Second order differential invariants of linear frames

Ján Brajerčík

Abstract. The aim of this paper is to characterize all second order tensor-valued and scalar differential invariants of the bundle of linear frames FX over an n-dimensional manifold X. These differential invariants are obtained by factorization method and are described in terms of bases of invariants. Second order natural Lagrangians of frames have been characterized explicitly; if $n = 1, 2, 3, 4$, the number of functionally independent second order natural Lagrangians is $N = 0, 6, 33, 104$, respectively.

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1 Introduction

Throughout this paper, a *left G-manifold* is a smooth manifold endowed with a left action of a Lie group G. A mapping between two left G-manifolds transforming G-orbits into G-orbits is said to be *G-equivariant*. As usual, we denote by $\mathbb{R}$ the field of real numbers. The r-th differential group L^r_n of $\mathbb{R}^n$ is the Lie group of invertible r-jets with source and target at the origin $0 \in \mathbb{R}^n$; the group multiplication in L^r_n is defined by the composition of jets. Note that $L^1_n = GL_n(\mathbb{R})$. For generalities on spaces of jets and their mappings, differential groups, their actions, etc., we refer to [6, 11, 13].

Let P and Q be two left L^r_n-manifolds, and U be an open, L^r_n-invariant set in P. A smooth L^r_n-equivariant mapping $F : U \to Q$ is called a *differential invariant*. If Q is the real line $\mathbb{R}$, endowed with the trivial action of L^r_n, an equivariant mapping F is called a *scalar invariant*.

Let X be an n-dimensional manifold. By an *r-frame* at a point $x \in X$ we mean an invertible r-jet with source $0 \in \mathbb{R}^n$ and target x. The set of r-frames together with its natural structure of a principal L^r_n-bundle with base X is denoted by F^rX, and is called the *bundle of r-frames* over X. For $r = 1$, we get the *bundle of linear frames*, and write $F^1X = FX$. If S is a left L^r_n-manifold, then the bundle with type fibre S, *associated* with F^rX is denoted by F^r_SX.

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If S is a left L^1_n-manifold, we denote by $T^r_n S$ the manifold of r-jets with source $0 \in \mathbb{R}^n$ and target in S. For finding differential invariants of frames it is convenient to realize FX as a bundle with type fibre L^1_n, associated with itself. Then, the r-jet prolongation $J^r FX$ of FX can be considered as a fibre bundle with type fibre $T^r_n L^1_n$, associated with $F^{r+1}X$.

For characterizing *natural* Lagrangians on $J^r FX$, i.e. Lagrangians invariant with respect to all *diffeomorphisms of* X, it is sufficient to describe all differential invariants defined on the type fiber $P = T^r_n L^1_n$ of $J^r FX$. The aim of this paper is to give explicit characterization of second order natural Lagrangians.

Most of differential invariants with values in Q appearing in differential geometry correspond with the case when Q is an L^1_n-manifold. These differential invariants can be described as follows. Let $K^{r,s}_n$ be the *kernel* of the canonical group morphism $\pi^{r,s}_n : L^r_n \to L^s_n$, where $r \geq s$. If L^r_n acts on Q via its subgroup L^1_n, each *continuous*, L^r_n-equivariant mapping $F : U \to Q$ has the form $F = f \circ \pi$, where $\pi : P \to P/K^{r,1}_n$ is the quotient projection, $P/K^{r,1}_n$ is the space of $K^{r,1}_n$-orbits, and $f : P/K^{r,1}_n \to Q$ is a continuous, L^1_n-equivariant mapping. Indeed, in this scheme $P/K^{r,1}_n$ is considered with the quotient topology, but is not necessarily a smooth manifold. The quotient projection π is continuous but not necessarily smooth. If $P/K^{r,1}_n$ has a smooth structure such that π is a submersion, we call π the basis of differential invariants on P (for more details of a basis, see [12]). The general concepts on equivariant mappings, related with a normal subgroup of a Lie group, and corresponding assertions with the proofs can be found in [4, 8].

A method based on this observation was first time applied to the problem of finding differential invariants of a linear connection in [8]. The initial problem was reduced to a more simple problem of the classical invariant theory (see e.g. [14, 15]) to describe all L^1_n-equivariant mappings from $P/K^{r,1}_n$ to L^1_n-manifolds. Our aim in this paper is to study invariants of linear frames by the same method, which allows us to simplify expressions of the action of L^r_n on P.

In this paper, we first introduce the domain of second order differential invariants with values in L^1_n-manifolds, which is, according to the prolongation theory of manifolds endowed with a Lie group action (see e.g. [6, 7, 11]), the L^3_n-manifold $P = T^2_n L^1_n$. Then we describe the *frame* action of L^3_n on $T^2_n L^1_n$. This action corresponds to the second jet prolongation of the frame lift of a diffeomorphism of X. Using a tensor decomposition, we also construct the corresponding orbit space of the normal subgroup $K^{3,1}_n$ of L^3_n. We show that this orbit space can be identified with Cartesian products of L^1_n with some tensor spaces over $\mathbb{R}^n$; in this way the differential invariants with values in L^1_n-manifolds are described in terms of their basis. Note that the second order differential invariants with values in L^2_n-manifolds can be easily obtained by the same manner.

These results are subsequently used for extension of the theory of the first order differential invariants of frames in [5], studied in terms of integrals of canonical differential system, to the second order case. Applying factorization method, we give an explicit characterization of second order *scalar invariants* of frames and *Lagrangians*, defined on $J^2 FX$, *invariant* with respect to all diffeomorphisms on X. In Section 8 we also introduce the concept of *canonical odd n-form* on FX, where $n = \dim X$, which gives an exact description of globally defined invariant object in the role of volume form. In [5], (ordinary) n-form was considered, which is only available over orientable manifolds. We also extend the remarks of the authors on the use of differential invariants as Lagrangians over *non-orientable* manifolds. For the theory of odd de Rham forms and odd base forms we refer to [9].

The calculations in this paper rely on the jet description of differential invariants. This is based on the existence of a Lie group (the differential group) whose invariants are exactly the differential invariants. Such description implies, in particular, that the arising theory is comparatively simpler than other versions of the theory of differential invariants.

Using another left action of L^1_n on itself, called *coframe* action, it is possible to obtain the corresponding *differential invariants of coframes*. Note that they can be obtained by the same method. For differential invariants of coframes, see [3]; it represents an extension of Ph.D. thesis of the first author, devoted to the second order case, to the third order case.

There are several types of invariance of Lagrangians on frame bundles. One of them is invariance with respect to the canonical action of L^1_n on J^rFX. All L^1_n-invariant Lagrangians on J^rFX are explicitly described in [2].

2 Jet prolongations of L^1_n manifolds

In this section, the general prolongation theory of left G-manifolds is applied to the case of the Lie group $G = L^1_n = GL_n(\mathbb{R})$. We use the prolongation formula derived in [7], and the terminology and notation of the book [11].

Recall that the r-th differential group L^r_n of $\mathbb{R}^n$ is the group of invertible r-jets with source and target at the origin $0 \in \mathbb{R}^n$. The group multiplication in L^r_n is defined by the composition of jets. Let $J^r_0\alpha \in L^r_n$, where $\alpha = (\alpha^i)$ is a diffeomorphism of a neighborhood U of the origin $0 \in \mathbb{R}^n$ into $\mathbb{R}^n$ such that $\alpha(0) = 0$. The *first canonical coordinates* $a^i_{j_1}$, $a^i_{j_1 j_2}$, $\dots$, $a^i_{j_1 j_2 \dots j_r}$, where $1 \leq i \leq n$, $1 \leq j_1 \leq j_2 \leq \dots \leq j_r \leq n$, on L^r_n are defined by

$$a^i_{j_1 j_2 \dots j_k}(J^r_0\alpha) = D_{j_1} D_{j_2} \dots D_{j_k} \alpha^i(0), \quad 1 \leq k \leq r. \tag{2.1}$$

We also define the *second canonical coordinates* $b^i_{j_1}$, $b^i_{j_1 j_2}$, $\dots$, $b^i_{j_1 j_2 \dots j_r}$, on L^r_n by

$$b^i_{j_1 j_2 \dots j_k}(J^r_0\alpha) = a^i_{j_1 j_2 \dots j_k}(J^r_0\alpha^{-1}), \quad 1 \leq k \leq r.$$

Indeed, $a^i_j b^j_k = \delta^i_k$ (the Kronecker symbol).

Let us consider a left L^1_n-manifold S, and denote by $T^r_n S$ the manifold of r-jets with source $0 \in \mathbb{R}^n$ and target in S. According to the general theory of prolongations of left G-manifolds, $T^r_n S$ has a (canonical) structure of a left L^{r+1}_n-manifold. To define this structure, denote by t_x the translation of $\mathbb{R}^n$ defined by $t_x(y) = y - x$. Consider elements $q \in T^r_n S$, $q = J^r_0\gamma$, and $a \in L^{r+1}_n$, $a = J^{r+1}_0\alpha$. Denoting $\bar{\alpha}_x = t_x \circ \alpha \circ t_{-\alpha^{-1}(x)}$, and $\bar{\alpha}(x) = J^1_0\bar{\alpha}_x$ we get an element of the group L^1_n. Then formula

$$a \cdot q = J^r_0(\bar{\alpha} \cdot (\gamma \circ \alpha^{-1})) \tag{2.2}$$

defines a left action of the differential group L^{r+1}_n on $T^r_n S$. We usually call formula (2.2) the *prolongation formula* for the action of the group L^1_n on S. The left L^{r+1}_n-manifold $T^r_n S$ is called the *r-jet prolongation* of the left L^1_n-manifold S.

3 Frames

Let X be an n-dimensional manifold. Recall that an *r-frame* at a point $x \in X$ is an invertible r-jet with source $0 \in \mathbb{R}^n$ an target at x. The set of r-frames, denoted by $F^r X$, will be considered with its natural structure of a principal L^r_n-bundle over X. We write $FX = F^1 X$; FX is the bundle of *linear* frames.

For computing differential invariants of frame bundles it is important to realize FX as a fibre bundle with type fibre L^1_n, *associated* with itself. Thus, the structure group of FX is the group $L^1_n = GL_n(\mathbb{R})$, with canonical coordinates (a^i_j), defined by (2.1). If (p^i_j) are the canonical coordinates on the type fibre L^1_n of fibre bundle FX, then the *left action* of the structure group L^1_n of FX on the type fibre L^1_n is represented by the group multiplication $L^1_n \times L^1_n \ni (J^1_0\alpha, J^1_0\eta) \to J^1_0(\alpha \circ \eta) \in L^1_n$. In the canonical coordinates, $p^i_j(J^1_0(\alpha \circ \eta)) = a^i_k(J^1_0\alpha)p^k_j(J^1_0\eta)$, which can be written simply by

$$\bar{p}^i_j = a^i_k p^k_j. \tag{3.1}$$

(3.1) is called the *frame action* of L^1_n on itself.

$J^r FX$ denotes the r-jet prolongation of FX. It follows from the general theory of jet prolongations of fibre bundles that $J^r FX$ can be considered as a fibre bundle over X with type fibre $T^r_n L^1_n$, associated with $F^{r+1}X$. Equations of the group action of L^{r+1}_n on $T^r_n L^1_n$ can be obtained from (2.2) and (3.1).

4 The second jet prolongation of the frame action

Now we derive an explicit expression for the action (2.2) of the group L^3_n on $T^2_n L^1_n$, associated with (3.1).

Let U be a neighborhood of the origin $0 \in \mathbb{R}^n$. Let α be a diffeomorphism of U onto $\alpha(U) \subset \mathbb{R}^n$ such that $\alpha(0) = 0$. Then $\bar{\alpha}(x) = J^1_0\bar{\alpha}_x$, where $\bar{\alpha}_x = t_x \circ \alpha \circ t_{-\alpha^{-1}(x)}$. Let $\gamma : U \to L^1_n$ be a mapping. For every $x \in \alpha(U)$ we denote $\psi(x) = \bar{\alpha}(x) \cdot \gamma(\alpha^{-1}(x))$, and the dot on the right hand side means the multiplication in the group L^1_n. In coordinates,

$$p^i_j(\psi(x)) = p^i_j(\bar{\alpha}(x) \cdot \gamma(\alpha^{-1}(x))) = a^i_s(\bar{\alpha}(x))p^s_j(\gamma(\alpha^{-1}(x))). \tag{4.1}$$

Note that in this formula,

$$a^i_s(\bar{\alpha}(x)) = D_s\alpha^i(\alpha^{-1}(x)). \tag{4.2}$$

Now the chart expression of the frame action is obtained by expressing the r-jet $J^r_0\psi = J^{r+1}_0\alpha \cdot J^r_0\gamma$ (2.2) in coordinates. Consider the case $r = 2$.

Lemma 1. *The group action of L^3_n on $T^2_n L^1_n$ induced by the frame action of L^1_n on L^1_n is defined by the equations*

$$\begin{aligned}
&\bar{p}^i_j = p^s_j a^i_s, \\
&\bar{p}^i_{j,k} = p^s_{j,t} a^i_s b^t_k + p^s_j a^i_{st} b^t_k, \\
&\bar{p}^i_{j,kl} = p^s_{j,tu} a^i_s b^u_l b^t_k + p^s_{j,t}(a^i_{su} b^u_k b^t_l + a^i_{su} b^u_l b^t_k + a^i_s b^t_{kl}) \\
&\qquad + p^s_j(a^i_{stu} b^u_l b^t_k + a^i_{st} b^t_{kl}).
\end{aligned} \tag{4.3}$$

Proof. First equation is obtained from (4.1) and (4.2), to get the remaining equations, we differentiate (4.1) twice, and then put $x = 0$. □

5 Differential invariants with values in L^1_n-manifolds

In this part we are interested in differential invariants $F : T^2_n L^1_n \to Q$, where Q is arbitrary left L^1_n-manifold. We define the homomorphism

$$\pi^{3,1}_n : L^3_n \to L^1_n, \qquad \pi^{3,1}_n(a^i_j, a^i_{jk}, a^i_{jkl}) = (a^i_j).$$

Notice that each differential invariant F with values in L^1_n-manifold satisfies

$$F(a \cdot q) = \pi^{3,1}_n(a) \cdot F(q),$$

for each $a \in L^3_n$, $q \in T^2_n L^1_n$.

Let $K^{3,1}_n$ denote the kernel of the canonical homomorphism $\pi^{3,1}_n$; $K^{3,1}_n$ is normal subgroup of L^3_n represented by elements in coordinates written as $(\delta^i_j, a^i_{jk}, a^i_{jkl})$. Now we restrict the action (4.3) to the subgroup $K^{3,1}_n$ of L^3_n. The following result is fundamental for the discussion of the corresponding orbit spaces.

Lemma 2. *The group action of $K^{3,1}_n$ on $T^2_n L^1_n$ induced by the frame action of L^1_n on L^1_n is defined by the equations*

$$\begin{aligned}
&\bar{p}^i_j = p^i_j,\\
(5.1)\qquad &\bar{p}^i_{j,k} = p^i_{j,k} + p^s_j a^i_{sk},\\
&\bar{p}^i_{j,kl} = p^i_{j,kl} + p^s_{j,l} a^i_{sk} + p^s_{j,k} a^i_{sl} + (p^i_{j,t} + p^s_j a^i_{st}) b^t_{kl} + p^s_j a^i_{skl}.
\end{aligned}$$

Proof. We take $a^i_j = b^i_j = \delta^i_j$ in (4.3). □

Corollary 1. *The action* (5.1) *is free.*

Now we describe *orbits* of the group actions (5.1). Let us introduce some notation. Using the second canonical coordinates on $T^2_n L^1_n$, we denote by q^i_j the inverse matrix of the matrix p^i_j; thus, $q^i_j : T^2_n L^1_n \to \mathbb{R}$ are functions such that $q^i_s p^s_j = \delta^i_j$.

We also use the special notation for *symmetrization* and *antisymmetrization* of indexed families of functions through selected indices. Symmetrization (resp. antisymmetrization) in some indices $j, k, l, \ldots$ is denoted by writing a bar (resp. a tilde) over these indices, i.e., by writing $\bar{j}, \bar{k}, \bar{l}, \ldots$ (resp. $\tilde{j}, \tilde{k}, \tilde{l}, \ldots$).

First, we state some auxiliary assertions on the Young decomposition of tensors of type $(0, 3)$. Let us have a tensor $\Delta = \Delta_{jkl}$ and let n be the dimension of the underlying vector space. We define

$$\begin{aligned}
&\mathcal{S}\Delta = \tfrac{1}{6}(\Delta_{jkl} + \Delta_{ljk} + \Delta_{klj} + \Delta_{jlk} + \Delta_{lkj} + \Delta_{kjl}),\\
(5.2)\qquad &\mathcal{Q}\Delta = \tfrac{1}{3}(\Delta_{jkl} + \Delta_{kjl} - \Delta_{lkj} - \Delta_{klj}) + \tfrac{1}{3}(\Delta_{jkl} + \Delta_{lkj} - \Delta_{kjl} - \Delta_{ljk}),\\
&\mathcal{A}\Delta = \tfrac{1}{6}(\Delta_{jkl} + \Delta_{ljk} + \Delta_{klj} - \Delta_{jlk} - \Delta_{lkj} - \Delta_{kjl}).
\end{aligned}$$

Note that (5.2) can be equivalently written by

$$\mathcal{S}\Delta = \Delta_{\bar{j}\bar{k}\bar{l}}, \qquad \mathcal{Q}\Delta = \tfrac{1}{3}(2\Delta_{jkl} - \Delta_{ljk} - \Delta_{klj}), \qquad \mathcal{A}\Delta = \Delta_{\tilde{j}\tilde{k}\tilde{l}}. \tag{5.3}$$

Lemma 3. (a) *If $n \geq 3$, a tensor $\Delta = \Delta_{jkl}$ has a unique decomposition*

$$\Delta = \mathcal{S}\Delta + \mathcal{Q}\Delta + \mathcal{A}\Delta,$$

such that $\mathcal{S}(\mathcal{S}\Delta) = \mathcal{S}\Delta$, $\mathcal{Q}(\mathcal{Q}\Delta) = \mathcal{Q}\Delta$, $\mathcal{A}(\mathcal{A}\Delta) = \mathcal{A}\Delta$, $\mathcal{S}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{S}\Delta) = \mathcal{A}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{A}\Delta) = \mathcal{S}(\mathcal{A}\Delta) = \mathcal{A}(\mathcal{S}\Delta) = 0$.

(b) *If $n = 2$, a tensor $\Delta = \Delta_{jkl}$ has a unique decomposition*

$$\Delta = \mathcal{S}\Delta + \mathcal{Q}\Delta,$$

such that $\mathcal{S}(\mathcal{S}\Delta) = \mathcal{S}\Delta$, $\mathcal{Q}(\mathcal{Q}\Delta) = \mathcal{Q}\Delta$, $\mathcal{S}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{S}\Delta) = 0$.

Proof. These assertions can be verified by a direct computation. □

Corollary 2. *For tensor $\Delta = \Delta_{jkl}$ symmetric in the indices k, l there is a unique decomposition*

$$\Delta = \mathcal{S}\Delta + \mathcal{Q}\Delta, \tag{5.4}$$

such that $\mathcal{S}(\mathcal{S}\Delta) = \mathcal{S}\Delta$, $\mathcal{Q}(\mathcal{Q}\Delta) = \mathcal{Q}\Delta$, $\mathcal{S}(\mathcal{Q}\Delta) = \mathcal{Q}(\mathcal{S}\Delta) = 0$.

Finally, we introduce the following functions on $T^2_n L^1_n$:

$$\begin{aligned}
&I^i_{jk}(p^a_b, p^a_{b,c}, p^a_{b,cd}) = q^l_{\tilde{j}} p^i_{l,\tilde{k}}, \\
&I^i_{jkl}(p^a_b, p^a_{b,c}, p^a_{b,cd}) = 2q^s_j p^i_{s,kl} - q^s_k p^i_{s,lj} - q^s_l p^i_{s,jk} \\
&\quad -3(q^t_{\bar{j}} p^m_{t,\bar{l}} q^s_{\bar{m}} p^i_{s,\bar{k}} + q^t_{\bar{j}} p^m_{t,\bar{k}} q^s_{\bar{m}} p^i_{s,\bar{l}}) \\
&\quad +p^i_{s,m}(2q^s_j q^t_{\bar{k}} p^m_{t,\bar{l}} - q^s_k q^t_{\bar{l}} p^m_{t,\bar{j}} - q^s_l q^t_{\bar{j}} p^m_{t,\bar{k}}).
\end{aligned} \tag{5.5}$$

It is obvious that the functions I^i_{jk} are antisymmetric in indices j, k, and the functions I^i_{jkl} are symmetric in indices k, l, which gives us $I^i_{jk} + I^i_{kj} = 0$, $I^i_{jkl} - I^i_{jlk} = 0$, respectively. Moreover, we have the identity $I^i_{jkl} + I^i_{ljk} + I^i_{klj} = 0$.

Lemma 4. *$K^{3,1}_n$-orbits in $T^2_n L^1_n$ induced by the frame action of L^1_n on L^1_n is defined by the equations*

$$p^i_j = c^i_j, \qquad I^i_{jk}(p^a_b, p^a_{b,c}, p^a_{b,cd}) = c^i_{jk}, \qquad I^i_{jkl}(p^a_b, p^a_{b,c}, p^a_{b,cd}) = c^i_{jkl},$$

where $c^i_j, c^i_{jk}, c^i_{jkl} \in \mathbb{R}$ are arbitrary constants such that $\det c^i_j \neq 0$.

Proof. Consider the action (5.1) of $K^{3,1}_n$ on $T^2_n L^1_n$, induced by the frame action of L^1_n on L^1_n, in standard notation given by $\bar{p}^i_j = a^i_k p^k_j$. Rewrite this action in the form

$$\bar{p}^i_j = p^i_j, \qquad \bar{p}^i_{j,k} = p^i_{j,k} + p^s_j a^i_{sk}, \qquad \bar{p}^i_{j,kl} = p^i_{j,kl} + \chi^i_{j,kl} + p^s_j a^i_{skl}, \tag{5.6}$$

where the functions

$$\chi^i_{j,kl} = p^s_{j,l} a^i_{sk} + p^s_{j,k} a^i_{sl} + (p^i_{j,t} + p^s_j a^i_{st}) b^t_{kl} \tag{5.7}$$

are symmetric in the indices k, l. From (5.6) we get

$$\bar{q}^i_j = q^i_j, \qquad a^i_{sk} = q^j_{\bar{s}}(\bar{p}^i_{j,\bar{k}} - p^i_{j,\bar{k}}), \qquad a^i_{skl} = q^j_{\bar{s}}(\bar{p}^i_{j,\bar{k}\bar{l}} - p^i_{j,\bar{k}\bar{l}} - \chi^i_{j,\bar{k}\bar{l}}). \tag{5.8}$$

Substituting the second equation of (5.8) to (5.6) we have

$$q^j_s(\bar{p}^i_{j,k} - p^i_{j,k}) = q^j_{\bar{s}}(\bar{p}^i_{j,\bar{k}} - p^i_{j,\bar{k}}),$$

which means that we compare the tensor on the left hand side with its symmetric part. It gives us $\bar{q}^l_{\tilde{j}}\bar{p}^i_{l,\tilde{k}} = q^l_{\tilde{j}}p^i_{l,\tilde{k}}$. Thus, for the functions I^i_{jk}, defined by (5.5), we have

$$I^i_{jk}(\bar{p}^a_b, \bar{p}^a_{b,c}, \bar{p}^a_{b,cd}) = I^i_{jk}(p^a_b, p^a_{b,c}, p^a_{b,cd}), \tag{5.9}$$

and the functions I^i_{jk} are invariant with respect to the action (5.1) of $K^{3,1}_n$ on $T^2_n L^1_n$.

Again, substituting (5.8) to (5.6) we have

$$q^s_j(\bar{p}^i_{s,kl} - p^i_{s,kl} - \chi^i_{s,kl}) = q^s_{\bar{j}}(\bar{p}^i_{s,\bar{k}\bar{l}} - p^i_{s,\bar{k}\bar{l}} - \chi^i_{s,\bar{k}\bar{l}}).$$

It means that we compare the tensor

$$\Delta^i_{jkl} = q^s_j(\bar{p}^i_{s,kl} - p^i_{s,kl} - \chi^i_{s,kl}), \tag{5.10}$$

symmetric in the subscripts k, l, on the left hand side, with its symmetric part $\mathcal{S}\Delta = \Delta^i_{\bar{j}\bar{k}\bar{l}}$. Using decomposition (5.4) for the tensor Δ given by (5.10), we get

$$\mathcal{Q}\Delta = 0. \tag{5.11}$$

Applying (5.3) to (5.11), and using (5.7), and (5.8), after long calculation we obtain that (5.11) is equivalent to

$$I^i_{jkl}(\bar{p}^a_b, \bar{p}^a_{b,c}, \bar{p}^a_{b,cd}) = I^i_{jkl}(p^a_b, p^a_{b,c}, p^a_{b,cd}), \tag{5.12}$$

which means that the functions I^i_{jkl}, defined by (5.5), are invariant with respect to the action (5.1) of $K^{3,1}_n$ on $T^2_n L^1_n$. □

6 Basis of the second order invariants

Now, from the assertions on equivariant mappings of manifolds (see [4, 8]) we can obtain the exact characteristics of basis of differential invariants with values in L^1_n-manifolds.

First, let us denote by S^0_n the vector subspace of the tensor product $\bigotimes^2 \mathbb{R}^{n*} = \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$, defined in the canonical coordinates on $\mathbb{R}$ by the equations

$$x_{jk} + x_{kj} = 0.$$

Similarly, S^1_n denotes the vector subspace of the tensor product $\bigotimes^3 \mathbb{R}^{n*} = \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$, defined by the equations

$$x_{jkl} - x_{jlk} = 0, \quad x_{jkl} + x_{ljk} + x_{klj} = 0.$$

We can summarize the discussion of Section 5 in the following theorem, describing differential invariants on $T^2_n L^1_n$ with values in L^1_n-manifolds.

Theorem 1. (a) *The frame action defines on $T^2_n L^1_n$ the structure of a left principal $K^{3,1}_n$-bundle.*

(b) *The quotient space $T^2_n L^1_n / K^{3,1}_n$ is canonically isomorphic to the space $L^1_n \times (\mathbb{R}^n \otimes S^0_n) \times (\mathbb{R}^n \otimes S^1_n)$.*

Proof. (a) Since we have already proved that the action (5.1) of $K^{3,1}_n$ on $T^2_n L^1_n$ is free (see Corollary 1), in order to show that $T^2_n L^1_n$ is a principal $K^{3,1}_n$-bundle it remains to show that the equivalence "$J^2_0\gamma \sim J^2_0\bar{\gamma}$ if and only if there exists an element $J^3_0\alpha \in K^{3,1}_n$ such that $J^2_0\bar{\gamma} = J^3_0\alpha \cdot J^2_0\gamma$" is a closed submanifold in $T^2_n L^1_n \times T^2_n L^1_n$. But using (5.1) with help of (5.9) and (5.12), we see that this submanifold is defined by the equations

$$p^i_j(J^2_0\bar{\gamma}) - p^i_j(J^2_0\gamma) = 0,$$

$$I^i_{jk}(J^2_0\bar{\gamma}) - I^i_{jk}(J^2_0\gamma) = 0,$$

$$I^i_{jkl}(J^2_0\bar{\gamma}) - I^i_{jkl}(J^2_0\gamma) = 0,$$

and is therefore closed.

(b) Let $J^2_0\gamma \in T^2_n L^1_n$ and let $[J^2_0\gamma]$ be the corresponding class of $J^2_0\gamma$ in quotient $T^2_n L^1_n / K^{3,1}_n$. We set

$$\begin{aligned} p^i_j([J^2_0\gamma]) &= p^i_j(J^2_0\gamma), \\ I^i_{jk}([J^2_0\gamma]) &= I^i_{jk}(J^2_0\gamma), \\ I^i_{jkl}([J^2_0\gamma]) &= I^i_{jkl}(J^2_0\gamma). \end{aligned} \tag{6.1}$$

Relations (5.1), (5.9), and (5.12) imply that coordinates defined by (6.1) do not depend on a representant of given class, and for two different classes there are different sets of numbers. It means that factor projection $\pi : T^2_n L^1_n \to T^2_n L^1_n / K^{3,1}_n$ can be expressed by

$$\pi = (p^i_j, I^i_{jk}, I^i_{jkl}).$$

Thus, canonical isomorphism of bundles maps the class from $T^2_n L^1_n / K^{3,1}_n$ with coordinates $(p^i_j, I^i_{jk}, I^i_{jkl})$, to the element of $L^1_n \times (\mathbb{R}^n \otimes S^0_n) \times (\mathbb{R}^n \otimes S^1_n)$ with the same canonical coordinates. $\square$

Theorems 1 says that every second order differential invariant of frames factorizes through the corresponding bundle projection. Consider the components of the isomorphisms defined by $p^i_j : T^2_n L^1_n \to L^1_n$, $I^i_{jk} : T^2_n L^1_n \to \mathbb{R}^n \otimes S^0_n$, and $I^i_{jkl} : T^2_n L^1_n \to \mathbb{R}^n \otimes S^1_n$. We have the following results.

Corollary 3. *The mappings p^i_j, I^i_{jk}, I^i_{jkl} represent a basis of second order invariants of frames with values in left L^1_n-manifold.*

7 Basis of scalar invariants

Note that if G is a Lie group, K is its normal subgroup, and P is a G-manifold then the quotient manifold $(P/K)/(G/K)$ is canonically isomorphic with P/G (see [1]).

This means that for finding scalar invariants of Lie group G, we can equivalently factorize P by normal subgroup K and subsequently by the factor group G/K. Thus, to obtain scalar invariants of L^3_n on $T^2_n L^1_n$ it is sufficient to consider L^1_n-equivariant mappings defined on $T^2_n L^1_n / K^{3,1}_n$.

Let us define some functions $\mathcal{I}^i_{jk}$, $\mathcal{I}^i_{jkl}$, on $T^2_n L^1_n$, by

$$
\begin{aligned}
\mathcal{I}^i_{jk} &= q^i_l p^s_k p^l_{j,s},\\
\mathcal{I}^i_{jkl} &= q^i_r (2p^s_k p^t_l p^r_{j,st} - p^s_l p^t_j p^r_{k,st} - p^s_j p^t_k p^r_{l,st}\\
&\quad -\tfrac{3}{2} q^u_m p^r_{u,s} (p^s_k p^t_l p^m_{j,t} + p^s_l p^t_k p^m_{j,t}) - \tfrac{5}{2} p^m_{j,s} p^s_k p^r_{l,m}\\
&\quad +\tfrac{1}{2} p^s_j p^r_{k,m} p^m_{l,s} + 2p^r_{j,m} p^s_k p^m_{l,s}).
\end{aligned}
\tag{7.1}
$$

We have the following

Theorem 2. *The functions $\mathcal{I}^i_{jk}$, $\mathcal{I}^i_{jkl}$ represent a basis of second order scalar invariants of frames.*

Proof. The group $L^1_n \simeq L^3_n / K^{3,1}_n$ acts in factor space $T^2_n L^1_n / K^{3,1}_n$, where the functions I^i_{jk} and I^i_{jkl} live, by

$$
\bar{I}^i_{jk} = a^i_r b^s_j b^t_k I^r_{st}, \quad \bar{I}^i_{jkl} = a^i_r b^s_j b^t_k b^u_l I^r_{stu},
\tag{7.2}
$$

respectively. Using relations $a^i_r = q^m_r \bar{p}^i_m$, and $b^s_j = \bar{q}^v_j p^s_v$, obtained from (3.1), in (7.2), we have

$$
\bar{q}^a_i \bar{p}^j_b \bar{p}^k_c \bar{I}^i_{jk} = q^a_r p^s_b p^t_c I^r_{st}, \quad \bar{q}^a_i \bar{p}^j_b \bar{p}^k_c \bar{p}^l_d \bar{I}^i_{jkl} = q^a_r p^s_b p^t_c p^u_d I^r_{stu},
$$

which describes L^1_n-invariant objets in $T^2_n L^1_n / K^{3,1}_n$. Using (5.5), we get

$$
q^a_r p^s_b p^t_c I^r_{st} = \mathcal{I}^a_{bc}, \quad q^a_r p^s_b p^t_c p^u_d I^r_{stu} = \mathcal{I}^a_{bcd},
$$

where $\mathcal{I}^a_{bc}$, $\mathcal{I}^a_{bcd}$ are given by (7.1). $\square$

Using factorization method, we are allowed to determine the number of independent L^3_n-invariant functions on $T^2_n L^1_n$ as the dimensions of the corresponding factor spaces. Thus, the number of functionally independent invariants $\mathcal{I}^i_{jk}$, $\mathcal{I}^i_{jkl}$ is given by

$$
\dim(T^1_n L^1_n / L^2_n) = \tfrac{1}{2}\, n^2 (n-1), \quad \dim(T^2_n L^1_n / L^3_n) = \tfrac{1}{3}\, n^2 (n^2 - 1),
$$

respectively. For instance of $n = 1, 2, 3, 4$, the number of independent L^3_n-invariant functions on $T^2_n L^1_n$ is $N = 0, 6, 33, 104$, respectively.

Let us consider a left action of the general linear group L^1_n on the real line $\mathbb{R}$ by

$$
L^1_n \times \mathbb{R} \ni (a, t) \mapsto |\det a^{-1}| \cdot t \in \mathbb{R}.
$$

The real line, endowed with this action, is an L^1_n-manifold, denoted by $\widetilde{\mathbb{R}}$.

We also introduce the function $\mathcal{I}_0$ defined on $T^2_n L^1_n$ by

$$
T^2_n L^1_n \ni q \mapsto \mathcal{I}_0(q) = |\det q^i_j(q)| \in \mathbb{R}.
$$

Lemma 5. *The function $\mathcal{I}_0 : T^2_n L^1_n \to \widetilde{\mathbb{R}}$ is a differential invariant.*

Proof. Obviously, the function $\mathcal{I}_0$ is smooth, and for every $a \in L^3_n$, and every $q \in T^2_n L^1_n$ we have $\mathcal{I}_0(a \cdot q) = |\det a^{-1}| \cdot \mathcal{I}_0(q)$. □

Corollary 4. *Every differential invariant $\mathcal{I}$ defined on $T^2_n L^1_n$, with values in $\widetilde{\mathbb{R}}$, is the product of some scalar invariant and the function $\mathcal{I}_0$.*

8 Canonical odd n-form on FX

In order to compare our results with [5], we recall in this Section the concept of volume form needed for integration on not necessarily orientable manifold.

Any chart (U, φ), $\varphi = (x^i)$, on X, induces the *fibred* chart (V, ψ), $\psi = (x^i, x^i_j)$, on FX. For every frame $\Xi \in V$ we have $\det x^i_j(\Xi) \neq 0$, and we can define some other coordinates y^j_k of Ξ by setting $x^i_j y^j_k = \delta^i_k$. We define a function $V \ni \Xi \mapsto |\det y^i_j(\Xi)| \in \mathbb{R}$, *associated* with the chart (V, ψ). With a chart (V, ψ) we also associate the object

$$\tilde{\omega}_{(V,\psi)} = |\det y^i_j| \cdot \tilde{\varphi} \otimes dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \tag{8.1}$$

where $\tilde{\varphi}$ is a *field of odd scalars* on X, associated with (U, φ) (see [9]). It is easily seen that (8.1) represent a globally defined odd base form on FX; we denote this form by $\tilde{\omega}$, and call it *canonical odd n-form on X*.

This form has the following properties:

1. For each frame field $\zeta : W \to FX$, where W is an open set on X, the pullback $\zeta^* \tilde{\omega}$ is an odd volume form on W.

2. The construction of $\tilde{\omega}$ does not depend on orientability of base manifold X. In the case of orientable and oriented manifolds X, $\tilde{\omega}$ is equivalent to an (ordinary) n-form on FX.

3. The form $\tilde{\omega}$ is diff X-*invariant*, i.e. $(F\alpha)^* \tilde{\omega} = \tilde{\omega}$ for all diffeomorphisms α of X, where $F\alpha$ is the canonical lift of α to FX.

It should be pointed out that odd n-forms $\zeta^* \tilde{\omega}$ may be used as volume forms for integration on the base manifold X. In particular, these forms naturally appears as a components of Lagrangians for variational problems for frame fields. We discuss these questions in the subsequent Section.

9 Second order natural Lagrangians of frames

Our aim in this Section is to characterize all Lagrangians on $J^2 FX$, invariant with respect to all diffeomorphisms of X. First we recall main concepts to this purpose.

We present basic definitions in full generality (for odd base forms). If the underlying manifold X is orientable, odd base forms may be replaced by ordinary forms.

Let us denote by μ the bundle projection $\mu : FX \to X$. The canonical jet projection $\mu^2 : J^2 FX \to X$ is, for every $J^2_x \zeta \in J^2 FX$, defined by $\mu^2(J^2_x \zeta) = x$. A second order Lagrangian for FX is any μ^2-horizontal n-form λ defined on the second jet prolongation $J^2 FX$ of FX. In a chart (V, ψ), $\psi = (x^i, x^i_j)$, on FX, and the *associated* chart (V^2,ψ^2), $\psi^2=(x^i,x^i_j,x^i_{j,k},x^i_{j,kl})$, on J^2FX, a Lagrangian λ has an expression

$$\lambda=\mathcal{L}\cdot\tilde{\varphi}\otimes\omega_0,$$

where $\omega_0=dx^1\wedge dx^2\wedge\ldots\wedge dx^n$, $\tilde{\varphi}$ is a field of odd scalars, and $\mathcal{L}:V^2\to\mathbb{R}$ is the component of λ with respect to (V,ψ) (the Lagrange function associated with (V,ψ)).

We say that a second order Lagrangian λ is *natural*, if for every diffeomorphism $\alpha:W\to X$, where W is an open set in X, the canonical lift $F\alpha$ of α to FX is an invariance transformation of λ, i.e.,

$$(J^2F\alpha)^*\lambda=\lambda$$

on the corresponding open set in J^2FX.

The following theorem is an application of a general result to the structure we consider in this paper (see [10, 11]).

Theorem 3. *Let X be an n-dimensional manifold. There exists a one-to-one correspondence between natural Lagrangians on J^2FX and differential invariants $\mathcal{I}:T^2_nL^1_n\to\widetilde{\mathbb{R}}$.*

We denote by $\mathcal{A}_{\mathrm{diff}X}$ the algebra of diff X-invariant functions on J^2FX. Define in any chart (V,ψ), $\psi=(x^i,x^i_j)$, on FX, functions $\mathcal{L}^i_{jk}$, $\mathcal{L}^i_{jkl}$, by

$$\begin{aligned}
\mathcal{L}^i_{jk}&=y^i_lx^s_{\bar{k}}x^l_{\bar{j},s},\\
\mathcal{L}^i_{jkl}&=y^i_r(2x^s_kx^t_lx^r_{j,st}-x^s_lx^t_jx^r_{k,st}-x^s_jx^t_kx^r_{l,st}\\
&\quad-\tfrac{3}{2}y^u_mx^r_{u,s}(x^s_kx^t_{\bar{l}}x^m_{\bar{j},t}+x^s_lx^t_{\bar{k}}x^m_{\bar{j},t})-\tfrac{5}{2}x^m_{j,s}x^s_{\bar{k}}x^r_{\bar{l},m}\\
&\quad+\tfrac{1}{2}x^s_jx^r_{\bar{k},m}x^m_{\bar{l},s}+2x^r_{j,m}x^s_{\bar{k}}x^m_{\bar{l},s})
\end{aligned}\tag{9.1}$$

(compare with (7.1)). The functions $\mathcal{L}^i_{jk}$, $\mathcal{L}^i_{jkl}$, in coordinates expressed by (9.1), are globally defined functions on J^2FX.

Corollary 5. *The functions $\mathcal{L}^i_{jk},\mathcal{L}^i_{jkl}\in\mathcal{A}_{\mathrm{diff}X}$, and every function $\mathcal{L}\in\mathcal{A}_{\mathrm{diff}X}$ can be locally written as a differentiable function of the functions $\mathcal{L}^i_{jk}$, $\mathcal{L}^i_{jkl}$, defined by* (9.1).

The following theorem is an immediate consequence of the invariance theory.

Theorem 4. *Every natural Lagrangian λ on J^2FX is of the form*

$$\lambda=\mathcal{L}\tilde{\omega},$$

where $\mathcal{L}\in\mathcal{A}_{\mathrm{diff}X}$ and $\tilde{\omega}$ is canonical odd n-form of FX.

Remark 1. Restricting ourselves to the *first order* natural Lagrangians, Theorem 4 corresponds with the known results published in [5].

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Author's address:

Ján Brajerčík
Department of Mathematics, University of Prešov,
Ul. 17 Novembra 1, 081 16 Prešov, Slovakia.
E-mail: brajerci@unipo.sk