Elementary work, Newton law and Euler-Lagrange equations

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Abstract. The aim of this paper is to show a geometrical connection between elementary mechanical work, Newton law and Euler-Lagrange ODEs or PDEs. The single-time case is wellknown, but the multitime case is analyzed here for the first time. Section 1 introduces the Newton law via a covariant vector or via a tensorial 1-form. Section 2 shows that the unitemporal Euler-Lagrange ODEs can be obtained from mechanical work and single-time Newton law . Section 3 describes the Noether First Integrals in the unitemporal Lagrangian dynamics. Section 4 shows that the multitemporal Euler-Lagrange PDEs can be obtained from the mechanical work and multitime Newton law . Section 5 describes the First Integrals in multitemporal anti-trace Lagrangian dynamics.

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Key words: mechanical work; Newton law; Euler-Lagrange ODEs or PDEs; submanifold.

1 Elementary work and Newton law

Let $y = (y^I)$, $I = 1, ..., N$, be an arbitrary point in R^N. In case of forces defined on R^N, the *elementary mechanical work* can be written as an 1-form $\omega = f_I(y)dy^I$. On a submanifold M of dimension n in R^N, described by the equations $y^I = y^I(x)$, $x = (x^i)$, $i = 1, ..., n$, we have $dy^I = \frac{\partial y^I}{\partial x^i}dx^i$. Consequently, it appears the pull-back

$$\omega = F_i(x)dx^i,\ F_i(x) = f_I(y(x))\frac{\partial y^I}{\partial x^i}(x).$$

Single-time Newton law . Introducing the time t, we can write the *unitemporal Newton law* on R^N as equality of 1-forms

$$f_I = m\delta_{IJ}\frac{d\dot{y}^J}{dt}.$$

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The representation of unitemporal Newton law on the submanifold M is

$$F_i = m\delta_{IJ}\frac{d\dot{y}^I}{dt}\frac{\partial y^J}{\partial x^i}. \tag{1.1}$$

Multitime Newton law . Introducing the multitime $t = (t^\alpha), \alpha = 1, ..., m$, we can write the *multitemporal (tensorial) Newton law* as equality of 1-forms

$$f_I = m\delta_{IJ}\delta^{\alpha\beta}\frac{\partial^2 y^J}{\partial t^\alpha \partial t^\beta}.$$

The representation multitemporal Newton law on the submanifold M is

$$F_i = m\delta_{IJ}\delta^{\alpha\beta}\frac{\partial^2 y^I}{\partial t^\alpha \partial t^\beta}\frac{\partial y^J}{\partial x^i}.$$

An anti-trace of the force F_i is the *Newton tensorial* 1*-form*

$$F^\sigma_{i\alpha} = m\delta_{IJ}\delta^{\sigma\beta}\frac{\partial^2 y^I}{\partial t^\alpha \partial t^\beta}\frac{\partial y^J}{\partial x^i}, \text{ with } F_i = F^\alpha_{i\alpha}. \tag{1.2}$$

2 Single-time Euler-Lagrange ODEs obtained from mechanical work

Looking at the Newton law (1.1) and using the operator $\frac{d}{dt}$, we observe the identity

$$\delta_{IJ}\frac{d\dot{y}^I}{dt}\frac{\partial y^J}{\partial x^i} = \frac{d}{dt}\left(\delta_{IJ}\dot{y}^I\frac{\partial y^J}{\partial x^i}\right) - \delta_{IJ}\dot{y}^I\frac{d}{dt}\frac{\partial y^J}{\partial x^i}$$

or otherwise

$$\delta_{IJ}\frac{d\dot{y}^I}{dt}\frac{\partial y^J}{\partial x^i} = \frac{d}{dt}\left(\delta_{IJ}\dot{y}^I\frac{\partial y^J}{\partial x^i}\right) - \delta_{IJ}\dot{y}^I\frac{\partial}{\partial x^i}\frac{dy^J}{dt}.$$

Consequently

$$\frac{F_i}{m} = \frac{d}{dt}\left(\delta_{IJ}\dot{y}^I\frac{\partial y^J}{\partial x^i}\right) - \delta_{IJ}\dot{y}^I\frac{\partial \dot{y}^J}{\partial x^i}.$$

Since $\dot{y}^I = \frac{\partial y^I}{\partial x^i}\dot{x}^i$, the Jacobian matrix satisfies $\frac{\partial y^I}{\partial x^i} = \frac{\partial \dot{y}^I}{\partial \dot{x}^i}$. Hence

$$\frac{F_i}{m} = \frac{d}{dt}\left(\delta_{IJ}\dot{y}^I\frac{\partial \dot{y}^J}{\partial \dot{x}^i}\right) - \delta_{IJ}\dot{y}^I\frac{\partial \dot{y}^J}{\partial x^i}$$

or

$$F_i = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^i}\left(\frac{m}{2}\delta_{IJ}\dot{y}^I\dot{y}^J\right)\right) - \frac{\partial}{\partial x^i}\left(\frac{m}{2}\delta_{IJ}\dot{y}^I\dot{y}^J\right).$$

If we use the kinetic energy $T = \frac{m}{2}\delta_{IJ}\dot{y}^I\dot{y}^J$, we can write

$$F_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} - \frac{\partial T}{\partial x^i}.$$

Now, we suppose that the pullback $\omega = F_i(x)dx^i$ is a completely integrable (closed) 1-form, i.e., it is associated to a conservative force. Setting $\omega = -dV = -\frac{\partial V}{\partial x^i}dx^i$, i.e.,

$F_i = -\frac{\partial V}{\partial x^i}$ and introducing the Lagrangian $L = T - V$, it follows the *Euler-Lagrange ODEs*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0,$$

whose solutions are the curves $x(t)$.

Particularly, the previous theory survive for any changing of coordinates.

Summing up, for single-time case, it appears the following

Theorem. *1) A constrained conservative movement is described by the Euler-Lagrange ODEs.*

2) For conservative systems, the Euler-Lagrange ODEs represents the invariant form of Newton law , with or without constraints.

3 First integrals in single-time Lagrangian dynamics

If $L(x(t), \dot{x}(t))$ is an autonomous Lagrangian, satisfying the regularity condition $\det\left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\right) \neq 0$ (see the *Legendrian duality*), then the Hamiltonian

$$H(x,p) = \dot{x}^i(x,p)\frac{\partial L}{\partial \dot{x}^i}(x, \dot{x}(x,p)) - L(x, \dot{x}(x,p))$$

or shortly

$$H(x,p) = p_i \dot{x}^i(x,p) - L(x,p)$$

is a first integral both for Euler-Lagrange and Hamilton equations. Which chances we have to find new first integrals?

Noether Theorem *Let $T(t,x)$ be the flow generated by the C^1 vector field $X(x) = (X^i(x))$. If the autonomous Lagrangian L is invariant under this flow, then the function*

$$I(x,\dot{x}) = \frac{\partial L}{\partial \dot{x}^i}(x,\dot{x})X^i(x)$$

is a first integral of the movement generated by the Lagrangian L.

Proof. We denote $x_s(t) = T(s, x(t))$. The invariance of L means

$$0 = \frac{dL}{ds}(x_s(t), \dot{x}_s(t))|_{s=0} = \frac{\partial L}{\partial \dot{x}^i}(x(t),\dot{x}(t))\frac{\partial X^i}{\partial x^j}(x(t))\dot{x}^j(t) + \frac{\partial L}{\partial x^i}(x(t),\dot{x}(t))X^i(x(t)).$$

Consequently, by the derivation formulas and by the Euler-Lagrange equations, we find

$$\frac{dI}{dt}(x(t),\dot{x}(t)) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}(x,\dot{x})\right)X^i(x) + \frac{\partial L}{\partial \dot{x}^i}(x,\dot{x})\frac{\partial X^i}{\partial x^j}(x)\dot{x}^j$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}(x,\dot{x}) - \frac{\partial L}{\partial x^i}(x,\dot{x})\right)X^i(x) = 0.$$

In this way, the function $I(x,\dot{x})$ is a first integral.

4 Multitime Euler-Lagrange PDEs obtained from the mechanical work

We start from the Newton law (1.2). Now we use the identity

$$\delta_{IJ}\delta^{\sigma\beta}\frac{\partial^2 y^I}{\partial t^\alpha \partial t^\beta}\frac{\partial y^J}{\partial x^i} = D_\alpha\left(\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial x^i}\right) - \delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}D_\alpha\left(\frac{\partial y^J}{\partial x^i}\right)$$

or otherwise

$$\frac{1}{m}F^\sigma_{i\alpha} = D_\alpha\left(\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial x^i}\right) - \delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial}{\partial x^i}\left(\frac{\partial y^J}{\partial t^\alpha}\right).$$

Since

$$\frac{\partial y^I}{\partial t^\gamma} = \frac{\partial y^I}{\partial x^i}\frac{\partial x^i}{\partial t^\gamma},$$

the Jacobian matrix satisfies

$$\frac{\partial y^I_\gamma}{\partial x^i_\lambda} = \frac{\partial y^I}{\partial x^i}\delta^\lambda_\gamma.$$

It follows

$$F^\sigma_{i\alpha}\delta^\lambda_\gamma = D_\alpha\left(m\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial x^i}\delta^\lambda_\gamma\right) - m\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial}{\partial x^i}\left(\frac{\partial y^J}{\partial t^\alpha}\delta^\lambda_\gamma\right)$$

or

$$F^\sigma_{i\alpha}\delta^\lambda_\gamma = D_\alpha\left(m\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J_\gamma}{\partial x^i_\lambda}\delta^\lambda_\gamma\right) - \frac{\partial}{\partial x^i}\left(\frac{m}{2}\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial t^\alpha}\delta^\lambda_\gamma\right)$$

or

$$F^\sigma_{i\alpha}\delta^\lambda_\gamma = D_\alpha\left(\frac{\partial}{\partial x^i_\lambda}\left(\frac{m}{2}\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial t^\gamma}\right)\right) - \frac{\partial}{\partial x^i}\left(\frac{m}{2}\delta_{IJ}\delta^{\sigma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial t^\alpha}\delta^\lambda_\gamma\right).$$

Contracting λ with α and σ with γ, we find

$$F^\alpha_{i\alpha} = F_i = D_\alpha\left(\frac{\partial}{\partial x^i_\alpha}\left(\frac{m}{2}\delta_{IJ}\delta^{\gamma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial t^\gamma}\right)\right) - \frac{\partial}{\partial x^i}\left(\frac{m}{2}\delta_{IJ}\delta^{\gamma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial t^\gamma}\right).$$

If we use the multitemporal kinetic energy

$$T = \frac{m}{2}\delta_{IJ}\delta^{\gamma\beta}\frac{\partial y^I}{\partial t^\beta}\frac{\partial y^J}{\partial t^\gamma},$$

then we can write

$$F_i = D_\alpha\frac{\partial T}{\partial x^i_\alpha} - \frac{\partial T}{\partial x^i}.$$

Now, we suppose that the pullback $\omega = F_i(x)dx^i$ is a completely integrable (closed) 1-form, i.e., it is associated to a conservative force. Setting $\omega = -dV = -\frac{\partial V}{\partial x^i}dx^i$, i.e., $F_i = -\frac{\partial V}{\partial x^i}$ and introducing the Lagrangian $L = T - V$, it follows the *multitemporal Euler-Lagrange PDEs*

$$D_\alpha\frac{\partial L}{\partial x^i_\alpha} - \frac{\partial L}{\partial x^i} = 0,$$

whose solutions are m-sheets $x(t)$.

Particularly, the previous theory survive for any changing of coordinates.

Summing up, for multitime case, it appears the following

Theorem. *1) A constrained conservative movement is described by the Euler-Lagrange PDEs.*

2) For conservative systems, the Euler-Lagrange PDEs represents the invariant form of Newton law , with or without constraints.

5 First integrals in multitime Lagrangian dynamics

An autonomous multitime Lagrangian is a function of the form $L(x, x_\gamma)$. We reconsider the *multitime anti-trace Euler-Lagrange PDEs* ([4], [5])

$$\frac{\partial L}{\partial x^i}\delta^\gamma_\beta - D_\beta \frac{\partial L}{\partial x^i_\gamma} = 0, \qquad (At-E-L)$$

in order to introduce *multitemporal anti-trace Hamilton PDEs.* Starting from the Lagrangian $L(x, x_\gamma(x,p))$, satisfying the regularity condition

$$\det\left(\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}\right) \neq 0$$

(see the *Legendrian duality*), define the *Hamiltonian*

$$H(x,p) = x^i_\alpha(x,p)\frac{\partial L}{\partial x^i_\alpha}(x, x_\gamma(x,p)) - L(x, x_\gamma(x,p))$$

or shortly

$$H(x,p) = p^\alpha_i x^i_\alpha(x,p) - L(x,p).$$

Theorem (multitime anti-trace Hamilton PDEs) *Let $x(\cdot)$ be a solution of multitemporal anti-trace Euler-Lagrange PDEs $(At-E-L)$. Define $p(\cdot) = (p^\alpha_i(\cdot))$ via Legendrian duality. Then the pair $x(\cdot), p(\cdot))$ is a solution of multitemporal anti-trace Hamilton PDEs*

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H}{\partial p^\beta_i}(x(t),p(t)),\ \ \frac{\partial p^\alpha_i}{\partial t^\beta}(t) = -\delta^\alpha_\beta \frac{\partial H}{\partial x^i}(x(t),p(t)). \qquad (At-H)$$

Moreover, if the Lagrangian $L(x, x_\gamma(x,p))$ is autonomous, then the Hamiltonian $H(x,p)$ is a first integral of the system (At-H).

Here we have a system of $nm(m+1)$ PDEs of first order with $n(1+m)$ unknown functions $x^i(\cdot)$, $p^\alpha_i(\cdot)$.

Proof.: We find

$$\frac{\partial}{\partial x^i}H(x,p) = -\frac{\partial}{\partial x^i}L(x, x_\gamma(x,p)).$$

By hypothesis $p_i^\alpha(t) = \frac{\partial L}{\partial x^i_\alpha}(x(t), x_\gamma(t))$ if and only if $\frac{\partial x^i}{\partial t^\alpha}(t) = x_\alpha(x(t), p(t))$. Consequently, multitemporal anti-trace Euler-Lagrange PDEs $(At - E - L)$ imply

$$\frac{\partial p_i^\alpha}{\partial t^\beta}(t) = \delta^\alpha_\beta \frac{\partial L}{\partial x^i}(x(t), x_\gamma(t))$$
$$= \delta^\alpha_\beta \frac{\partial L}{\partial x^i}(x(t), x_\gamma(x(t), p(t))) = -\delta^\alpha_\beta \frac{\partial H}{\partial x^i}(x(t), p(t)),$$

i.e., we find the multitemporal anti-trace Hamilton PDEs on the second place,

$$\frac{\partial p_i^\alpha}{\partial t^\beta}(t) = -\delta^\alpha_\beta \frac{\partial H}{\partial x^i}(x(t), p(t)).$$

Moreover, the equality $\frac{\partial H}{\partial p_i^\alpha}(x, p) = x^i_\alpha(x, p)$ produces $\frac{\partial H}{\partial p_i^\alpha}(x(t), p(t)) = x^i_\alpha(x(t), p(t))$. On the other hand, $p_i^\alpha(t) = \frac{\partial L}{\partial x^i_\alpha}(x(t), x_\gamma(t))$ and so $x_\alpha(t) = x_\alpha(x(t), p(t))$. In this way, it appears the multitemporal anti-trace Hamilton PDEs on the first place,

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H}{\partial p_i^\beta}(x(t), p(t)).$$

Since the Hamiltonian is autonomous, using multitemporal anti-trace Hamilton PDEs, we find

$$D_\gamma H = \frac{\partial H}{\partial x^i}\frac{\partial x^i}{\partial t^\gamma} + \frac{\partial H}{\partial p_i^\lambda}\frac{\partial p_i^\lambda}{\partial t^\gamma} = 0.$$

If the Lagrangian is autonomous, then the Hamiltonian is a first integral both for multitemporal anti-trace Euler-Lagrange PDEs and multitemporal anti-trace Hamilton PDEs. Which chances we have to find new first integrals?

Theorem *Let $T(t, x)$ be the m-flow generated by the C^1 vector fields $X_\alpha(x) = (X^i_\alpha(x))$. If the autonomous Lagrangian L is invariant under this flow, then the function*

$$I(x, x_\gamma) = \frac{\partial L}{\partial x^i_\beta}(x, x_\gamma) X^i_\beta(x)$$

is a first integral of the movement generated by the Lagrangian L via multitemporal anti-trace Euler-Lagrange PDEs.

Proof. We denote $x_s(t) = T(s, x(t))$. The invariance of L means

$$0 = D_\alpha L(x_s(t), x_{s\gamma}(t))|_{s=0} = \frac{\partial L}{\partial x^i_\beta}(x(t), x_\gamma(t))\frac{\partial X^i_\beta}{\partial x^j}(x(t)) x^j_\alpha(t) + \frac{\partial L}{\partial x^i}(x(t), x_\gamma(t)) X^i_\alpha(x(t)).$$

Consequently, by derivation formulas and by multitemporal anti-trace Euler-Lagrange PDEs, we find

$$D_\alpha I(x(t), x_\gamma(t)) = \left(D_\alpha \frac{\partial L}{\partial x^i_\beta}(x, x_\gamma)\right) X^i_\beta(x) + \frac{\partial L}{\partial x^i_\beta}(x, x_\gamma)\frac{\partial X^i_\beta}{\partial x^j}(x) x^j_\alpha$$
$$= \left(D_\alpha \frac{\partial L}{\partial \dot{x}^i_\beta}(x, x_\gamma) - \frac{\partial L}{\partial x^i}(x, x_\gamma)\delta^\beta_\alpha\right) X^i_\beta(x) = 0.$$

In this way, the function $I(x, x_\gamma)$ is a first integral.

6 Conclusion

The results explained in the previous sections show that the Euler-Lagrange ODEs or PDEs, for the Lagrangian $L = T - V$, can be obtained using the elementary mechanical work, Newton law and techniques from differential geometry. On the other hand, the Euler-Lagrange ODEs or PDEs are usually introduced via variational calculus [16]. It follows that the conservative Newton law is invariant representable as Euler-Lagrange equations.

Other results regarding the multitemporal Euler-Lagrange or Hamilton PDEs can be found in our papers [2]-[15].

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