ(L,K)) \geq (\leq)\tilde{V}_p(I^\circ L)^{\frac{(n-j-1)}{n-1}}\tilde{V}_p(I^\circ K)^{\frac{j}{n-1}}.$$

with equality if and only if K is a dilation of L. Hence

$$\tilde{V}_p(I^\circ K) \geq (\leq)\tilde{V}_p(I^\circ L).$$

with equality if and only if K is a dilation of L.

Similarly, take L for M in (3.13) and use again Theorem 3.1, we get

$$\tilde{V}_p(I^\circ K) \leq (\geq)\tilde{V}_p(I^\circ L),$$

with equality if and only if K is a dilation of L. Hence

$$\tilde{V}_p(I^\circ K) = \tilde{V}_p(I^\circ L).$$

and K is a dilation of L, in view of intersection bodies are centered, then there exist $\lambda > 0$ such that $K = \lambda L$, for $0 \leq i < n-1$, therefore $\lambda = 1$.

Exactly the same sort of argument shows that condition (3.14) implies that K and L must be translates.

4 The Aleksandrov-Fenchel inequality for L_p-dual mixed volumes of star duality of mixed intersection bodies

The following Lemmas will be required to prove Theorem 4.1.

Lemma 4.1 ([27]). *If $K_1, \ldots, K_n \in \varphi^n$, then*

$$\tilde{V}(K_1,\ldots,K_n)^r \leq \prod_{j=1}^{r} \tilde{V}(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_n),$$

with equality if and only if $K_1, \ldots, K_n$ are all dilations of each other.

Lemma 4.2 *If $K_1, \ldots, K_n \in \varphi^n$, then*

$$\tilde{V}_p(I^\circ(K_1,\ldots,K_{n-1})) = \frac{1}{n}\int_{S^{n-1}} \tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)^{-p} dS(u).$$

From (2.5) and (2.3), it follows.

Lemma 4.3 *If $f_i \in C(S^{n-1})$ and $f_i > 0 (i = 1, 2, \ldots, m)$, then*

$$\int_{S^{n-1}} f_1(u)\cdots f_m(u)dS(u) \leq \prod_{i=1}^{m} \|f_i(u)\|_m,$$

with equality if and only if all f_i are proportional.

The following Aleksandrov-Fenchel inequality for L_p-dual mixed volumes of star duality of mixed intersection bodies will be proved:

Theorem 4.1 *Let $K_1, \ldots, K_{n-1} \in \varphi^n$, $-\infty < p < 0$, and $1 < r \leq n-1$, then*

$$\tilde{V}_p(I^\circ(K_1,\ldots,K_{n-1}))^r \leq \prod_{j=1}^{r} \tilde{V}_p(I^\circ(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_{n-1})), \tag{4.1}$$

with equality if and only if $K_1, \dots, K_{n-1}$ *are all dilations of each other.*

Proof From Lemma 4.1, we obtain that

(4.2)
$$\tilde{v}(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)^p \geq \left(\prod_{j=1}^{r} \tilde{v}(\underbrace{K_j \cap E_u, \dots, K_j \cap E_u}_{r}, K_{r+1} \cap E_u, \dots, K_{n-1} \cap E_u) \right)^{\frac{p}{r}},$$

with equality if and only if $K_1 \cap E_u, \dots, K_{n-1} \cap E_u$ are all dilations of each other, it follows if and only if $K_1, \dots, K_{n-1}$ are all dilations of each other. From (4.2) and in view of Lemma 4.2 and Lemma 4.3, we obtain that

$$\tilde{V}_p(I^\circ(K_1, \dots, K_{n-1})) = \frac{1}{n} \int_{S^(n-1)} \tilde{v}(K_1 \cap E_u, \dots, K_{n-1} \cap E_u)^{-p} dS(u)$$

$$\leq \frac{1}{n} \int_{S^u} \left(\prod_{j=1}^{r} \tilde{v}(\underbrace{K_j \cap E_u, \dots, K_j \cap E_u}_{r}, K_{r+1} \cap E_u, \dots, K_{n-1} \cap E_u) \right)^{\frac{-p}{r}} dS(u)$$

$$\leq \left(\prod_{j=1}^{r} \frac{1}{n} \int_{S^{n-1}} \tilde{v}(\underbrace{K_j \cap E_u, \dots, K_j \cap E_u}_{r}, K_{r+1} \cap E_u, \dots, K_{n-1} \cap E_u)^{-p} dS(u) \right)^{\frac{1}{r}}$$

$$= \left(\prod_{j=1}^{r} \tilde{V}_p(I^\circ\, (\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_{n-1})) \right)^{\frac{1}{r}}.$$

In view of the equality conditions of (4.2) and inequality in Lemma 4.3, it follows that the equality holds if and only if $K_1, \dots, K_{n-1}$ are all dilations of each other. □

Taking for $p = -(n-i)$, $i < n$ in (4.1), (4.1) changes to the following result.

Corollary 4.1 *If* $K_1, \dots, K_{n-1} \in \varphi^n$, $0 \leq i < n, 0 < j < n-1$ *and* $0 < r \leq n-1$ *then*

(4.3)
$$\tilde{W}_i(I(K_1, \dots, K_{n-1}))^r \leq \prod_{j=1}^{r} \tilde{W}_i(I(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_{n-1})),$$

with equality if and only if $K_1, \dots, K_{n-1}$ *are all dilations of each other.*

This is just a dual form of the following inequality which was given by Lutwak [24].

The Aleksandrove-Fenchel inequality for mixed projection bodies. If $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, $0 \leq i < n, 1 < j < n-1$ and $0 < r \leq n-1$, then

$$W_i(\Pi(K_1, \dots, K_{n-1}))^r \geq \prod_{j=1}^{r} W_i(\Pi(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_{n-1})).$$

Corollary 4.2 *If* $K_1, \dots, K_{n-1} \in \varphi^n$ *and* $-\infty < p < 0$, *then*

$$\tilde{V}_p(I^\circ(K_1, K_2, K_3, \dots, K_{n-1}))^2 \leq \tilde{V}_p(I^\circ(K_1, K_1, K_3, \dots, K_{n-1}))\tilde{V}_p(I^\circ(K_2, K_2, K_3, \dots, K_{n-1})).$$

This is just a dual form of the following inequality which was given by Lutwak [26].
If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then

$$V(\Pi^*(K_1, K_2, K_3, \ldots, K_{n-1}))^2 \leq V(\Pi^*(K_1, K_1, K_3, \ldots, K_{n-1}))V(\Pi^*(K_2, K_2, K_3, \ldots, K_{n-1})).$$

From the cases $r = n-1$ of inequality (4.1), it follows

Corollary 4.3 *If* $K_1, \ldots, K_{n-1} \in \varphi^n$ *and* $-\infty < p < 0$, *then*

$$\tilde{V}_p(I^\circ(K_1, \ldots, K_{n-1}))^{n-1} \leq \tilde{V}_p(I^\circ K_1) \cdots \tilde{V}_p(I^\circ K_{n-1}), \tag{4.4}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ *are all dilations of each other.*

This is just a dual form of the following inequality which was given by Lutwak [26].

If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then

$$V(\Pi^*(K_1, \ldots, K_{n-1}))^{n-1} \leq V(\Pi^* K_1) \cdots V(\Pi^* K_{n-1}),$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are homothetic of each other. Please see references [31] and [21] about similar recent results.

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Authors' addresses:

Zhao Chang-jian
Department of Information and Mathematics Sciences,
College of Science, China Jiliang University, Hangzhou 310018, P.R.China.
E-mail: chjzhao@yahoo.com.cn chjzhao@163.com

Mihály Bencze (Corresponding author)
Str. Hărmanului 6, 505600 Săcele-Négyfalu, Jud. Braşov, Romania.
E-mail: benczemihaly@yahoo.com aedit@metanet.ro