On the integrability of orthogonal distributions in Poisson manifolds

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Abstract. In this article we study conditions for the integrability of the distribution defined on a regular Poisson manifold as the orthogonal complement (with respect to a pseudo-Riemannian metric) to the tangent spaces of the leaves of a symplectic foliation. Integrability criteria in terms of Casimir covectors and in terms of the Nijenhuis Tensor defined by the orthogonal distribution are obtained. Examples of integrability and non-integrability of this distribution are provided.

M.S.C. 2000: 53D17, 58D27, 53B21.
Key words: Poisson manifold; integrability; Nijenhuis tensor.

1 Introduction

Let (M^n, P) be a regular Poisson manifold. Denote by $\mathcal{S} = \{S_m | m \in M\}$ the symplectic foliation of M by symplectic leaves (of constant dimension k). Denote by $T(\mathcal{S})$ the sub-bundle of $T(M)$ of tangent spaces to the symplectic leaves (the association $x \to T_x(\mathcal{S})$ is an integrable distribution on M which we will also denote by $T(\mathcal{S})$). Let M be endowed with a pseudo-Riemannian metric g such that the restriction of g to each symplectic leaf is non-degenerate.

Let $\mathcal{N}_m = S_m^\perp$ be the subspace of $T_m(M)$ that is g-orthogonal to S_m. The association $m \to \mathcal{N}_m$ defines a distribution $\mathcal{N}$ which is transversal and complemental to the distribution $T(\mathcal{S})$. The restriction of the metric g to $\mathcal{N}$ is non-degenerate and has constant signature. In general, the distribution $\mathcal{N}$ is not integrable.

If the metric g is Riemannian, and if the Poisson tensor is parallel with respect to the Levi-Civita connection $\nabla = \nabla^g$ defined by g, ie: $\nabla P = 0$, then it is a classical result of A. Lichnerowicz ([13], Remark 3.11) that the distribution $\mathcal{N}$ is integrable, and the restriction of the metric g to the symplectic leaves defines, together with the symplectic structure $\omega_S = P|_S^{-1}$, a **Kähler structure** on symplectic leaves.

Integrability of the distribution $\mathcal{N}$ depends strongly on the foliation $\mathcal{S}$ and its "transversal topology" (see [11]). Thus, in general it is more a question of the theory of bundles with Ehresmann connections rather than that of Poisson geometry. Yet

Balkan Journal of Geometry and Its Applications, Vol.15, No.2, 2010, pp. 49-60.

in some instances, it is useful to have integrability conditions in terms of the Poisson structure P, and to relate integrability of the distribution $\mathcal{N}$ with other structures of the Poisson manifold - Casimir functions, Poisson vector fields, etc.

Our interest in this question was influenced by our study of the representation of a dynamical system in metriplectic form, i.e. as a sum of a Hamiltonian vector field (with respect to a Poisson structure, and a gradient one (with respect to a metric g), see [2, 4, 12, 1], or, [6] for a more general approach. Integrability of the distribution $\mathcal{N}$ guarantees that in the geometrical (local) splitting of the space M as a product of a symplectic leaf and a transversal submanifold with Casimir functions c^i as local coordinates (see [14]), the transversal submanifold can be chosen to be invariant under the gradient flow (with respect to the metric g) of the functions c^i.

As a result one can separate observables of the system into Casimirs undergoing pure gradient (dissipative) evolution from those (along symplectic leaves) which undergo the mix of Hamiltonian and gradient evolutions. Such a separation leads to an essential simplification of the description of the transversal dynamics in metriplectic systems.

In Section 2 we introduce necessary notions and notations. In Section 3 we obtain necessary and sufficient conditions on the metric g and the tensor P for the distribution $\mathcal{N}$ to be integrable. We derive these conditions in terms of covariant derivatives of the Poisson Tensor, in terms of covariant derivatives of Casimir covectors, and as conditions on the nullity of the Nijenhuis Torsion of the (1,1)-tensor $A^{\mu}_{\nu} = P^{\mu\kappa}g_{\kappa\nu}$. As a corollary we prove that the distribution $\mathcal{N}$ is integrable if parallel translation (via the Levi-Civita connection of the metric g) in the direction of $\mathcal{N}$ preserves the symplectic distribution $T(\mathcal{S})$.

In Section 4 we present integrability conditions in Darboux-Weinstein coordinates: the distribution $\mathcal{N}$ is integrable if and only if the following symmetry conditions are fulfilled for Γ: $\Gamma_{JIs} = \Gamma_{JIs}$, where $\Gamma_{\alpha\beta\gamma} = g_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma}$, and where capital Latin letters I, J indicate the transversal coordinates ,while small Latin letters indicate coordinates along symplectic leaves.

In Section 5 we describe an example of non-integrability of $\mathcal{N}$ and refer to the integrability of $\mathcal{N}$ for linear Poisson structures on dual spaces $\mathfrak{g}^*$ of semi-simple Lie algebras $\mathfrak{g}$, with the metric induced by the Killing form, as well as for the dual $\mathfrak{se}(3)^*$ to the Lie algebra $\mathfrak{se}(3)$ of Euclidian motions in $\mathbb{R}^3$ with the simplest non-degenerate ad^*-invariant metric(s).

2 Orthogonal distribution of Poisson manifold with a pseudo-Riemannian metric

Let (M^n, P) be a regular Poisson manifold ([13]). We will be use local coordinates x^α in the domains $U \subset M$ with the corresponding local frame $\{\frac{\partial}{\partial x^\alpha}\}$ and the dual coframe dx^α. Let g be a pseudo-Riemannian metric on M as above, and let Γ denote the Levi-Civita connection associated with g. The tensor $P^{\tau\sigma}(x)$ defines a mapping

$$0 \to C(M) \to T^*(M) \xrightarrow{P} T(\mathcal{S}) \to 0$$

where $C(M) \subset T^*(M)$ is the kernel of P and $T(\mathcal{S})$ is (as defined above) the tangent distribution of the symplectic foliation $\{\mathcal{S}^k\}$. The space $C(M)$ is a sub-bundle of the cotangent bundle T^*M consisting of Casimir covectors. Locally, $C(M)$ is generated by differentials of functionally independent Casimir functions $c^i(x)$, $i = 1, \ldots, n-k$ satisfying the condition $P^{\tau\sigma} dc^i_\sigma = 0$ (in this paper we assume the condition of summation by repeated indices).

We denote by $\mathcal{N}$ the **distribution defined as the g-orthogonal complement $T(\mathcal{S})^\perp$ to $T(\mathcal{S})$ in $T(M)$**. Then we have, at every point x a decomposition into a direct sum of distributions (sub-bundles)

$$T_x M = T_x(\mathcal{S}) \oplus \mathcal{N}_x.$$

The assignment $x \to \mathcal{N}_x$ defines a **transverse connection** for the foliation $\mathcal{S}$, or, more exactly, for the bundle $(M, \pi, M/\mathcal{S})$ over the **space of leaves** $M/\mathcal{S}$, whenever one is defined (see below). We are interested in finding necessary and sufficient conditions on P and g under which the distribution $\mathcal{N}$ is integrable. By the Frobenius theorem, integrability of $\mathcal{N}$ is equivalent to the involutivity of the distribution $\mathcal{N}$ with respect to the Lie bracket of $\mathcal{N}$-valued vector fields (sections of the sub-bundle $\mathcal{N} \subset T(M)$).

Let $\omega^i = \omega^i_\mu dx^\mu$ ($i \leq d = n\text{-}k$) be a local basis for $C(M)$. For any α in $T^*(M)$, let $\alpha^\sharp$ denote the image of α under the bundle isomorphism $\sharp : T^*(M) \to TM$ of index lifting induced by the metric g. The inverse isomorphism (index lowering) will be conventionally denoted by $\flat : T(M) \to T^*M$. We introduce the following vector fields: $\xi_i = (\omega^i)^\sharp \in T(M)$.

Lemma 1. *The vectors ξ_i form a (local) basis for $\mathcal{N}$.*

Proof. Since g is non-degenerate, the vectors ξ_i are linearly independent and span a subspace of TM of dimension d. For any vector $\eta \in TM$,

$$\langle \xi_i, \eta \rangle_g \;=\; g_{\mu\nu}\xi^\mu_i \eta^\nu \;=\; g_{\mu\nu} g^{\mu\lambda}\omega^i_\lambda \eta^\nu \;=\; \omega^i_\nu \eta^\nu \;=\; \omega^i(\eta).$$

So the vector η is g-orthogonal to all ξ_i if and only if η is annihilated by each ω^i. That is, $\eta \in Ann(C(M)) = \{\lambda \in T(M) \,|\, \omega^j(\lambda) = 0,\ \forall\, j \leq d\}$. Since $Ann(C(M)) = T(S)$, we see that the linear span of $\{\xi_i\}^\perp$ is $T(\mathcal{S})$. □

Recall (see [10]) that the **curvature** (Frobenius Tensor) of the "transversal connection" $\mathcal{N}$ is the bilinear mapping $\mathfrak{R}_\mathcal{N} : T(M) \times T(M) \to T(\mathcal{S})$ defined as

$$\mathfrak{R}_\mathcal{N}(\gamma, \eta) = v([h\gamma, h\eta]), \tag{2.1}$$

where $h : T(M) \to \mathcal{N}$ is g-orthogonal projection onto $\mathcal{N}$, and $v : T(M) \to T(\mathcal{S})$ is g-orthogonal projection onto $T(\mathcal{S})$.

It is known (see [9]) that $\mathcal{N}$ is integrable if and only if the curvature $\mathfrak{R}_\mathcal{N}$ defined above is identically zero on $TM \times TM$.

3 Integrability criteria

Condition that the curvature (2.1) is identically zero is equivalent to $v([\gamma, \eta]) = 0$, for all $\gamma, \eta \in \mathcal{N}$. If we write the vectors γ, η in terms of the basis $\{\xi_i\}$, then we have

$$\begin{aligned} v([\gamma^i\xi_i, \eta^j\xi_j]) &= v\left(\gamma^i(\xi_i \cdot \eta^j)\xi_j - \eta^j(\xi_j \cdot \gamma^i)\xi_i + \gamma^i\eta^j[\xi_i, \xi_j]\right) \\ &= \gamma^i\eta^j v([\xi_i, \xi_j]), \quad \text{since } \ v(\xi_k) = 0\ \forall\, k. \end{aligned}$$

Thus $\mathfrak{R} = 0$ if and only if $v([\xi_i, \xi_j]) = 0$ for all $i, j \leq d$.

Consider the linear operator $A : T(M) \to T(M)$ defined by the (1,1)-tensor field $A^\tau_\mu = P^{\tau\sigma} g_{\sigma\mu}$. Since g is non-degenerate we have $Im A = T(\mathcal{S})$. Since each basis vector $\xi_i \in \mathcal{N}$ is of the form $\xi^\mu_i = g^{\mu\nu} \omega^i_\nu$ with $\omega^i \in ker P$, we also have

$$A^\tau_\mu \xi^\mu_i \;=\; P^{\tau\sigma} g_{\sigma\mu} g^{\mu\nu} \omega^i_\nu \;=\; P^{\tau\nu} \omega^i_\nu = 0.$$

Therefore $\mathcal{N} \subset ker A$, and by comparing dimensions we see that $\mathcal{N} = ker A$. Notice that operator A and the orthonormal projector v have the same image and kernel. We conclude that

$$\mathfrak{R} = 0 \;\Leftrightarrow\; A[\xi_i, \xi_j] = 0, \;\; \forall\, i, j \leq d.$$

We now prove the main result of this section.

Theorem 1. *Let ω^i, $0 \leq i \leq d$ be a local basis for $C(M)$ and let $(\omega^i)^\sharp = \xi_i$ be the corresponding local basis of $\mathcal{N}$. Let ∇ be the Levi-Civita covariant derivative on TM corresponding to the metric g. Then the following statements are equivalent:*

1. *The distribution $\mathcal{N}$ is integrable.*
2. *For all $i, j \leq d$, and all $\tau \leq n$,*
$$P^{\tau\sigma}(\nabla_{\xi_i} \omega^j_\sigma - \nabla_{\xi_j} \omega^i_\sigma) = 0.$$
3. *For all $i, j \leq d$, and all $\tau \leq n$,*
$$g^{\lambda\alpha}(\nabla_\lambda P)^{\tau\sigma}(\omega^i \wedge \omega^j)_{\sigma\alpha} = 0,$$
where $\nabla_\lambda = \nabla_{\partial/\partial x^\lambda}$.
4. *For all $i, j \leq d$, and all $\tau \leq n$,*
$$P^{\tau\sigma} g_{\sigma\lambda}(\nabla_{\xi_i} \xi^\lambda_j - \nabla_{\xi_j} \xi^\lambda_i) = 0.$$
5. *The sub-bundle $C(M)$ is invariant under the following skew-symmetric bracket on 1-forms generated by the bracket of vector fields:*
$$[\alpha, \beta]_g = [\alpha^\sharp, \beta^\sharp]^\flat$$
i.e. if $\alpha, \beta \in \Gamma(C(M))$, then $[\alpha, \beta]_g \in \Gamma(C(M))$.

Proof. Since the Levi-Civita connection of g is torsion-free, we know that

$$[\xi_i, \xi_j] = \nabla_{\xi_i} \xi_j - \nabla_{\xi_j} \xi_i.$$

Therefore, in a local chart (x^α),

$$A^\tau_\lambda [\xi_i, \xi_j]^\lambda = A^\tau_\lambda (\nabla_{\xi_i} \xi^\lambda_j - \nabla_{\xi_j} \xi^\lambda_i) = P^{\tau\sigma} g_{\sigma\lambda} (\nabla_{\xi_i} \xi_j - \nabla_{\xi_j} \xi_i) = \\ = P^{\tau\sigma}(\nabla_{\xi_i} \omega^j_\sigma - \nabla_{\xi_j} \omega^i_\sigma).$$

In the last step we have used the fact that lifting and lowering of indices by the metric g commutes with the covariant derivative ∇ defined by the Levi-Civita connection of g.

Recalling from the discussion before the Theorem that the integrability of the distribution $\mathcal{N}$ is equivalent to the nullity of $A[\xi_i, \xi_j]$ for all $i, j \leqq d$, we see that statements 1.,2., and 4. are equivalent.

To prove the equivalence of these statements to the statement 3. we notice that

$$\begin{aligned} P^{\tau\sigma}(\nabla_{\xi_i}\omega^j_\sigma - \nabla_{\xi_j}\omega^i_\sigma) &= P^{\tau\sigma}(\xi_i^\lambda \nabla_\lambda \omega^j_\sigma - \xi_j^\lambda \nabla_\lambda \omega^i_\sigma) = \\ &= P^{\tau\sigma} g^{\lambda\alpha}(\omega^i_\alpha \nabla_\lambda \omega^j_\sigma - \omega^j_\alpha \nabla_\lambda \omega^i_\sigma) = g^{\lambda\alpha}(\nabla_\lambda P)^{\tau\sigma}(\omega^i_\sigma \omega^j_\alpha - \omega^i_\alpha \omega^j_\sigma) = \\ &= g^{\lambda\alpha}(\nabla_\lambda P)^{\tau\sigma}(\omega^i \wedge \omega^j)_{\sigma\alpha}. \end{aligned}$$

Here, at the third step we have used the following equality

$$P^{\tau\sigma} g^{\lambda\alpha} \omega^i_\alpha \nabla_\lambda \omega^j_\sigma = -g^{\lambda\alpha}(\nabla_\lambda P)^{\tau\sigma} \omega^i_\sigma \omega^j_\alpha,$$

since $P^{\tau\sigma}\omega^j_\sigma = 0$ (similarly for the second term).

To prove the equivalence of the condition 5. with the other statements we act as follows. Let $\alpha = \alpha_i \omega^i$ and $\beta = \beta_j \omega^j$ be any two sections of sub-bundle $C(M) \subset T^*(M)$. Then $\alpha^\sharp = \sum_i \alpha_i \xi_i$ and $\beta^\sharp = \sum_i \beta_j \xi_j$. So we have

$$\begin{aligned} [\alpha, \beta]_g &= \nabla_{\beta_j \xi_j}(\alpha_i \omega^i) - \nabla_{\alpha_i \xi_i}(\beta_j \omega^j) \\ &= \beta_j \left[\alpha_i \nabla_{\xi_j} \omega^i + \frac{\partial \alpha_i}{\partial x^k} \xi_j^k \omega^i \right] - \alpha_i \left[\beta_j \nabla_{\xi_i} \omega^j + \frac{\partial \beta_j}{\partial x^k} \xi_i^k \omega^j \right] \\ &= \alpha_i \beta_j (\nabla_{\xi_j} \omega^i - \nabla_{\xi_i} \omega^j) + \beta^\sharp(\alpha_i) \omega^i - \alpha^\sharp(\beta_j) \omega^j \\ &= \alpha_i \beta_j (\nabla_{\xi_j} \omega^i - \nabla_{\xi_i} \omega^j) + (\sum_j \beta_j \xi_j^k) \frac{\partial \alpha_i}{\partial x^k} \omega^i - (\sum_i \alpha_i \xi_i^k) \frac{\partial \beta_j}{\partial x^k} \omega^j \\ &= \alpha_i \beta_j [\omega^i, \omega^j]_g + (\beta^\sharp(\alpha_i) - \alpha^\sharp(\beta_i)) \omega^i. \end{aligned} \tag{3.1}$$

At the last step we have used the following (recall that $\omega^i = (\xi_i)^\flat$)

$$\nabla_{\xi_j}\omega^i - \nabla_{\xi_i}\omega^j = (\nabla_{\xi_j}\xi_i - \nabla_{\xi_i}\xi_j)^\flat = [\xi_i, \xi_j]^\flat = [\omega^{i\,\sharp}, \omega^{j\,\sharp}]^\flat = [\omega^i, \omega^j]_g.$$

The second term in the right side of (3.1) is always in the kernel $C(M)$ of P, thus applying P to both sides yields

$$P^{\tau\sigma}([\alpha, \beta]_g)_\sigma = \alpha_i \beta_j P^{\tau\sigma}([\omega^i, \omega^j]_g)_\sigma = \alpha_i \beta_j P^{\tau\sigma} g_{\sigma\lambda}(\nabla_{\xi_i}\xi_j^\lambda - \nabla_{\xi_j}\xi_i^\lambda).$$

Therefore, condition 4. above holds if and only if the space of sections of the bundle $C(M)$ of Casimir covectors is invariant under the bracket $[-,-]_g$. □

Corollary 1. *If $(\nabla_{(\omega^i)^\sharp}P)^{\tau\sigma}\omega^j_\sigma = 0$ for all σ, i and j, i.e. if $\nabla_{(\omega^i)^\sharp}P|_{C(M)} = 0$ for all i, then the distribution $\mathcal{N}$ is integrable.*

Proof. In the proof of the equivalence of statements 1. and 2. with statement 3. in the Theorem, it was shown that

$$\begin{aligned} P^{\tau\sigma}(\nabla_{\xi_i}\omega^j_\sigma - \nabla_{\xi_j}\omega^i_\sigma) &= g^{\lambda\alpha}(\nabla_\lambda P)^{\tau\sigma}(\omega^i_\alpha \omega^j_\sigma - \omega^j_\alpha \omega^i_\sigma) = \\ &= g^{\lambda\alpha}\omega^i_\alpha(\nabla_\lambda P)^{\tau\sigma}\omega^j_\sigma - g^{\lambda\alpha}\omega^j_\alpha(\nabla_\lambda P)^{\tau\sigma}\omega^i_\sigma = \xi_i^\lambda(\nabla_\lambda P)^{\tau\sigma}\omega^j_\sigma - \xi_j^\lambda(\nabla_\lambda P)^{\tau\sigma}\omega^i_\sigma = \\ &= (\nabla_{\xi_i}P)^{\tau\sigma}\omega^j_\sigma - (\nabla_{\xi_j}P)^{\tau\sigma}\omega^i_\sigma. \end{aligned}$$

Since $\xi_i = (\omega^i)^\sharp$ for each i, if $\nabla_{(\omega^i)^\sharp} P|_{C(M)} = 0$, then condition 3. of the Theorem is fulfilled. □

The following criteria specify the part of the A. Lichnerowicz condition that P is g-parallel (see [13]) ensuring the integrability of the distribution $\mathcal{N}$:

Corollary 2. *If* $\nabla_{\alpha^\sharp} : T(M) \to T(M)$ *preserves the tangent sub-bundle* $T(\mathcal{S})$ *to the symplectic leaves for every* $\alpha \in C(M)$*, then* $\mathcal{N}$ *is integrable.*

Proof. If the parallel translation $\nabla_{\alpha^\sharp}$ along the trajectories of the vector field $\xi = \alpha^\sharp$ preserves $T(\mathcal{S})$, then it also preserves its g-orthogonal complement $\mathcal{N}$, and hence the dual to parallel translation in the cotangent bundle will preserve sub-bundle $C(M) = \mathcal{N}^\flat$ (see Lemma 1). That is,

$$P^{\tau\sigma} \nabla_{\alpha^\sharp} \beta_\sigma = 0$$

for any β in $C(M)$. Writing this equality in the form $(\nabla_{\alpha^\sharp} P)^{\tau\sigma} \beta_\sigma = 0$ and using the previous Corollary we get the result. □

Remark 1. *Lichnerowicz's condition, i.e. the requirement that* $\nabla P = 0$*, guarantees much more than the integrability of the distribution* $\mathcal{N}$ *and, therefore, the local splitting of* M *into a product of a symplectic leaf* S *and complemental manifold* N *with zero Poisson tensor. It also guarantees regularity of the Poisson structure, and reduction of the metric* g *to the block diagonal form* $g = g_S + g_{\mathcal{N}}$*, with the corresponding metrics on the symplectic leaves and maximal integral manifolds* N_m *of* $\mathcal{N}$ *being independent on the complemental variables (i.e. the metric* g_S *on the symplectic leaves is independent from the coordinates* y *along* N_m*). Furthermore, the condition* $\nabla P = 0$ *also ensures the independence of the symplectic form* ω_S *from the transversal coordinates* y *(see [13], Remark 3.11). Finally from* $\nabla^{g_S} \omega_S = 0$ *follows the existence of a* g_S*-parallel Kahler metric on the symplectic leaves.*

Corollary 3. *Let* $\nabla_\lambda \omega^i = 0$ *for all* λ, i *(i.e. the 1-forms* $\omega^i = dc^i$ *are* ∇^g*-covariant constant). Then*

i) The distribution $\mathcal{N}$ *is integrable,*

ii) the vector fields ξ_i *are Killing vector fields of the metric* g*, and*

iii) the Casimir functions c^i *are harmonic:* $\Delta_g c^i = 0$.

Proof. The first statement is a special case of condition 3. in the Theorem above. To prove the second, we calculate the Lie derivative of g in terms of the covariant derivative $\nabla \omega^i$,

(3.2)

$$(\mathfrak{L}_{\xi_i} g)_{\sigma\lambda} = g_{\gamma\lambda} \nabla_\sigma \xi_i^\gamma + g_{\sigma\gamma} \nabla_\lambda \xi_i^\gamma = \nabla_\sigma \omega^i_\lambda + \nabla_\lambda \omega^i_\sigma = \frac{\partial \omega^i_\lambda}{\partial x^\sigma} + \frac{\partial \omega^i_\sigma}{\partial x^\lambda} - \omega^i_\gamma (\Gamma^\gamma_{\sigma\lambda} + \Gamma^\gamma_{\sigma\lambda}) =$$
$$= \frac{\partial^2 c^i}{\partial x^\sigma x^\lambda} + \frac{\partial c^i}{\partial x^\lambda x^\sigma} - 2\omega^i_\gamma \Gamma^\gamma_{\sigma\lambda} = 2\frac{\partial^2 c^i}{\partial x^\sigma x^\lambda} - 2\omega^i_\gamma \Gamma^\gamma_{\sigma\lambda} = 2\nabla_\lambda \omega^i_\sigma.$$

Thus, if the condition of the Corollary is fulfilled, ξ_i are Killing vector fields. The third statement follows from

$$\Delta_g c^i = div_g(\xi_i = (dc^i)^\sharp) = \frac{1}{2}Tr_g(\mathcal{L}_{\xi_i}g) = \frac{1}{2}g^{\lambda\mu}(\mathcal{L}_{\xi_i}g)_{\lambda\mu}.$$

□

3.1 Nijenhuis Tensor

Conventionally the integrability of different geometrical structures presented by a $(1,1)$-tensor field can be characterized in terms of the corresponding Nijenhuis tensor. Thus, it is interesting to see the relation of our criteria presented above to the nullity of the corresponding Nijenhuis tensor.

Definition 1. *Given any* $(1,1)$ *tensor field* J *on* M*, there exists a tensor field* N_J *of type* $(1,2)$ *(called the Nijenhuis torsion of* J*) defined as follows (see [9], Sec.1.10):*

$$N_J(\xi,\eta) = [J\xi, J\eta] - J[J\xi,\eta] - J[\xi, J\eta] + J^2[\xi,\eta]$$

for all vector fields ξ,η*.*

If J is an *almost product structure*, i.e. $J^2 = Id$, then $N_J = 0$ is equivalent to the integrability of J. In fact, given such a structure on M, we can define projectors $v = (1/2)(Id + J)$ and $h = (1/2)(Id - J)$ onto complementary distributions $Im(v)$ and $Im(h)$ in TM such that at each point $x \in M$,

$$T_xM = Im(v)_x \oplus Im(h)_x.$$

It is known (see [9], Sec.3.1) that J is integrable if and only if $Im(v)$ and $Im(h)$ are integrable, and that the following equivalences hold:

$$N_J = 0 \leftrightarrow N_h = 0 \leftrightarrow N_v = 0.$$

Consider now the two complementary distributions $T(\mathcal{S})$ and $\mathcal{N}$ discussed above. Suppose that v is g-orthogonal projection onto the distribution $T(\mathcal{S})$, and h is g-orthogonal projection onto $\mathcal{N}$. Applying these results in this setting we see that that the distribution $\mathcal{N}$ is integrable if and only if $N_v = 0$.

Since $v^2 = v$, and since any $\xi \in T(M)$ can be expressed as $\xi = v\xi + h\xi$, we have

$$\begin{aligned} N_v(\xi,\eta) &= [v\xi, v\eta] - v[v\xi, v\eta + h\eta] - v[v\xi + h\gamma, v\eta] + v[v\gamma + h\gamma, v\eta + h\eta] \\ &= (Id - v)[v\xi, v\eta] + v[h\gamma, h\eta] = h[v\xi, v\eta] + v[h\xi, h\eta] \end{aligned}$$

for all ξ and η in $T(M)$. Since $T(\mathcal{S})$ is integrable we have $[v\xi, v\eta] \in T(\mathcal{S})$, and so

$$N_v(\xi,\eta) = v[h\xi, h\eta].$$

As a result, we can restrict ξ and η to be sections of the distribution $\mathcal{N}$ to get the following integrability condition for $\mathcal{N}$ in terms of (1,1)-tensor v:

$$\mathcal{N} \text{ is integrable } \leftrightarrow \; N_v(\xi,\eta)^\mu = v^\mu_\nu[\xi,\eta]^\nu \overset{*}{=} -\partial_j v^\mu_\nu(\xi\wedge\eta)^{j\nu} = 0,$$

for all μ and all $\xi, \eta \in \Gamma(\mathcal{N})$. The equality $(*)$ on the right is proved in the same way as the similar result for the action of $P^{\tau\sigma}$ in the proof of statement (3) of Theorem 1.

The tensor $A^\mu_\nu = g_{\nu\sigma}P^{\sigma\mu}$ discussed above can be considered to be a linear mapping from $T(M)$ to $T(\mathcal{S})$, but since A is not idempotent, it does not define a projection. However, the tensors A and v, having the same kernel and image are related in the sense that the integrability of $\mathcal{N}$ is also equivalent to

$$A^\mu_\nu[\xi,\eta]^\nu = -\partial_\sigma A^\mu_\nu(\xi\wedge\eta)^{\sigma\nu} = 0,$$

for all sections ξ and η of the distribution $\mathcal{N}$ (using the same argument as for the tensor v^μ_σ above).

In fact, since the linear mappings A, v of $T_m(M)$ have the same kernel and image for all $m \in M$, there exists a (non-unique) pure gauge automorphism $D : T(M) \to T(M)$ of the tangent bundle (i.e. inducing the identity mapping of the base M and, therefore, defined by a smooth (1,1)-tensor field D^μ_ν) such that $A^\mu_\sigma = D^\mu_\nu v^\nu_\sigma$. For any couple ξ and η of sections from $\Gamma(\mathcal{N})$, we have

$$\begin{aligned} A^\mu_\sigma[\xi,\eta]^\sigma &= -\partial_\nu A^\mu_\sigma(\xi\wedge\eta)^{\nu\sigma} = -\partial_\nu D^\mu_\kappa(v^\kappa_\sigma(\xi\wedge\eta)^{\nu\sigma} + D^\mu_\kappa\partial_\nu v^\kappa_\sigma(\xi\wedge\eta)^{\nu\sigma} = \\ &= -D^\mu_\kappa\partial_\nu v^\kappa_\sigma(\xi\wedge\eta)^{\nu\sigma} = D^\mu_\kappa N_v(\xi,\eta). \end{aligned}$$

This proves

Theorem 2. *There exists a (not unique) invertible linear automorphism D of the bundle $T(M)$ such that for all couples of vector fields $\xi, \eta \in \Gamma(\mathcal{N})$*

$$A[\xi,\eta] = D(N_v(\xi,\eta)).$$

Thus, $N_v|_{\mathcal{N}\times\mathcal{N}} \equiv 0$ iff $A[\xi,\eta] = 0$ for all $\xi,\eta \in \Gamma(\mathcal{N})$.

4 Local criteria for integrability

Since M is regular, any point in M has a neighborhood in which the Poisson tensor P has, in Darboux-Weinstein (DW) coordinates (y^A, x^i), the following canonical form (see [14])

$$P = \begin{pmatrix} 0_{p\times p} & 0_{p\times 2k} \\ 0_{2k\times p} & \begin{pmatrix} 0_k & -I_k \\ I_k & 0_k \end{pmatrix} \end{pmatrix}$$

We will use Greek indices λ, μ, τ for general local coordinates, small Latin i, j, k for the canonical coordinates along symplectic leaves and capital Latin indices A, B, C for transversal coordinates. In these DW-coordinates we have, since P is constant,

$$(\nabla_\lambda P)^{\tau\sigma} = P^{j\sigma}\Gamma^\tau_{j\lambda} - P^{j\tau}\Gamma^\sigma_{j\lambda}.$$

Using the structure of the Poisson tensor we get, in matrix form,

$$(\nabla_\lambda P)^{\tau\sigma} = \begin{pmatrix} 0_{p\times p} & P^{js}\Gamma^T_{j\lambda} \\ -P^{it}\Gamma^s_{j\lambda} & P^{js}\Gamma^t_{j\lambda} - P^{jt}\Gamma^s_{j\lambda} \end{pmatrix},$$

where the index τ takes values (T, t), and the index σ takes values (S, s), transversally and along the symplectic leaf respectively.

In DW-coordinates we choose $\omega^\tau = dy^\tau$ as a basis for the co-distribution $C(M)$. Now we calculate (using the symmetry of the Levi-Civita connection Γ)

$$(\nabla_\lambda P)^{\tau\sigma}(dy^I \wedge dy^J)_{\alpha\sigma} = -\delta^I_\alpha P^{j\tau}\Gamma^J_{j\lambda} + \delta^J_\alpha P^{j\tau}\Gamma^I_{j\lambda},$$

so that

$$g^{\lambda\alpha}(\nabla_\lambda P)^{\tau\sigma}(dy^I \wedge dy^J)_{\alpha\sigma} = P^{j\tau}[g^{J\lambda}\Gamma^J_{j\lambda} - g^{I\lambda}\Gamma^I_{j\lambda}].$$

This expression is zero if $\tau = T$, so the summation goes by $\tau = t$ only.

Substituting the Poisson Tensor in its canonical form we get the integrability criteria 3. of Theorem 1 in the form

$$g^{J\lambda}\Gamma^I_{\lambda t} - g^{I\lambda}\Gamma^J_{\lambda t} = 0, \ \forall\ I, J, t.$$

Using the metric g to lower indices, we finish the proof of the following

Theorem 3. *Let (y^I, x^i) be local DW-coordinates in M. Use capital Latin indices for transversal coordinates y along $\mathcal{N}$ and small Latin indices for coordinates x along symplectic leaves. Then the distribution $\mathcal{N}$ is integrable if and only if $\Gamma_{JIt} = \Gamma_{IJt},\ \forall\ I, J, t$.*

5 Examples

5.1. 4-dim example of non-integrable N. Here we construct an (example of the lowest possible dimension where the distribution $\mathcal{N}_g$ is not integrable. $(M = \mathbb{R}^4, P)$ will be a 4-d Poisson manifold and $rank(P) = 2$ at all points of M.

Let P^{ij} be the Poisson tensor given in the **global** coordinates x^α by the following 4×4 matrix: $P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. Let $\omega^1 = dx^1$, and $\omega^2 = dx^2$. Then $\{\omega^1, \omega^2\}$ is a basis for the kernel $C(M)$ of P, and

$$(\omega^1 \wedge \omega^2)_{\alpha\sigma} = \begin{cases} 1, & \alpha = 1,\ \sigma = 2 \\ -1, & \alpha = 2,\ \sigma = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

Let now g be an arbitrary pseudo-Riemannian metric defined on $M = \mathbb{R}^4$ by a non-degenerate symmetric (0,2)-tensor $g_{\lambda\mu}$. The corresponding g-orthogonal distribution is denoted by $\mathcal{N}$ and the Levi-Civita connection of the metric g by ∇.

Consider $\nabla_\lambda P^{\tau\sigma} = \partial_\lambda P^{\tau\sigma} + P^{\tau\mu}\Gamma^\sigma_{\lambda\mu} + P^{\sigma\mu}\Gamma^\tau_{\lambda\mu}$. Since P is constant, the first term of this expression is always zero. Furthermore, since each ω^k is in the kernel of P, we see that the third term in this expression will contract to zero with $(\omega^1 \wedge \omega^2)_{\alpha\sigma}$. Therefore, using (5.1) we get

$$g^{\lambda\alpha}\nabla_\lambda P^{\tau\sigma}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = g^{\lambda\alpha}P^{\tau\mu}\Gamma^\sigma_{\lambda\mu}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = g^{\lambda 1}P^{\tau\mu}\Gamma^2_{\lambda\mu} - g^{\lambda 2}P^{\tau\mu}\Gamma^1_{\lambda\mu}.$$

The only values of τ for which $P^{\tau\mu} \neq 0$ are $\tau = 3$ and $\tau = 4$. We consider each case individually:

$$\tau = 3 : g^{\lambda\alpha}\nabla_\lambda P^{\tau\sigma}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = g^{\lambda 1}P^{34}\Gamma^2_{\lambda 4} - g^{\lambda 2}P^{34}\Gamma^1_{\lambda 4} =$$

$$= g^{\lambda 1}\Gamma^2_{\lambda 4} - g^{\lambda 2}\Gamma^1_{\lambda 4} = \frac{1}{2}(g^{\lambda 1}g^{2\delta} - g^{\lambda 2}g^{1\delta})(g_{\lambda\delta,4} + g_{4\delta,\lambda} - g_{\lambda 4,\delta}) = g^{\lambda 1}g^{2\delta}(g_{4\delta,\lambda} - g_{4\lambda,\delta}).$$

$$\tau = 4 : g^{\lambda\alpha}\nabla_\lambda P^{\tau\sigma}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = g^{\lambda 1}P^{43}\Gamma^2_{\lambda 3} - g^{\lambda 2}P^{43}\Gamma^1_{\lambda 3} = -g^{\lambda 1}\Gamma^2_{\lambda 3} + g^{\lambda 2}\Gamma^1_{\lambda 3} =$$

$$= \frac{1}{2}(-g^{\lambda 1}g^{2\delta} + g^{\lambda 2}g^{1\delta})(g_{\lambda\delta,3} + g_{3\delta,\lambda} - g_{\lambda 3,\delta}) = g^{\lambda 1}g^{2\delta}(g_{3\lambda,\delta} - g_{3\delta,\lambda}).$$

Thus, the integrability condition takes the form of the following system of equations

$$\begin{cases} g^{\lambda 1}g^{2\delta}(g_{3\lambda,\delta} - g_{3\delta,\lambda}) = 0, \\ g^{\lambda 1}g^{2\delta}(g_{4\delta,\lambda} - g_{4\lambda,\delta}) = 0. \end{cases} \tag{5.2}$$

equivalent to the symmetry conditions given in Theorem 3. Both expressions in (5.2) are zero if g is diagonal or block-diagonal with 2x2 matrix blocks. For these types of metric, the transversal distribution $\mathcal{N}$ is integrable.

On the other hand, let $g = \begin{pmatrix} 1 & 0 & f & 0 \\ 0 & 1 & 0 & 0 \\ f & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where $f(x)$ satisfies to the condition $\partial_2 f \neq 0$. This symmetrical matrix has $1, 1, 1+f, 1-f$ as its eigenvalues. Thus g determines the Riemannian metric in the region $|f| < 1$, and the second of the conditions (9): $g^{\lambda\alpha}\nabla_\lambda P^{\tau\sigma}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = 0$ fails since, for $\tau = 4$ we have:

$$g^{\lambda\alpha}\nabla_\lambda P^{4\sigma}(\omega^1 \wedge \omega^2)_{\alpha\sigma} = g^{\lambda 1}g^{2\delta}(g_{3\lambda,\delta} - g_{3\delta,\lambda}) = g^{\lambda 1}g_{3\lambda,2} = g^{11}g_{31,2} = \partial_2 f \neq 0.$$

As an example of such a function f for which both conditions (i.e. conditions $|f| < 1$ and $\partial_2 f \neq 0$) are fulfilled in the whole space $\mathbb{R}^4$ we can take the function $f(x^1, \ldots, x^4) = \frac{1}{\pi}tan^{-1}(x^2)$ where the principal branch of $tan^{-1}(x)$ is chosen (taking values between $-\pi/2$ and $\pi/2$).

We can also see that the distribution $\mathcal{N}$ is not integrable by a direct computation. Vector fields $\xi_1 = \partial_1 + f\partial_3$, $\xi_2 = \partial_2$ form the local basis vectors for $\mathcal{N}$. Their Lie bracket is $[\xi_1, \xi_2] = \partial_2 f \partial_3$, which is not in the span of $\{\xi_1, \xi_2\}$ (since $\partial_2 f \neq 0$). Thus, distribution $\mathcal{N}$ is not integrable.

Remark 2. *The very possibility to choose a global metric g such that the distribution $\mathcal{N}_g$ is integrable is determined mostly by the topological properties of the "bundle" of leaves of the symplectic foliation, i.e. the existence of a zero curvature Ehresmann connection. More specifically, one can prove the following*

Proposition 1. *Let $(M, \pi, B; (F, \omega))$ be a symplectic fibration (see [3]) with the model symplectic fiber (F, ω), base B and the total Poisson space M. If the bundle (M, π, B) is topologically non-trivial, then the Poisson manifold $(M, P = \omega_b^{-1}$ cannot be endowed with a (global) pseudo-Riemannian metric g such that the orthogonal distribution $\mathcal{N}_g$ would be integrable.*

For its proof see [5].

Thus, if we take an arbitrary nontrivial bundle over a simply-connected manifold B, it can not have a nonlinear connection of zero curvature. An example is the *tangent bundle* $(T(\mathbf{C}P(2)), \pi, \mathbf{C}P(2))$ *over* $B = \mathbf{C}P(2)$*, where the standard symplectic structure on* $B = \mathbf{C}P(2)$ *determines the (constant) symplectic structure along the fibers.*

5.2. Examples of integrability of N**.** Linear Poisson structures (Surio-Kostant-Kirillov bracket) in the dual space $\mathfrak{g}^*$ to the semi-simple Lie algebras $\mathfrak{g}$ endowed with the metric induced by the Killing form (dual Killing metric), see [8], deliver a family of examples where orthogonal (to the coadjoint orbits, see [7]) distribution N is integrable. More specifically, in the paper [5] it is shown that restriction of the Poisson structure and the dual Killing metric to the subspace of regular elements $\mathfrak{g}^*_{reg}$ has the integrable orthogonal distribution $\mathcal{N}$ in the following cases:

1. $\mathfrak{g}$-compact semi-simple Lie algebra, $M = \mathfrak{g}^*_{reg}$ with the dual Killing metric. Furthermore, via the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ using Killing metric, each connected component (Weyl Chamber) of the Lie algebra $\mathfrak{t}$ of a maximal torus $T \subset G$ is a maximal integral surface of the distribution $\mathcal{N}$ at each point x.

2. $\mathfrak{g}$-real semi-simple Lie algebra, $M = \mathfrak{g}^*_{reg}$ with the dual Killing metric. Maximal integral submanifolds of $\mathcal{N}$ are images under the identification $i_K : \mathfrak{g} \equiv \mathfrak{g}^*$ of (the regular parts of) Cartan subalgebras of $\mathfrak{g}$.

3. Let $\mathfrak{g} = e(3)$ be the Euclidian Lie algebra in dimension 3. For any choice of (constant) non-degenerate ad-invariant metric on the subspace $M = \mathfrak{g}^*_{reg}$ of 4d co-adjoint orbits of dual space $e(3)^*$, the distribution $\mathcal{N}$ is integrable.

6 Conclusion

In this work we discuss necessary and sufficient conditions for the distribution $\mathcal{N}$ on a regular Poisson manifold (M, P) defined as orthogonal complement of tangent to symplectic leaves with respect to some (pseudo-)Riemannian metric g on M to be integrable. We present these conditions in different forms, including a condition in terms of a symmetry of Christoffel coefficients of the Levi-Civita connection of the metric g and get some Corollaries, one of which specifies the part of the Lichnerowicz ($\nabla P = 0$) condition ensuring integrability of $\mathcal{N}$ (see [13], 3.11). We present examples of non-integrable $\mathcal{N}$ (the model 4d case and the case of a nontrivial symplectic fibration). In the preprint [5] we have proved integrability of $\mathcal{N}$ on the regular part of the dual space $\mathfrak{g}^*$ of a real semi-simple Lie algebra $\mathfrak{g}$ and the same in the case of the 3d Euclidian Lie algebra $\mathfrak{e}(3)$ with a linear Poisson structure.

As the case of a symplectic fibration shows, the integrability of $\mathcal{N}$ is possible only on a topologically trivial bundle. Thus, it would be interesting to study maximal integral submanifolds of $\mathcal{N}$ in the case of nontrivial symplectic bundles. In particular, it would be interesting to get conditions on the metric g under which these maximal integral submanifolds would have maximal possible dimension.

Acknowledgment. Second author would like to express his gratitude to Professor V.Guillemin for the useful discussion during the PNGS meeting at Portland, OR.

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