Higher order Grassmann fibrations and the calculus of variations

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. Geometric structure of global integral variational functionals on higher order tangent bundles and Grassmann fibrations are investigated. The theory of Lepage forms is extended to these structures. The concept of a Lepage form allows us to introduce the Euler-Lagrange distribution for variational functionals, depending on velocities, in a similar way as in the calculus of variations on fibred manifolds. Integral curves of this distribution include all extremal curves of the underlying variational functional. The generators of the Euler-Lagrange distribution, defined by the Lepage forms of the first order, are found explicitly.

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1 Introduction

Our main objective in this paper is the geometric structure of the variational theory on higher order velocity spaces and Grassmann fibrations. We consider global integral variational functionals for *curves* and 1-*dimensional submanifolds* of a given manifold. A specific feature of this theory is the absence of the concept of a *Lagrangian*, a basic element of the classical calculus of variations and the variational theory on fibred manifolds; in this paper the role of a Lagrangian is played by a differential 1-form on a velocity manifold, called a *Lepage form*.

We introduce Lepage forms, and derive a geometric (coordinate-free) first variation formula on the *higher order velocity spaces*. Our definition extends properties of the Cartan and Lepage forms, used in classical mechanics and the global variational theory on fibred manifolds. Main tools we use are properties of forms on the manifolds of velocities, and independence of the variational integral on parametrization. These notions are naturally characterized via the theory of jets and higher order contact elements. We also describe the *Euler-Lagrange distribution*, related with a Lepage form, whose integral curves include all extremals of the variational integral. In particular, we derive chart expressions for a Lepage form as well as the generators of

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the Euler-Lagrange destribution on the *first order* Grassmann fibration in terms of *adapted coordinates*. The reader can easily understand that the presented theory can be extended to variational functionals for submanifolds of dimension greater than 1.

Throughout this paper, R is the field of real numbers. We denote by Y a fixed smooth manifold of dimension $m+1$. T^rY is the manifold of velocities of order r over Y (r-jets $J_0^r\zeta$ with source $0 \in$ R and target $\zeta(0) \in T^rY$,) and $\tau^{r,s} : T^rY \to T^sY$ are the canonical jet projections. ImmT^rY denotes an open submanifold of regular velocities in T^rY (r-jets of immersions), and L^r is the differential group of order r of R , consisting of regular r-jets $J_0^r\alpha \in$ ImmT^rR such $\alpha(0) = 0$, with the group multiplication the composition of jets. G^rY is the Grassmann fibration of order r over Y (the quotient manifold ImmT^rY/L^r); $\rho^{r,s} : G^rY \to G^sY$ denotes the canonical projection of Grassmann fibrations. For simplicity, we restrict our coordinate considerations to lower order cases $r = 1, 2$.

For the general theory of jets and contact elements the reader is referred to the papers [3, 5, 6]. The theorems, presented in this paper, namely the structure theory of Lepage forms and a new description of extremals in terms of a distribution, are an extension of the variational calculus on fibred manifolds as explained in [1, 2, 4, 7].

2 Velocities and Grassmann fibrations

Given a chart (V, ψ), $\psi = (y^K)$, on a manifold Y of dimension $m+1$, the associated charts on the manifolds of regular velocities ImmT^1Y and ImmT^2Y are denoted by (V^1, ψ^1), $\psi^1 = (y^K, \dot{y}^K)$, and (V^2, ψ^2), $\psi^2 = (y^K, \dot{y}^K, \ddot{y}^K)$. Each index L defines a partition of the index set, $\{L\}$, $\{1, 2, \cdots, L-1, L+1, \cdots, m, m+1\}$, and the *subordinate charts* on ImmT^1Y and ImmT^2Y, denoted by $(V^{1,L}, \psi^{1,L})$, $\psi^{1,L} = (y^L, \dot{y}^L, y^\sigma, \dot{y}^\sigma)$, and $(V^{2,L}, \psi^{2,L})$, $\psi^{2,L} = (y^L, \dot{y}^L, \ddot{y}^L, y^\sigma, \dot{y}^\sigma, \ddot{y}^\sigma)$; recall that the sets $V^{1,L}$ and $V^{2,L}$ are defined by

$$\dot{w}^L \neq 0. \tag{2.1}$$

The corresponding *second subordinate charts* are denoted by $(V^{1,L}, \chi^{1,L})$, $\chi^{1,L} = (w^L, \dot{w}^L, w^\sigma, w_1^\sigma)$, and $(V^{2,L}, \chi^{2,L})$, $\chi^{2,L} = (w^L, \dot{w}^L, \ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma)$. The second subordinate charts can be introduced by the transformation equations.

Lemma 1. *The transformation equations between the charts* $(V^{2,L}, \psi^{2,L})$ *and* $(V^{2,L}, \chi^{2,L})$ *are*

$$\begin{aligned} &y^L = w^L,\ \dot{y}^L = \dot{w}^L,\ \ddot{y}^L = \ddot{w}^L \\ &y^\sigma = w^\sigma,\ \dot{y}^\sigma = w_1^\sigma \dot{w}^L,\ \ddot{y}^\sigma = w_2^\sigma (\dot{w}^L)^2 + w_1^\sigma \ddot{w}^L. \end{aligned} \tag{2.2}$$

Let L^r be the *differential group* of order r of the real line R; in the context of this work, L^r describes the change of parameter in variational functionals for curves in Y. L^r acts canonically on T^rY to the right by composition of jets,

$$T^rY \times L^r \ni (J_0^r\zeta, J_0^r\alpha) \to J_0^r\zeta \circ J_0^r\alpha = J_0^r(\zeta \circ \alpha) \in T^rY. \tag{2.3}$$

Clearly, this group action restricts to the submanifold of *regular* velocities ImmT^rY. Recall that the *canonical coordinates* $a_1, a_2, \cdots, a_r$ on L^r are the functions on L^r defined by $a_k(J_0^r\alpha) = D^k\alpha(0)$.

Lemma 2. (a) *The group multiplication* $(J_0^2\alpha, J_0^2\beta) \to J_0^2(\alpha\circ\beta)$ *in the group* L^2 *is expressed by the equations*

$$\begin{aligned} &a_1(J_0^r\alpha\circ J_0^r\beta) = a_1(J_0^r\alpha)\cdot a_1(J_0^r\beta),\\ &a_2(J_0^r\alpha\circ J_0^r\beta) = a_2(J_0^r\alpha)\cdot (a_1(J_0^r\beta))^2 + a_1(J_0^r\alpha)\cdot a_2(J_0^r\beta). \end{aligned} \tag{2.4}$$

(b) *The group action (2.3), restricted to the set* $ImmT^2Y$, *is expressed in a subordinate chart* $(V^{2,L},\chi^{2,L})$, $\chi^{2,L} = (w^L, \dot w^L, \ddot w^L, w^\sigma, w_1^\sigma, w_2^\sigma)$, *by the equations*

$$\begin{aligned} &w^L(J_0^2\zeta\circ J_0^2\alpha) = w^L(J_0^2\zeta),\\ &\dot w^L(J_0^2\zeta\circ J_0^2\alpha) = \dot w^L(J_0^2\zeta)a_1(J_0^2\alpha),\\ &\ddot w^L(J_0^2\zeta\circ J_0^2\alpha) = \dot w^L(J_0^2\zeta)a_2(J_0^2\alpha) + \ddot w^L(J_0^2\zeta)a_1(J_0^2\alpha)^2,\\ &w^\sigma(J_0^2\zeta\circ J_0^2\alpha) = w^\sigma(J_0^2\zeta),\\ &w_1^\sigma(J_0^2\zeta\circ J_0^2\alpha) = w_1^\sigma(J_0^2\zeta),\\ &w_2^\sigma(J_0^2\zeta\circ J_0^2\alpha) = w_2^\sigma(J_0^2\zeta). \end{aligned} \tag{2.5}$$

Lemma 3. *Suppose we have two charts on* $Y, (V,\psi), \psi = (y^K)$, *and* $(V,\bar\psi), \bar\psi = (\bar y^K)$ *such that* $V\cap\bar V \neq \varnothing$, *and the transformation equations*

$$\bar y^M = f^M(y^L, y^\sigma),\ \bar y^\nu = f^\nu(y^L, y^\sigma). \tag{2.6}$$

Then the transformation equations between the subordinate charts $(V^{1,L},\chi^{1,L})$, $\chi^{1,L} = (w^L, \dot w^L, w^\sigma, w_1^\sigma)$ *and* $(\bar V^{1,M},\bar\chi^{1,M})$, $\bar\chi^{1,M} = (\bar w^M, \dot{\bar w}^M, \bar w^\sigma, \bar w_1^\sigma)$, *are*

$$\begin{aligned} &\bar w^M = f^M(w^L, w^\sigma),\ \bar w^\nu = f^\nu(w^L, w^\sigma),\ \dot{\bar w}^M = \frac{\partial f^M}{\partial w^L}\dot w^L + \frac{\partial f^M}{\partial w^\sigma}\dot w^L w_1^\sigma,\\ &\bar w_1^\nu = \frac{1}{\dfrac{\partial f^L}{\partial w^L} + \dfrac{\partial f^L}{\partial w^\tau}w_1^\tau}\left(\frac{\partial f^\nu}{\partial w^L} + \frac{\partial f^\nu}{\partial w^\sigma}w_1^\sigma\right). \end{aligned} \tag{2.7}$$

Let G^rY be the Grassmann fibration of order r over Y. We denote by $[J_0^r\zeta]$ the L^r-*orbit* of a regular velocity $J_0^r\zeta$, and by $\rho^{r,s} : G^rY \to G^sY$ the canonical projection. For every index L the pair $(\tilde V^{2,L},\tilde\chi^{2,L})$, where $\tilde V^{2,L} = (\rho^{2,0})^{-1}(V), \tilde\chi^{2,L} = (\tilde w^L, \tilde w^\sigma, \tilde w_1^\sigma, \tilde w_2^\sigma)$, and for all $J_0^2\zeta \in V^{2,L}$

$$\begin{aligned} &\tilde w^L([J_0^2\zeta]) = w^L(J_0^2\zeta),\\ &\tilde w^\sigma([J_0^2\zeta]) = w^\sigma(J_0^2\zeta),\ \tilde w_1^\sigma([J_0^2\zeta]) = w_1^\sigma(J_0^2\zeta),\ \tilde w_2^\sigma([J_0^2\zeta]) = w_2^\sigma(J_0^2\zeta), \end{aligned} \tag{2.8}$$

is a chart on G^3Y.

Let I be an open interval, containing $0 \in$ R, and let $\gamma : I \to Y$ be a curve. Denote by tr_{t_0} the translation $t \to t - t_0$ of R. γ defines the *r-jet prolongation*

$$I \ni t \to (T^r\gamma)(t) = J_0^r(\gamma\circ\mathrm{tr}_{-t}) \in T^rY. \tag{2.9}$$

Further on, we suppose that γ is an *immersion* such that for a chart on Y, $\gamma(I) \subset V$ and $T^r\gamma(I) \subset V^{r,L}$ for some L.

Lemma 4. *The 2-jet prolongation $T^2\gamma$ is expressed by*

$$
\begin{aligned}
&w^L \circ T^2\gamma = w^L\gamma,\ \dot{w}^L \circ T^2\gamma = D(w^L\gamma),\ \ddot{w}^L \circ T^2\gamma = D^2(w^L\gamma),\\
&w^\sigma \circ T^2\gamma = w^\sigma\gamma,\ w_1^\sigma \circ T^2\gamma = \frac{D(w^\sigma\gamma)}{D(w^L\gamma)},\\
&w_2^\sigma \circ T^2\gamma = \frac{1}{(D(w^L\gamma))^2}\left(D^2(w^\sigma\gamma) - \frac{D^2(w^L\gamma)}{D(w^L\gamma)}D(w^\sigma\gamma)\right).
\end{aligned}
\tag{2.10}
$$

The r-jet prolongation of an immersion $\gamma : I \to Y$ defines a curve in G^rY,

$$I \ni t \to G^r\gamma(t) = [T^r\gamma(t)] \in G^rY, \tag{2.11}$$

called the *Grassmann prolongation* of γ of order r.

We examine the behavior of the mapping $T^2\gamma$ under *reparametrizations*. Let J be an open interval, containing the origin 0, and let $\mu : J \to I$ be a diffeomorphism. μ is defined by an equation

$$t = \mu(s). \tag{2.12}$$

Setting for every $s \in J$

$$\mu_s(t) = \mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s}(t) = -\mu(s) + \mu(s+t), \tag{2.13}$$

we get another diffeomorphism $\mu_s : J_s \to I_s$ of open intervals, containing 0. Since $D\mu_s(t) = D\mu(s+t)$ and $D^2\mu_s(t) = D^2\mu(s+t)$, μ_s satisfies

$$\mu_s(0) = 0,\ D\mu_s(0) = D\mu(s),\ D^2\mu_s(0) = D^2\mu(s). \tag{2.14}$$

In particular, the 2-jet $J_0^2\mu_s$ is an element of the differential group L^2 for all s, whose canonical coordinates are $a_1(s) = D\mu(s), a_2(s) = D^2\mu(s)$. μ induces a differentiable mapping $s \to J_0^2\mu_s$ of the domain J of μ into L^2; μ also induces a diffeomorphism

$$\mathrm{Imm}T^2Y \ni J_0^2\zeta \to J_0^2\zeta \circ J_0^2\mu_s \in \mathrm{Imm}T^2Y, \tag{2.15}$$

defined by the canonical action of L^2 on $\mathrm{Imm}T_0^2Y$. The mappings $s \to J_0^2\mu_s$ and $J_0^2\zeta \to J_0^2\zeta \circ J_0^2\mu_s$ are said to be *associated* with μ.

A diffeomorphism $\mu : J \to I$ assigns to the immersion γ an immersion $\gamma \circ \mu : J \to Y$, and its 2-jet prolongation $T^2(\gamma \circ \mu)$.

Lemma 5. (a) *The mapping $s \to T^2(\gamma \circ \mu)(s)$ satisfies*

$$T^2(\gamma \circ \mu)(s) = T^2\gamma(\mu(s)) \circ J_0^2\mu_s. \tag{2.16}$$

(b) *The mapping $s \to T^2(\gamma \circ \mu)(s)$ is expressed in the chart $(V^{2,L}, \chi^{2,L})$, $\chi^{2,L} = (w^L, w_1^L, w_2^L, w^\sigma, w_1^\sigma, w_2^\sigma)$ by*

$$
\begin{aligned}
&w^L(T^2(\gamma \circ \mu)(s)) = w^L(T^2\gamma(\mu(s))),\\
&w_1^L(T^2(\gamma \circ \mu)(s)) = w_1^L(T^2\gamma(\mu(s)))a_1(J_0^2\mu_s),\\
&w_2^L(T^2(\gamma \circ \mu)(s)) = w_1^L(T^2\gamma(\mu(s)))a_2(J_0^2\mu_s) + w_2^L(T^2\gamma(\mu(s)))a_1(J_0^2\mu_s)^2,\\
&w^\sigma(T^2(\gamma \circ \mu)(s)) = w^\sigma(T^2\gamma(\mu(s))),\\
&w_1^\sigma(T^2(\gamma \circ \mu)(s)) = w_1^\sigma(T^2\gamma(\mu(s))),\\
&w_2^\sigma(T^2(\gamma \circ \mu)(s)) = w_2^\sigma(T^2\gamma(\mu(s))).
\end{aligned}
\tag{2.17}
$$

There exists a bijection between the set of diffeomorphisms $\alpha : (-a, a) \to \mathrm{R}$ such that $\alpha(0) = 0$, and the set of diffeomorphisms $\mu : (-a + t_0, a + t_0) \to \mathrm{R}$ such that $\mu(t_0) = 0$. Given μ, we denote by t_μ the centre of the domain of μ, and set

$$\alpha = \mu \circ \mathrm{tr}_{-t_\mu}. \tag{2.18}$$

Then we have $\alpha^{-1} = \mathrm{tr}_{t_\mu} \circ \mu^{-1}$, and $\mu^{-1} = \mathrm{tr}_{-t_\mu} \circ \alpha^{-1}$. Using this correspondence, we have $\mu_s = \mathrm{tr}_{\mu(s)}\mu\mathrm{tr}_{-s} = \mathrm{tr}_{\mu(s)}\alpha\mathrm{tr}_{t_\mu}\mathrm{tr}_{-s}$ hence, since the 2-jet of a translation is equal to the identity element $J_0^2\mathrm{id}_\mathrm{R}$ of the group L^2,

$$T^2(\gamma \circ \mu)(s) = T^2\gamma(\alpha(\mathrm{tr}_{t_\mu}(s))) \circ J_0^2\alpha_s. \tag{2.19}$$

Let α be an isomorphism of Y. For any curve γ in Y, defined on an open interval $I \subset R$, $\alpha \circ \gamma$ is a curve in Y, defined on I, with values in Y. Let $P \in T^rY$, $P = J_0^r\zeta$. Setting $T^r\alpha(J_0^r\zeta) = J_0^r(\alpha\zeta)$, we get an r-jet, depending on P only. The mapping $T^rY \ni P \to T^r\alpha(P) \in T^rY$ is a diffeomorphism, called the *r-jet prolongation* of α. Clearly, $\tau^{r,s} \circ T^r\alpha = T^s\alpha \circ \tau^{r,s}$ for all $s = 0, 1, 2, \cdots, r$. This construction can immediately be modified for vector fields by means of flows; we denote by $T^r\xi$ the *r-jet prolongation* of a vector field ξ on Y.

Lemma 6. *Let ξ be a vector field on Y, and let*

$$\xi = \xi^K \frac{\partial}{\partial y^K} \tag{2.20}$$

be the chart expression for ξ in a chart (V, ψ), $\psi = (y^K)$. Then in a subordinate chart $(V^{2,L}, \chi^{2,L})$, $\chi^{2,L} = (w^L, \dot{w}^L, \ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma)$,

$$T^2\xi = \xi^L \frac{\partial}{\partial w^L} + \dot{\xi}^L \frac{\partial}{\partial \dot{w}^L} + \ddot{\xi}^L \frac{\partial}{\partial \ddot{w}^L} + \xi^\nu \frac{\partial}{\partial w^\nu} + \xi_1^\nu \frac{\partial}{\partial w_1^\nu} + \xi_2^\nu \frac{\partial}{\partial w_2^\nu}, \tag{2.21}$$

where

$$\begin{aligned}
\dot{\xi}^L &= \left(\frac{\partial \xi^L}{\partial w^L} + \frac{\partial \xi^L}{\partial w^\sigma} w_1^\sigma\right) \dot{w}^L, \\
\ddot{\xi}^L &= \left(\frac{\partial^2 \xi^L}{\partial (w^L)^2} + \frac{\partial^2 \xi^L}{\partial w^L \partial w^\sigma} w_1^\sigma + \frac{\partial \xi^L}{\partial w^\nu} w_2^\nu\right) (\dot{w}^L)^2 \\
&+ \left(\frac{\partial^2 \xi^L}{\partial w^\nu \partial w^L} + \frac{\partial^2 \xi^L}{\partial w^\nu \partial w^\sigma} w_1^\sigma\right) w_1^\nu (\dot{w}^L)^2 + \left(\frac{\partial \xi^L}{\partial w^L} + \frac{\partial \xi^L}{\partial w^\sigma} w_1^\sigma\right) \ddot{w}^L, \\
\xi_1^\nu &= \frac{\partial \xi^\nu}{\partial w^L} + \frac{\partial \xi^\nu}{\partial w^\sigma} w_1^\sigma, \\
\xi_2^\nu &= \frac{\partial^2 \xi^\nu}{\partial (w^L)^2} + 2\frac{\partial^2 \xi^\nu}{\partial w^L \partial w^\sigma} w_1^\sigma + \frac{\partial^2 \xi^\nu}{\partial w^\lambda \partial w^\sigma} w_1^\lambda w_1^\sigma + \frac{\partial \xi^\nu}{\partial w^\sigma} w_2^\sigma.
\end{aligned} \tag{2.22}$$

3 The calculus of variations on velocity manifolds

In this section we introduce basic geometric concepts of the calculus of variations for the *first order* variational functionals on velocity spaces; higher order theory can be developed along the same lines.

Choose a velocity $P \in T^3Y$ and a representative ζ of P; then $P = J^3_0\zeta$. ζ defines a mapping $T^2\zeta$ of a neighbourhood of $0 \in \mathrm{R}$ into T^2Y and the tangent mapping at 0, $T_0T^2\zeta : T_0\mathrm{R} \to T_{J^2_0\zeta}T^2Y$, sending a vector $\xi \in T_0\mathrm{R}$ to a vector of T^2Y at the point $T^2\zeta(0) = J^2_0\zeta = \tau^{3,2}(J^3_0\zeta)$. Express ξ is in the canonical basis of the 1-dimensional vector space $T_0\mathrm{R}$ as $\xi = \xi_0 \cdot (d/dt)_0$ and define a vector field δ *along the projection* $\tau^{3,2}$ by

$$\delta(J^3_0\zeta) = T_0T^2\zeta \cdot \left(\frac{d}{dt}\right)_0 . \tag{3.1}$$

The vector field δ induces a mapping $\eta \to h\eta$, defined on differential 1-forms on T^2Y, with values in the module of functions on T^3Y by

$$h\eta(J^3_0\zeta) = \eta(J^2_0\zeta)\left(\delta(J^3_0\zeta)\right). \tag{3.2}$$

In particular, if f is a function on an open set $W \subset T^2Y$, then the formula $\delta(f) = h(df)$ defines a function $\delta(f)$ on the set $(\tau^{3,1})^{-1}(W) \subset T^3Y$.

Lemma 7. *Let η be a 1-form, let $(V, \psi), \psi = (y^K)$, be a chart on Y, and let γ be a curve with values in V.*

(a) *δ has a chart expression*

$$\delta = \dot{y}^K \frac{\partial}{\partial y^K} + \ddot{y}^K \frac{\partial}{\partial \dot{y}^K}. \tag{3.3}$$

(b) *If η is expressed as $\eta = A_k dy^K + B_K d\dot{y}^K$, then*

$$h\eta = A_K\dot{y}^K + B_K\ddot{y}^K. \tag{3.4}$$

We call δ the *formal derivative morphism*; the function $\delta(f)$ is called the *formal derivative* of f. The mapping h is called the *horizontalization*.

Remark 1. From the definitions (3.1) and (3.2) we easily derive the formulas

$$hdw^L = \dot{w}^L, \quad hd\dot{w}^L = \ddot{w}^L, \quad hdw^\sigma = \dot{w}^L w^\sigma_1, \quad hdw^\sigma_1 = \dot{w}^L w^\sigma_2. \tag{3.5}$$

Let W be an open set in Y, and suppose we have a 1-form η, defined on the set $(\tau^{r,0})^{-1}(W) \subset \mathrm{Imm}T^rY$. We say that η is *contact*, if $T^r\zeta^*\eta = 0$ for all immersions ζ, defined on an open interval in R, with values in W. The ideal of the exterior algebra of differential forms on the set $(\tau^{r,0})^{-1}(W)$, locally generated by contact 1-forms, is called the *contact ideal*, and is denoted by Ω^1_cW. By a *contact k-form* we mean any k-form, belonging to the contact ideal.

Lemma 8. *Let W be an open set in Y, let η be a 1-form on $(\tau^{2,0})^{-1}(W)$, and let $(V, \psi), \psi = (y^K)$, be an chart on Y such that $V \subset W$. Then the following conditions are equivalent:*

(a) *η is a contact form.*

(b) *For every subordinate chart $(V^{2,L}, \psi^{2,L}), \psi^{2,L} = (y^L, \dot{y}^L, \ddot{y}^L, y^\sigma, \dot{y}^\sigma, \ddot{y}^\sigma)$,*

$$\eta = \dot{A}_L\dot{\eta}^L + A_\sigma\eta^\sigma + \dot{A}_\sigma\dot{\eta}^\sigma, \tag{3.6}$$

where

$$\dot{\eta}^L = d\dot{y}^L - \frac{\ddot{y}^L}{\dot{y}^L} dy^L, \ \eta^\sigma = dy^\sigma - \frac{\dot{y}^\sigma}{\dot{y}^L} dy^L, \ \dot{\eta}^\sigma = d\dot{y}^\sigma - \frac{\ddot{y}^\sigma}{\dot{y}^L} dy^L. \tag{3.7}$$

(c) *For every subordinate chart* $(V^{2,L}, \chi^{2,L})$, $\chi^{2,L} = (w^L, w_1^L, w_2^L, w^\sigma, w_1^\sigma, w_2^\sigma)$,

$$\eta = B_L^1 \omega_1^L + B_\sigma^0 \omega_0^\sigma + B_\sigma^1 \omega_1^\sigma, \tag{3.8}$$

where

$$\omega_1^L = dw_1^L - \frac{w_2^L}{w_1^L} dw^L, \ \omega_0^\sigma = dw^\sigma - w_1^\sigma dw^L, \ \omega_1^\sigma = dw_1^\sigma - w_2^\sigma dw^L. \tag{3.9}$$

(d) η *belongs to the kernel of the horizontalization* h, *i.e.*, $h\eta = 0$.

Clearly, the forms (3.7), and the forms (3.8) are linearly independent.

Suppose we have a 1-form η, defined on $\mathrm{Imm}T^1Y$. Let I be an open interval, and let $\gamma : I \to Y$ be an immersion. Any *compact* subinterval K of I defines the *variational integral*, associated with η,

$$\eta_K(\gamma) = \int_K (T^1\gamma)^* \eta. \tag{3.10}$$

The mapping η_K is the *integral variational functional*, associated with the η.

The function $h\eta$ is the *Lagrange function*, associated with η. The following gives us a description of variational functionals in terms of Lagrange functions.

Lemma 9. *Let* η *be a* 1*-form on* T^1Y, *and let* $\gamma : I \to Y$ *be an immersion, defined on an open interval* $I \subset \mathrm{R}$. *Then*

$$T^r\gamma^*\eta = (h\eta \circ T^{r+1}\gamma) \cdot dt. \tag{3.11}$$

Lemma 9 says that the Lagrange function L_η, associated with η, is given by

$$L_\eta(J_0^{r+1}\zeta) = h\eta \circ T^{r+1}(\zeta \circ \mathrm{tr}_t)(t). \tag{3.12}$$

Let $C_K^2 Y$ denote the set of curves in Y of class C^2, defined on a compact interval $K \subset \mathrm{R}$. We have for every isomorphism α of Y and every curve $\gamma \in C_K^2 Y$

$$\eta_K(\alpha\gamma) = \int_K (T^1(\alpha\gamma))^* \eta. \tag{3.13}$$

But by definitions, $T^1(\alpha\gamma) = T^1\alpha \circ T^1\gamma$, so (3.13) reduces to

$$\eta_K(\alpha\gamma) = \int_K (T^1\alpha \circ T^1\gamma)^* \eta = \int_K T^1\gamma^* T^1\alpha^* \eta. \tag{3.14}$$

Consequently, the variational functional $C_K^2 Y \ni \gamma \to \eta_K(\alpha\gamma) \in \mathrm{R}$ (3.13) satisfies

$$\eta_K(\alpha\gamma) = (T^1\alpha^*\eta)_K(\gamma), \tag{3.15}$$

and coincides with the variational functional, associated with the form $T^1\alpha^*\eta$.

This property of variational functionals can be transfered to vector fields. Let ξ be a vector field on Y and α_s^ξ its flow. Then for all sufficiently small s, $\eta_K(\alpha_s^\xi \gamma) = ((T^1\alpha_s^\xi)^* \eta)_K(\gamma)$. Differentiating, we have

$$\left(\frac{d\eta_K(\alpha_s^\xi \gamma)}{ds} \right)_0 = \int_K T^1\gamma^* \partial_{T^1\xi}\eta, \tag{3.16}$$

where $\partial_{T^1\xi}\eta$ is the *Lie derivative* of the form η by the vector field $T^1\xi$, the 1-jet prolongation of ξ. The mapping

$$C_K^2 Y \ni \gamma \to (\partial_{T^1\xi}\eta)_K(\gamma) = \int_K (T^1\gamma)^* \partial_{T^1\xi}\eta \in \mathrm{R} \tag{3.17}$$

is called the *first variation* of the variational functional η_K by the vector field ξ.

Note that the Lie derivative $\partial_{T^1\xi}\eta$ under the integral sign in (3.17) can be decomposed as $\partial_{T^1\xi}\eta = i_{T^1\xi} d\eta + d i_{T^1\xi}\eta$. The form η is said to be a *Lepage form*, if the 2-form $d\eta$ belongs to the contact ideal $\Omega_c^1 Y$.

Theorem 1. (The structure of Lepage forms) *The following conditions are equivalent:*

(a) *η is a Lepage form on $ImmT^1Y$.*

(b) *η has in any subordinate chart $(V^{1,L}, \psi^{1,L})$ an expression*

$$\eta = P_L dy^L + \frac{\partial P_L}{\partial \dot{y}^\sigma} \dot{y}^L \eta^\sigma + dF_L, \tag{3.18}$$

where F_L and P_L are function on $V^{1,L}$, and P_L satisfies

$$\frac{\partial P_L}{\partial \dot{y}^L} \dot{y}^L + \frac{\partial P_L}{\partial \dot{y}^\sigma} \dot{y}^\sigma = 0. \tag{3.19}$$

(c) *η has in a subordinate chart $(V^{1,L}, \chi^{1,L})$ an expression*

$$\eta = P_L dw^L + \frac{\partial P_L}{\partial w_1^\sigma} \omega^\sigma + dF, \tag{3.20}$$

where P_L and F are functions on $V^{1,L}$ such that

$$\frac{\partial P_L}{\partial w_1^L} = 0. \tag{3.21}$$

To demonstrate basic ideas of the proof, we show that (b) follows from (a). Consider the chart expression

$$\eta = A_L dy^L + A_\sigma \eta^\sigma + B_L d\dot{y}^L + B_\sigma d\dot{y}^\sigma, \tag{3.22}$$

where

$$\eta^\sigma = dy^\sigma - \frac{\dot{y}^\sigma}{\dot{y}^L} dy^L \tag{3.23}$$

(Lemma 8). Now

$$(3.24)\quad d\eta = \left(\frac{\partial A_L}{\partial y^\tau}dy^\tau + \frac{\partial A_L}{\partial \dot{y}^L}d\dot{y}^L + \frac{\partial A_L}{\partial \dot{y}^\tau}d\dot{y}^\tau\right) \wedge dy^L + dA_\sigma \wedge \eta^\sigma + A_\sigma d\eta^\sigma$$
$$+ \left(\frac{\partial B_L}{\partial y^L}dy^L + \frac{\partial B_L}{\partial y^\tau}dy^\tau\right) \wedge d\dot{y}^L + \left(\frac{\partial B_\sigma}{\partial y^L}dy^L + \frac{\partial B_\sigma}{\partial y^\tau}dy^\tau\right) \wedge d\dot{y}^\sigma$$
$$(3.25)\quad + \frac{\partial B_L}{\partial \dot{y}^\tau}d\dot{y}^\tau \wedge d\dot{y}^L + \left(\frac{\partial B_\sigma}{\partial \dot{y}^L}d\dot{y}^L + \frac{\partial B_\sigma}{\partial \dot{y}^\tau}d\dot{y}^\tau\right) \wedge d\dot{y}^\sigma.$$

The conditions that $d\eta$ be generated by the contact forms η^σ imply

$$(3.26)\quad \frac{\partial B_L}{\partial \dot{y}^\sigma} - \frac{\partial B_\sigma}{\partial \dot{y}^L} = 0,\ \frac{\partial B_L}{\partial \dot{y}^\tau} - \frac{\partial B_\tau}{\partial \dot{y}^\sigma} = 0.$$

Integrating these conditions we get

$$(3.27)\quad B_L = \frac{\partial F}{\partial \dot{y}^L},\ B_\sigma = \frac{\partial F}{\partial \dot{y}^\sigma}$$

for a function F. Then, however,

$$(3.28)\quad dF = \frac{\partial F}{\partial y^L}dy^L + \frac{\partial F}{\partial y^\sigma}dy^\sigma + B_L d\dot{y}^L + B_\sigma d\dot{y}^\sigma,$$

so we get $\eta = \tilde{A}_L dy^L + \tilde{A}_\sigma \eta^\sigma + dF$, where

$$(3.29)\quad \tilde{A}_L = A_L - \frac{\partial F}{\partial y^L} - \frac{\partial F}{\partial y^\sigma}\frac{\dot{y}^\sigma}{\dot{y}^L},\ \tilde{A}_\sigma = A_\sigma - \frac{\partial F}{\partial y^\sigma}.$$

From this expression we have

$$(3.30)\quad d\eta = \frac{\partial \tilde{A}_L}{\partial y^\tau}\eta^\tau \wedge dy^L + d\tilde{A}_\sigma \wedge \eta^\sigma + \frac{\partial \tilde{A}_L}{\partial \dot{y}^L}d\dot{y}^L \wedge dy^L + \left(\tilde{A}_\sigma - \frac{\partial \tilde{A}_L}{\partial \dot{y}^\sigma}\right) d\eta^\sigma.$$

But $d\eta$ is generated by the contact forms η^σ, from which we get (b).

Suppose we have a Lepage form η on the manifold of velocities $\mathrm{Imm}T^1Y$. We wish to describe a distribution Δ_η on $\mathrm{Imm}T^1Y$, defined by differential 1-forms $i_\Xi d\eta$, where Ξ runs through vector fields on $\mathrm{Imm}T^1Y$; Δ_η is the *Euler-Lagrange distribution* associated with η.

We give an explicit characterization of the Euler-Lagrange distribution of a Lepage form η in terms of the second subordinate charts. We know that

$$(3.31)\quad \eta = \mathcal{L}_L dw^L + \frac{\partial \mathcal{L}_L}{\partial w_1^\sigma}\omega^\sigma + d\mathcal{F}_L$$

in a subordinate chart $(V^{1,L}, \chi^{1,L})$, $\chi^{1,L} = (w^L, w_1^L, w^\sigma, w_1^\sigma)$, where $\mathcal{F}_L$ and $\mathcal{L}_L$ are functions on $V^{1,L}$ such that $\mathcal{L}_L = \mathcal{L}_L(w^L, w^\sigma, w_1^\sigma)$ (Theorem 1, (c)).

Theorem 2. (The Euler-Lagrange distribution) *In a second subordinate chart, the Euler-Lagrange distribution Δ_η is generated by the 1-forms*

$$(3.32)\quad \frac{\partial^2 \mathcal{L}_L}{\partial w_1^\nu \partial w_1^\sigma}\omega^\sigma,\quad \left(-\frac{\partial \mathcal{L}_L}{\partial w^\sigma} + \frac{\partial^2 \mathcal{L}_L}{\partial w^L \partial w_1^\sigma} + \frac{\partial^2 \mathcal{L}_L}{\partial w^\nu \partial w_1^\sigma}w_1^\nu\right)\omega^\sigma,\quad \frac{\partial^2 \mathcal{L}_L}{\partial w_1^\nu \partial w_1^\sigma}dw_1^\nu +$$
$$+ \left(\frac{\partial^2 \mathcal{L}_L}{\partial w^\sigma \partial w_1^\nu} + \frac{\partial^2 \mathcal{L}_L}{\partial w^\nu \partial w_1^\sigma}\right)\omega^\nu + \left(-\frac{\partial \mathcal{L}_L}{\partial w^\sigma} + \frac{\partial^2 \mathcal{L}_L}{\partial w^L \partial w_1^\sigma} + \frac{\partial^2 \mathcal{L}_L}{\partial w^\nu \partial w_1^\sigma}w_1^\nu\right)dw^L.$$

To prove the theorem, we contract the form $d\eta$ with a vector field

$$\Xi = \Xi^L \frac{\partial}{\partial w^L} + \Xi^\nu \frac{\partial}{\partial w^\nu} + \Xi_1^L \frac{\partial}{\partial w_1^L} + \Xi_1^\nu \frac{\partial}{\partial w_1^\nu}, \tag{3.33}$$

and obtain the generators of the distribution by calculating the coefficients in the chart expression for the form $i_\Xi d\eta$ at the components Ξ^L, Ξ^σ, and Ξ_1^ν.

Theorem 2 describes important special cases. In particular, if the matrix

$$\frac{\partial^2 \mathcal{L}_L}{\partial w_1^\nu \partial w_1^\sigma} \tag{3.34}$$

is non-singular, then each integral curve of the Euler-Lagrange distribution is *holonomic*, i.e., is the prolongation of a curve in the manifold Y. In this case the generators of the Euler-Lagrange distribution reduce to

$$\omega^\sigma, \ \left(-\frac{\partial \mathcal{L}_L}{\partial w^\sigma} + \frac{\partial^2 \mathcal{L}_L}{\partial w^L \partial w_1^\sigma} + \frac{\partial^2 \mathcal{L}_L}{\partial w^\nu \partial w_1^\sigma} w_1^\nu \right) dw^L + \frac{\partial^2 \mathcal{L}_L}{\partial w_1^\nu \partial w_1^\sigma} dw_1^\nu. \tag{3.35}$$

4 Parameter invariance

Suppose we have a 1-form η on $\mathrm{Imm}T^1Y$. Consider the variational integral (3.10). We are interested in the case when the number $\eta_K(\gamma)$ is independent of parametrization of the set $\gamma(I)$. This is characterized by the following theorems.

Theorem 3. *Let η be a 1-form on T^1Y, let $\gamma : I \to Y$ be an immersion, J an open interval, and $\mu : J \to I$ a diffeomorphism. The following conditions are equivalent:*

(a) *For any two compact intervals $L \subset J$ and $K \subset I$ such that $\mu(L) = K$,*

$$\eta_K(\gamma) = \eta_L(\gamma \circ \mu). \tag{4.1}$$

(b) *η satisfies*

$$(T^1\gamma)^*\eta = (\mu^{-1})^* T^1(\gamma \circ \mu)^*\eta. \tag{4.2}$$

Condition (4.2) is called the *invariance condition*; we say that η and γ *satisfy the invariance condition*, if (4.2) holds for all diffeomorphisms μ. We say that η is *parameter-invariant*, if (4.2) holds for all γ and μ.

Consider the variational functional (3.10). The form $(T^1\gamma)^*\eta$ has at every point $t_0 \in I$ an expression

$$(T^1\gamma)^*\eta(t_0) = \mathcal{L} \circ T^2\gamma(t_0) \cdot dt(t_0). \tag{4.3}$$

We can now give a version of the invariance condition in terms of the Lagrange function $\mathcal{L}$.

Lemma 10. *Let the immersion γ and the diffeomorphism μ be given. Then the following two conditions are equivalent:*

(1) *η and γ satisfy the invariance condition* (4.2).

(2) *For all* $s \in K$,

$$\mathcal{L}\left(J_0^2(\gamma \circ \mathrm{tr}_{-\mu(s)})\right) \cdot D\mu_s(0) = \mathcal{L}\left(J_0^2(\gamma \circ \mathrm{tr}_{-\mu(s)}) \circ J_0^2\mu_s\right). \tag{4.4}$$

The following is a criterion of invariance of the form η under changes of parametrization; the criterion says that the Lagrange function $\mathcal{L}$, associated with η, should be L^2-*equivariant.*

Theorem 4. *η satisfies the invariance condition if and only if*

$$\mathcal{L}\left(J_0^2\gamma\right) \cdot D\alpha(0) = \mathcal{L}\left(J_0^2\gamma \circ J_0^2\alpha\right) \tag{4.5}$$

for all $J_0^2\alpha \in L^2$.

Remark 2. According to Lemma 2, the group action of L^2 on $\mathrm{Imm}T^2Y$ is in a chart $(V,\psi), \psi = (y^K)$, given by the equations $\bar{y}^K = y^K, \dot{\bar{y}}^K = a\dot{y}^K, \ddot{\bar{y}}^K = a\ddot{y}^K + b\dot{y}^K$, where a and b are the canonical coordinates on L^2. Hence condition (3.32) can be expressed as $\mathcal{L}(y^K, a\dot{y}^K, a\ddot{y}^K + b\dot{y}^K) = a\mathcal{L}(y^K, \dot{y}^K, \ddot{y}^K)$.

Remark 3. Condition (4.5) can also be expressed in a subordinate chart $(V^{2,L}, \chi^{2,L})$, $\chi^{2,L} = (w^L, \dot{w}^L, \ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma)$. Since the group action of L^2 in this chart is given by the equations $\bar{w}^L = w^L, \dot{\bar{w}}^L = a\dot{w}^L, \ddot{\bar{w}}^L = b\dot{w}^L + a^2\ddot{w}^L, \bar{w}^\sigma = w^\sigma, \bar{w}_1^\sigma = w_1^\sigma, \bar{w}_2^\sigma = w_2^\sigma$, where a and b are the canonical coordinates on L^2, we have $\mathcal{L}(w^L, a\dot{w}^L, b\dot{w}^L + a^2\ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma) = a\mathcal{L}(w^L, \dot{w}^L, \ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma)$.

We are now in a position to give a complete description of Lepage forms on $\mathrm{Imm}T^1Y$ that satisfy the invariance condition.

Theorem 5. *Let η be a 1-form on $\mathrm{Imm}T^1Y$. The following two conditions are equivalent:*

(a) *η is a Lepage form, and satisfies the invariance condition.*

(b) *In every subordinate chart $(V^{1,L}, \chi^{1,L})$, $\chi^{1,L} = (w^L, \dot{w}^L, w^\sigma, w_1^\sigma)$ has an expression*

$$\eta = P_L dw^L + \frac{\partial P_L}{\partial w_1^\sigma}\omega^\sigma + dF_L, \tag{4.6}$$

where P_L and F_L are functions on the set $V^{1,L}$ such that

$$\frac{\partial P_L}{\partial \dot{w}^L} = 0, \ \frac{\partial F_L}{\partial \dot{w}^L} = 0. \tag{4.7}$$

Express the Lepage form η as in Theorem 1, by

$$\eta = P_L dw^L + \frac{\partial P_L}{\partial w_1^\sigma}\omega^\sigma + dF, \tag{4.8}$$

and compute the corresponding Lagrange function $\mathcal{L} = h\eta$. We have, using the formulas $hdw^L = \dot{w}^L, hd\dot{w}^L = \ddot{w}^L, hdw^\sigma = \dot{w}^L w_1^\sigma$, and $hdw_1^\sigma = \dot{w}^L w_2^\sigma$ (2.10),

$$\mathcal{L} = \left(P_L + \frac{\partial F}{\partial w^L}\right)\dot{w}^L + \frac{\partial F}{\partial \dot{w}^L}\ddot{w}^L + \frac{\partial F}{\partial w^\sigma}\dot{w}^L w_1^\sigma + \frac{\partial F}{\partial w_1^\sigma}\dot{w}^L w_2^\sigma. \tag{4.9}$$

But $\mathcal{L}(w^L, a\dot{w}^L, b\dot{w}^L + a^2\ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma) = a\mathcal{L}(w^L, \dot{w}^L, \ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma)$ for all $a, b \in$ R, $a \neq 0$, since η satisfies the invariance condition (Theorem 3), and $P_L = P_L(w^L, w^\sigma, w_1^\sigma)$ since η is Lepage; then (a) implies $\partial F/\partial \dot{w}^L = 0$.

Remark 4. From Theorem 5 we conclude that Theorem 2 remains valid for the Euler-Lagrange distribution of parameter-invariant variational problems.

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