The S-curvature of homogeneous (α, β)-metrics

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Abstract. In this paper we give a formula of the S-curvature of homogeneous (α, β)-metrics. Then we use this formula to deduce a formula of the mean Berwald curvature E_{ij} of Randers metrics.

M.S.C. 2000: 22E46, 53C30.
Key words: (α, β) metric; S-curvature; mean Berwald curvature.

1 Introduction

The purpose of this paper is to give an explicit formula for the S-curvature of homogeneous (α, β)-metrics in Finsler geometry, for fundamental theory of Finsler geometry we refer to [1] and [4]. Let M be a connected smooth manifold and α be a Riemannian metric on M. Then a Randers metric on M with the underlying Riemannian metric α is a Finsler metric of the form $F = \alpha + \beta$, where β is a smooth 1-form on M satisfying $||\beta_x||_\alpha < 1$, $\forall x \in M$, here $||\beta||_\alpha$ denote the length of the 1-form under the Riemannian metric α. This kind of spaces was first studied by G. Randers in [11] and was then named after him. Randers metrics are most closely related to Riemannian metrics among the class of Finsler metrics. The Finsler metrics of the form $F = \alpha\phi(s)$ are called (α, β)-metric, here $s = \frac{\beta}{\alpha}$, and $\phi(s)$ is a function of s. It is clear that (α, β)-metrics are the generalization of Randers metrics, in fact, if we set $\phi(s) = 1 + s$, then $F = \alpha(1 + \frac{\beta}{\alpha}) = \alpha + \beta$.

There are several interesting curvatures in Finsler geometry, among them the flag curvature is the most important one, which is the natural generalization of sectional curvature in Riemannian geometry. On the other hand, Z. Shen introduced the notion of S-curvature of a Finsler space in [12]. It is a quantity to measure the rate of change of the volume form of a Finsler space along the geodesics. S-curvature is a non-Riemannian quantity, i.e., any Riemann manifold has vanishing S-curvature. It is a very interesting fact that the S-curvature and the flag curvature are subtly related with each other. We now recall the definition of S-curvature. Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{e_i\}$ of V, let

$$\sigma_F = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(y^i e_i) < 1\}},$$

Balkan Journal of Geometry and Its Applications, Vol.15, No.2, 2010, pp. 39-48.
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where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{e_i\}$. But it is easily seen that

$$\tau(y) = \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F}, \quad y \in V\backslash\{0\}$$

is independent of the choice of the basis. $\tau = \tau(y)$ is called *the distortion* of (V, F). Now let (M, F) be a Finsler space. Let $\tau(x, y)$ be the distortion of the Minkowski norm F_x on T_xM. For $y \in T_xM - \{0\}$, let $\tau(t)$ be the geodesic with $\tau(0) = x$ and $\dot{\tau}(0) = y$. Then the quantity

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]|_{t=0}$$

is called *the S-curvature* of the Finsler space (M, F). A Finsler space (M, F) is said to have *almost isotropic S-curvature* if there exists a smooth function $c(x)$ on M and a closed 1-form η such that:

$$S(x, y) = (n + 1)(c(x)F(y) + \eta(y)), \quad x \in M,\ y \in T_xM.$$

If in the above equation $\eta = 0$, then (M, F) is said to have *isotropic S-curvature.* If $\eta = 0$ and $c(x)$ is a constant, then (M, F) is said to have *constant S-curvature.*

In this paper we will obtain an explicit formula of the S-curvature of homogeneous (α, β)-metrics, without using local coordinate systems. As an application, we prove that a homogeneous (α, β)-metric has isotropic S-curvature if and only if it has vanishing S-curvature. We also give an explicit formula of the mean Berwald curvature E_{ij} of homogeneous Randers metrics.

2 The S-curvature

In this section we will compute the S-curvature of a G-invariant homogeneous (α, β) metric $F = \alpha\phi(s)$ on the coset space G/H of a Lie group G, where $s = \frac{\beta}{\alpha}$. Since $(G/H, F)$ is homogeneous, we only need to compute this at the origin $o = H$. By [2], in a local coordinate system, the S-curvature of the (α, β) metric $F = \alpha\phi(s)$ with the underlying Riemann metric α can be expressed as

$$S = (2\Psi - \frac{f'(b)}{bf(b)})(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0),$$

where

$$Q = \tfrac{\phi'}{\phi - s\phi'}, \qquad \Delta = 1 + sQ + (b^2 + s^2)Q', \qquad \Psi = \tfrac{Q'}{2\Delta},$$

$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'',$$

$$r_{ij} = \tfrac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \tfrac{1}{2}(b_{i;j} - b_{j;i}), \quad b^i = b_j a^{ji},$$

$$r_j = b^i r_{ij},\ \ s_j = b^i s_{ij},\ \ r_{00} = r_{ij}y^i y^j,\ \ s_0 = s_i y^i,\ \ r_0 = r_i y^i,$$

and the function $f(b)$ in the formula is defined as follows. The *Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$* is defined by

$$\sigma_{BH}(x) = \frac{\omega_n}{Vol\{(y^i) \in \mathbb{R} : F(x, y^i\frac{\partial}{\partial x^i}) < 1\}},$$

and the *Holmes-Thompson volume form*, $dV_{HT} = \sigma_{HT}(x)dx$ is defined by

$$\sigma_{HT}(x) = \frac{1}{\omega_n} \int_{\{(y^i)\in\mathbb{R}^n | F(x, y^i \frac{\partial}{\partial x^i})<1\}} \det(g_{ij})dy,$$

where Vol denotes the Euclidean volume, $g_{ij} = \frac{\partial^2}{\partial y^i \partial y^j}[F^2]$, and

$$\omega_n = Vol(B^n(1)) = \frac{1}{n} Vol(S^{n-1}) = \frac{1}{n} Vol(S^{n-2}) \int_0^\pi sin^{n-2}(t)dt.$$

When $F = \sqrt{g_{ij}(x)y^i y^j}$ is a Riemannian metric, both volume forms reduce to the same Riemannian volume form $dV_{HT} = dV_{BH} = \sqrt{\det(g_{ij}(x))}dx$. Now for the (α, β) metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, $b = ||\beta_x||_\alpha$, let $dV = dV_{BH}$ or dV_{HT}. Then

$$f(b) = \begin{cases} \dfrac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(b\cos t)^n} dt}, & \text{if } dV = dV_{BH}, \\ \\ \dfrac{\int_0^\pi (\sin^{n-2} t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2} t dt}, & \text{if } dV = dV_{HT}, \end{cases}$$

where $T(s) = \phi(\phi - s\phi')^{n-2}\{(\phi - s\phi') + (b^2 - s^2)\phi''\}$. Then the volume form dV is given by $dV = f(b)dV_\alpha$, where $dV_\alpha = \sqrt{\det(\alpha_{ij})}dx$, denote the Riemannian volume form of α.

In [2] the authors showed that if $b = ||\beta_x||_\alpha$ is a constant, then $r_0 + s_0 = 0$. So in this case we have

$$S = -\alpha^{-1} \frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0).$$

Now we will deduce a formula of the S-curvature of homogeneous (α, β)-metrics. Recall that the group $I(M, F)$ of isometries of a Finsler space (M, F) is a Lie transformation group of M ([7]). If $I(M, F)$ acts transitively on M, then (M, F) is called homogeneous. Let $(G/H, F)$ be a homogeneous (α, β) metric of the form $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$ with α a G-invariant Riemannian metric on G/H and β a G-invariant vector field on G/H. As pointed out in [6], to β corresponds a unique vector u in $T_o(G/H)$ which is fixed under the linear isotropy representation of H on $T_o(G/H)$ and $o = H$ is the origin of G/H. It is clear that $b = ||\beta_x||_\alpha$ is a constant. Therefore the S-curvature can be expressed as

$$S = -\alpha^{-1} \frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0). \qquad (*)$$

Also, G/H is a reductive homogeneous manifold in the sense of Nomizu ([10], see also [9]), i.e, the Lie algebra of G has a decomposition:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \text{(direct sum of subspaces)} \qquad (2.1)$$

such that $Ad(\mathfrak{m}) \subset \mathfrak{m}$, $\forall h \in H$. Then we can identify $\mathfrak{m}$ with the tangent space of (G/H) at the origin $o = H$ and in this way β corresponds a vector in $\mathfrak{m}$ which is invariant under the adjoint action of H on $\mathfrak{m}$. In the following we still denote this vector by u.

Let $\langle , \rangle$ be the corresponding inner product on $\mathfrak{m}$. We now deduce some results concerning the Levi-Civita connection of $(G/H, \alpha)$ which will be useful to compute the S-curvature. In literature, there are several versions of the formula of the connection for Killing vector fields. Since we are interested in the differential of (left) invariant vector fields on G/H, we adopt the formula in [9]. Given $v \in g$, we can define a one-parameter transformation group φ_t ,$t \in \mathbb{R}$ of G/H by

$$\phi_t(gH) = (\exp(tv)g)H, \quad g \in G.$$

Then φ_t generates a vector field on G/H which is a Killing vector field (this is called *the fundamental vector field generated by* v in [9]). We denote this vector field by $\bar{v}$. The following formula is a direct consequence of the formula in [9, Vol.2, page 201] (see also [13]):

$$\langle (\nabla_{\bar{v}_1} \bar{v}_2)|_o, w\rangle = \frac{1}{2}(-\langle [v_1, v_2]_m, w\rangle + \langle [w, v_1]_m, v_2\rangle + \langle [w, v_2]_m, v_1\rangle), \tag{2.2}$$

where $v_1, v_2, w \in \mathfrak{m}$, and $[v_1, v_2]_m$ denote the protection of $[v_1, v_2]$ to $\mathfrak{m}$ with respect to the decomposition (2.1).

To apply the formula (2.2) to our study, we need to deduce some formula for the connection in a local coordinate system. Let $u_1 \cdots u_m$ be an orthonormal basis of $\mathfrak{m}$ with respect to $\langle \cdot, \cdot \rangle$. Then by [8] there exists a neighborhood U of o in G/H such that the mapping

$$(\exp x^1 u_1 \exp x^2 u_2 \cdots \exp x^m u_m)H \mapsto (x^1, x^2 \cdots x^m)$$

defines a local coordinate system on U.

In the following we adopt the computation of r_{00} and s_0 on $(G/H, F)$ in [5]. Let $(u_1, u_2, \cdots, u_m)$ be an orthonormal basis of $(\mathfrak{m}, \langle \cdot, \cdot \rangle)$ such that $u_{\mathfrak{m}} = \frac{u}{|u|}$. Recall that u is the vector in $\mathfrak{m}$ corresponding to the 1-from β. Let $(U, (x^1, x^2 \cdots x^n))$ be the local coordinate system defined as above. Then in [5] it was shown that

$$\begin{aligned} s_0(y)|_o = y^l s_l(o) = c y^l s_{nl}(o) &= \frac{1}{2} c^2 y^l \langle [u_n, u_l]_m, u_n\rangle \\ &= \frac{1}{2}\langle [c u_n, y^l u_l]_m, c u_n\rangle = \frac{1}{2}\langle [u, y]_m, u\rangle, \end{aligned}$$

and

$$r_{ij}(o) = -\frac{c}{2}(\langle [u_n, u_i]_m, u_j\rangle + \langle [u_n, u_j]_m, u_i\rangle).$$

Moreover,

$$\begin{aligned} r_{00}|_o = r_{ij}(o) y^i y^j &= -\frac{c}{2}(\langle [u_n, u_i]_m, u_j\rangle + \langle [u_n, u_j]_m, u_i\rangle) y^i y^j \\ &= -\frac{c}{2}(\langle [u_n, y^i u_i]_m, y^j u_j\rangle + \langle [u_n, y^j u_j]_m, y^i u_i\rangle) \\ &= -\frac{c}{2}(\langle [u_n, y]_m, y\rangle + \langle [u_n, y]_m, y\rangle) = -c\langle [u_n, y]_m, y\rangle. \end{aligned}$$

Substituting the above into the formula (∗) we obtain the formula of S-curvature:

$$\begin{aligned} S(o,y) &= -\frac{1}{\alpha(y)}\frac{\Phi}{2\Delta^2}(r_{ij}(o)y^iy^j - 2\alpha(y)Qs_0(y)) \\ &= -\frac{1}{\alpha(y)}\frac{\Phi}{2\Delta^2}(-c\langle[u,y]_{\mathfrak{m}},y\rangle - \alpha(y)Q\langle[u,y]_{\mathfrak{m}},u\rangle). \end{aligned}$$

Summarizing, we get

Theorem 2.1 *Let $F = \alpha\phi(s)$ be a G-invariant (α,β)-metric on the reductive homogeneous manifold G/H with a decomposition of the Lie algebra*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

Then the S-curvature of F has the form

$$S(o,y) = -\frac{1}{\alpha(y)}\frac{\Phi}{2\Delta^2}(-c\langle[u,y]_{\mathfrak{m}},y\rangle - \alpha(y)Q\langle[u,y]_{\mathfrak{m}},u\rangle), \quad y \in \mathfrak{m},$$

where u is the vector in $\mathfrak{m}$ corresponding to the 1-form β, and we have identified $\mathfrak{m}$ with the tangent space of G/H at the origin $o = H$.

As a direct application of the formula we have

Corollary 2.2 *Let $(G/H, F)$ be as in Theorem 2.1. Then F has isotropic S-curvature if and only if F has vanishing S-curvature.*

Proof. We only need to prove the direct implication. Suppose F has isotropic S-curvature:

$$S(x,y) = (n+1)c(x)F(y), \quad x \in G/H, y \in T_x(G/H).$$

Setting $x = o$ and $y = u$ and using the formula in Theorem 2.1, we get $c(o) = 0$. Hence $S(o,y) = 0$, $\forall y \in T_o(G/H)$. Since F is a homogeneous metric, we must have $S = 0$ everywhere. □

3 Mean Berwald curvature of homogeneous Randers metrics

In this section we apply the results in Section 2 to give a formula of mean Berwald curvature of homogeneous Randers metrics.

The mean Berwald curvature (E-curvature) is an important non-Riemannian quantity defined by (see [3])

$$E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right),$$

where $G^m = G^m(x,y)$ are the spray coefficients. We know that

$$S = \frac{\partial G^m}{\partial y^m} - (\ln \sigma(x))_{x^k} y^k$$

here $(\ln\sigma(x))_{x^k}$ is the function of x because $\ln\sigma(x)$ is the function of x. Hence

$$0=\frac{\partial^2}{\partial y^i\partial y^j}[(\ln\sigma(x))_{x^k}y^k]$$

This means that

$$\frac{\partial^2 S}{\partial y^i\partial y^j}=\frac{\partial^2}{\partial y^i\partial y^j}\left[\frac{\partial G^m}{\partial y^m}-(\ln\sigma(x))_{x^k}y^k\right]=\frac{\partial^2}{\partial y^i\partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right)=2E_{ij}$$

Now we compute

$$\frac{\partial^2 S}{\partial y^i\partial y^j}(o,y)=\frac{\partial^2 S(o,y)}{\partial y^i\partial y^j}=2E_{ij}(o,y).$$

By Theorem 2.1 we have

$$\frac{\partial^2 S(o,y)}{\partial y^i\partial y^j}=\frac{\partial^2}{\partial y^i\partial y^j}\left(\frac{c\Phi}{2\Delta^2\alpha(y)}\langle[u,y]_m,y\rangle\right)+\frac{\partial^2}{\partial y^i\partial y^j}\left(\frac{\Phi Q}{2\Delta^2}\langle[u,y]_m,u\rangle\right).$$

Before the computation, we recall that

$$\frac{\partial s}{\partial y}=\frac{1}{\alpha}\left(b_m-s\frac{y_m}{\alpha}\right)\ ,\ \frac{\partial\alpha}{\partial y^m}=\frac{y_m}{\alpha},$$

where $y_m=a_{mj}y^j$. Since $u_1,u_2,\cdots,u_m$ in Section 2 is an orthonormal basis, we have $a_{mj}|_o=\delta^m_j$. Therefore at the origin we have $y_m=y^m$. Now we consider the special case of the Randers metric $\phi=1+s$. Let $\psi=\phi-s\phi'$. Then we have

$$\begin{aligned}
&Q=\frac{\phi'}{\psi}=1,\quad Q'=0,\quad \Psi=\frac{Q'}{2\Delta}=0,\\
&\Delta=1+sQ+(b^2-s^2)Q'=1+s,\qquad \Theta=\frac{Q-sQ'}{2\Delta}=\frac{1}{2\Delta}=\frac{1}{2(1+s)},\\
&\Phi=-(Q-sQ')\{n\Delta+1+sQ\}-(b^2-s^2)(1+sQ)Q''=-(n+1)(1+s).
\end{aligned}$$

Letting $c=1$ we get

$$\begin{aligned}
S(o,y)&=\frac{1}{\alpha(y)}\frac{\Phi}{2\Delta^2}(-c\langle[u,y]_m,y\rangle-\alpha(y)Q\langle[u,y]_m,u\rangle)\\
&=-\frac{n+1}{2(1+s)\alpha(y)}\langle[u,y]_m,y\rangle-\frac{n+1}{2(1+s)}\langle[u,y]_m,u\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
2E_{ij}(o,y)=\frac{\partial^2 S(o,y)}{\partial y^i\partial y^j}&=\frac{\partial^2}{\partial y^i\partial y^j}\left(-\frac{n+1}{2(1+s)\alpha(y)}\langle[u,y]_m,y\rangle-\frac{n+1}{2(1+s)}\langle[u,y]_m,u\rangle\right)\\
&=\frac{\partial^2}{\partial y^i\partial y^j}\left(-\frac{n+1}{2(1+s)\alpha(y)}\langle[u,y]_m,y\rangle\right)-\frac{\partial^2}{\partial y^i\partial y^j}\left(\frac{n+1}{2(1+s)}\langle[u,y]_m,u\rangle\right).
\end{aligned}$$

Now

$$
\begin{aligned}
&\frac{\partial^2}{\partial y^i \partial y^j}\left(-\frac{n+1}{2(1+s)\alpha(y)}\langle [u,y]_m, y\rangle\right)\\
=&-\frac{n+1}{2}\left\{\langle [u,y]_m, y\rangle \frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{1}{(1+s)\alpha(y)}\right)+\frac{1}{(1+s)\alpha(y)}\frac{\partial^2\langle [u,y]_m, y\rangle}{\partial y^i \partial y^j}\right.\\
&\left.+\frac{\partial\langle [u,y]_m, y\rangle}{\partial y^j}\frac{\partial}{\partial y^i}\left(\frac{\Phi}{(1+s)\alpha(y)}\right)+\frac{\partial\langle [u,y]_m, y\rangle}{\partial y^i}\frac{\partial}{\partial y^j}\left(\frac{1}{(1+s)\alpha(y)}\right)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial\langle [u,y]_m, y\rangle}{\partial y^i} &= \langle [u,u_i]_m, y\rangle + \langle [u,y]_m, u_i\rangle,\\
\frac{\partial^2\langle [u,y]_m, y\rangle}{\partial y^i \partial y^j} &= \langle [u,u_i]_m, u_j\rangle + \langle [u,u_j]_m, u_i\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial y^j}\left(\frac{1}{(1+s)\alpha(y)}\right) &= -\frac{\frac{\partial s}{\partial y^j}\alpha(y)+(1+s)\frac{\partial \alpha}{\partial y^j}}{(1+s)^2\alpha^2(y)}\\
&= -\frac{\frac{1}{\alpha(y)}(b_j - s\frac{y^j}{\alpha(y)})\alpha(y)+(1+s)\frac{y^j}{\alpha(y)}}{(1+s)\alpha(y)}\\
&= -\frac{(b_j - s\frac{y^j}{\alpha(y)})+(1+s)\frac{y^j}{\alpha(y)}}{(1+s)^2\alpha^2(y)} = -\frac{b_j+\frac{y^j}{\alpha(y)}}{(1+s)^2\alpha^2(y)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{1}{(1+s)\alpha(y)}\right) = \frac{\partial}{\partial y^i}\left(-\frac{b_j+\frac{y^j}{\alpha(y)}}{(1+s)^2\alpha^2(y)}\right)\\
=&-\frac{\frac{\partial}{\partial y^i}(b_j+\frac{y^j}{\alpha(y)})(1+s)^2\alpha^2(y)-(b_j+\frac{y^j}{\alpha(y)})\frac{\partial}{\partial y^i}[(1+s)^2\alpha^2(y)]}{(1+s)^4\alpha^4(y)}\\
=&-\frac{\frac{\partial}{\partial y^i}(\frac{y^j}{\alpha(y)})(1+s)^2\alpha^2(y)-(b_j+\frac{y^j}{\alpha(y)})2(1+s)\alpha(y)[\frac{\partial s}{\partial y^i}\alpha^2(y)+s\frac{\partial\alpha(y)}{\partial y^i}]}{(1+s)^4\alpha^4(y)}\\
=&-\frac{1}{(1+s)^2\alpha^2(y)}\frac{\frac{\partial y^j}{\partial y^i}\alpha(y)-y^j\frac{\partial\alpha(y)}{\partial y^i}}{\alpha^2(y)}\\
&+\frac{2}{(1+s)\alpha^3(y)}\left(b_j+\frac{y^j}{\alpha(y)}\right)\left[\frac{1}{\alpha(y)}(b_j-s\frac{y^i}{\alpha(y)})\alpha(y)+s\frac{y^i}{\alpha(y)}\right]\\
=&-\frac{1}{(1+s)^2\alpha^4(y)}\left[\delta_i^j\alpha(y)-y^j\frac{y^i}{\alpha(y)}\right]+\frac{2}{(1+s)\alpha^3(y)}\left(b_j+\frac{y^j}{\alpha(y)}\right)b_i\\
=&-\frac{\delta_i^j}{(1+s)^2\alpha^3(y)}+\frac{y^iy^j}{(1+s)^2\alpha^5(y)}+\frac{2b_i(b_j+\frac{y^j}{\alpha(y)})}{(1+s)\alpha^3(y)}.
\end{aligned}
$$

Therefore we have

$$\begin{aligned}
&\frac{\partial^2}{\partial y^i \partial y^j}\left(-\frac{n+1}{2(1+s)\alpha(y)}\langle [u,y]_m, y\rangle\right)\\
&=\left\{-\frac{\delta_i^j}{(1+s)^2\alpha^3(y)}+\frac{y^iy^j}{(1+s)^2\alpha^5(y)}+\frac{2b_i(b_j+\frac{y^j}{\alpha(y)})}{(1+s)\alpha^3(y)}\right\}\langle [u,y]_m, y\rangle\\
&\quad+\frac{1}{(1+s)\alpha(y)}\langle [u,u_i]_m, u_j\rangle+\langle [u,u_j]_m, u_i\rangle\\
&\quad-\frac{b_j+\frac{y^j}{\alpha(y)}}{(1+s)^2\alpha^2(y)}\langle [u,u_i]_m, y\rangle+\langle [u,y]_m, u_i\rangle-\frac{b_i+\frac{y^i}{\alpha(y)}}{(1+s)^2\alpha^2(y)}\langle [u,u_j]_m, y\rangle+\langle [u,y]_m, u_j\rangle.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{\partial^2}{\partial y^i \partial y^j}\left(-\frac{n+1}{2(1+s)}\langle [u,y]_m, u\rangle\right)=-\frac{n+1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\langle [u,y]_m, y\rangle}{(1+s)}\right)\\
&=-\frac{n+1}{2}\left\{\frac{1}{1+s}\frac{\partial^2\langle [u,y]_m, u\rangle}{\partial y^i\partial y^j}+\langle [u,y]_m, u\rangle\frac{\partial^2\frac{1}{1+s}}{\partial y^i\partial y^j}\right.\\
&\quad\left.+\frac{\partial\frac{1}{1+s}}{\partial y^i}\frac{\partial\langle [u,y]_m, u\rangle}{\partial y^j}+\frac{\partial\frac{1}{1+s}}{\partial y^j}\frac{\partial\langle [u,y]_m, u\rangle}{\partial y^i}\right\},
\end{aligned}$$

where

$$\frac{\partial\langle [u,y]_m, u\rangle}{\partial y^j}=\langle [u,u_j]_m, u\rangle,\qquad \frac{\partial^2\langle [u,y]_m, u\rangle}{\partial y^i\partial y^j}=\frac{\partial}{\partial y^i}\langle [u,u_j]_m, u\rangle=0,$$

$$\frac{\partial\frac{1}{1+s}}{\partial y^j}=-\frac{\frac{1}{\alpha(y)}(b_j-s\frac{y^j}{\alpha(y)})}{(1+s)^2}.$$

Hence

$$\begin{aligned}
&\frac{\partial^2\frac{1}{1+s}}{\partial y^i\partial y^j}=\frac{\partial}{\partial y^i}\left(-\frac{\frac{1}{\alpha(y)}(b_j-s\frac{y^j}{\alpha(y)})}{(1+s)^2}\right)\\
&=-\frac{\frac{\partial}{\partial y^i}[\frac{1}{\alpha(y)}(b_j-s\frac{y^j}{\alpha(y)})](1+s)^2-\frac{1}{\alpha(y)}(b_j-s\frac{y^j}{\alpha(y)})\frac{\partial(1+s)}{\partial y^i}}{(1+s)^4}\\
&=-\frac{\alpha(y)\frac{\partial}{\partial y^i}(b_j-s\frac{y^j}{\alpha(y)})-(b_j-s\frac{y^j}{\alpha(y)})\frac{\partial\alpha(y)}{\partial y^i}}{(1+s)^2\alpha^2(y)}+\frac{2(1+s)}{(1+s)^4\alpha(y)}\left(b_j-s\frac{y^j}{\alpha(y)}\right)\frac{\partial s}{\partial y^i}\\
&=-\frac{1}{(1+s)^2}\left[\frac{1}{\alpha(y)}\left(-\frac{\partial s}{\partial y^i}\frac{y^i}{\alpha(y)}-s\frac{\partial\frac{y^j}{\alpha(y)}}{\partial y^i}\right)-\frac{1}{\alpha^2(y)}\left(b_j-s\frac{y^j}{\alpha(y)}\right)\frac{y^i}{\alpha(y)}\right]\\
&\quad+\frac{2}{(1+s)^3\alpha(y)}\left(b_j-s\frac{y^j}{\alpha(y)}\right)\frac{1}{\alpha(y)}\left(b_i-s\frac{y^i}{\alpha(y)}\right)\\
&=-\frac{1}{(1+s)^2}\left\{\frac{1}{\alpha(y)}\left[-\frac{1}{\alpha(y)}\left(b_i-s\frac{y^i}{\alpha(y)}\right)\frac{y^j}{\alpha(y)}-s\frac{\frac{\partial y^j}{\partial y^i}\alpha(y)-y^j\frac{\partial\alpha}{\partial y^i}}{\alpha^2(y)}\right.\right.\\
&\quad\left.\left.-\frac{1}{\alpha^2(y)}\left(b_j-s\frac{y^j}{\alpha(y)}\right)\frac{y^i}{\alpha(y)}\right]+\frac{2}{(1+s)\alpha^2(y)}\left(b_j-s\frac{y^j}{\alpha(y)}\right)\left(b_i-s\frac{y^i}{\alpha(y)}\right)\right\}
\end{aligned}$$

$$= -\frac{1}{(1+s)^2}\left[-\frac{1}{\alpha^3(y)}\left(b_i - s\frac{y^i}{\alpha(y)}\right)y^j - \frac{s}{\alpha^3(y)}\left(\alpha(y)\delta^i_j - y^j\frac{y^i}{\alpha(y)}\right)\right.$$
$$\left. -\frac{y^i}{\alpha^3(y)}\left(b_j - s\frac{y^j}{\alpha(y)}\right) + \frac{2}{(1+s)\alpha^2(y)}\left(b_j - s\frac{y^j}{\alpha(y)}\right)\left(b_i - s\frac{y^i}{\alpha(y)}\right)\right].$$

So,

$$\frac{\partial^2}{\partial y^i \partial y^j}\left(-\frac{n+1}{2(1+s)}\langle[u,y]_m,u\rangle\right)$$
$$= -\frac{\langle[u,y]_m,u\rangle}{(1+s)^2}\left[-\frac{1}{\alpha^3(y)}\left(b_i - s\frac{y^i}{\alpha(y)}\right)y^j - \frac{s}{\alpha^3(y)}\left(\alpha(y)\delta^i_j - y^j\frac{y^i}{\alpha(y)}\right)\right.$$
$$\left. -\frac{y^i}{\alpha^3(y)}\left(b_j - s\frac{y^j}{\alpha(y)}\right) + \frac{2}{(1+s)\alpha^2(y)}\left(b_j - s\frac{y^j}{\alpha(y)}\right)\left(b_i - s\frac{y^i}{\alpha(y)}\right)\right]$$
$$-\frac{\frac{1}{\alpha(y)}(b_j - s\frac{y^j}{\alpha(y)})}{(1+s)^2}\langle[u,u_i]_m,u\rangle - \frac{\frac{1}{\alpha(y)}(b_i - s\frac{y^i}{\alpha(y)})}{(1+s)^2}\langle[u,u_j]_m,u\rangle.$$

Finally we have

$$2E_{ij}(o,y) = \frac{\partial^2 S(o,y)}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j}\left(-\frac{n+1}{2(1+s)\alpha(y)}\langle[u,y]_m,y\rangle\right) - \frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{n+1}{2(1+s)}\langle[u,y]_m,u\rangle\right)$$
$$= -\frac{\langle[u,y]_m,u\rangle}{(1+s)^2}\left[-\frac{1}{\alpha^3(y)}\left(b_i - s\frac{y^i}{\alpha(y)}\right)y^j - \frac{s}{\alpha^3(y)}\left(\alpha(y)\delta^i_j - y^j\frac{y^i}{\alpha(y)}\right)\right.$$
$$\left. -\frac{y^i}{\alpha^3(y)}\left(b_j - s\frac{y^j}{\alpha(y)}\right) + \frac{2}{(1+s)\alpha^2(y)}\left(b_j - s\frac{y^j}{\alpha(y)}\right)\left(b_i - s\frac{y^i}{\alpha(y)}\right)\right]$$
$$-\frac{\frac{1}{\alpha(y)}(b_j - s\frac{y^j}{\alpha(y)})}{(1+s)^2}\langle[u,u_i]_m,u\rangle - \frac{\frac{1}{\alpha(y)}(b_i - s\frac{y^i}{\alpha(y)})}{(1+s)^2}\langle[u,u_j]_m,u\rangle$$
$$+\frac{1}{(1+s)\alpha(y)}(\langle[u,u_i]_m,u_j\rangle + \langle[u,u_j]_m,u_i\rangle)$$
$$-\frac{b_j + \frac{y^j}{\alpha(y)}}{(1+s)^2\alpha^2(y)}(\langle[u,u_i]_m,y\rangle + \langle[u,y]_m,u_i\rangle)$$
$$-\frac{b_i + \frac{y^i}{\alpha(y)}}{(1+s)^2\alpha^2(y)}(\langle[u,u_j]_m,y\rangle + \langle[u,y]_m,u_j\rangle)$$
$$+\left[-\frac{\delta^j_i}{(1+s)^2\alpha^3(y)} + \frac{y^i y^j}{(1+s)^2\alpha^5(y)} + \frac{2b_i(b_j + \frac{y^j}{\alpha(y)})}{(1+s)\alpha^3(y)}\right]\langle[u,y]_m,y\rangle.$$

Acknowledgement. The present work was supported by NSFC (no. 10971104) of China.

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