L^2-preserving Schrodinger heat flow under the Ricci flow

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Abstract. In the present paper, we study the L^2-preserved Schrödinger heat flow under the Ricci flow on closed manifolds. First, we establish the global existence and the uniqueness of the solution to the heat flow under the Ricci flow. Next, we prove an elliptic type gradient estimate of smooth positive solutions to the heat flow and get a Harnack inequality.

M.S.C. 2000: 53C21.
Key words: Ricci flow; L^2-preserving Schrödinger heat flow; elliptic type gradient estimate; Harnack inequality.

1 Introduction

Non-local heat flow was studied by many people, which arises in geometry such that the flow preserves some L^p norm in the sense that some geometrical quantity is preserved in the geometric heat flow, we can refer to [1, 15, 2, 8] and the references therein.

Recently, the L^2 preserving heat flow which have positive solutions on closed manifolds was considered in [2] and [8], where they studied the global existence, the uniqueness and the gradient estimates of L^2 norm preserving heat flow such as

$$\partial_t u = \triangle u + \lambda(t) u$$

with

$$\lambda(t) = \frac{\int_M |\nabla u|^2 \ dx}{\int_M u^2 \ dx}.$$

On the other hand, in [11], Perelman showed that the first eigenvalue of the operator $-\triangle + \frac{1}{4}R$ is monotone along the Ricci flow

$$\partial_t g = -2Ric, \quad (x,t) \in [0,T) \tag{1.1}$$

coupled with the conjugate heat equation and then ruled out nontrivial steady or expanding breathers on closed manifolds. In fact, from then on, many studies on this topics appeared in [12, 3, 4, 7, 5, 14] and the reference therein.

Balkan Journal of Geometry and Its Applications, Vol.15, No.2, 2010, pp. 113-125.

In this paper, we shall consider the following L^2-preserving Schrödinger heat flow which have positive solutions on closed manifolds with the metric evolving under Ricci flow (1.1):

$$\partial_t u = \triangle u + \lambda(t)u + cRu,\ in\ M \times [0,T),$$
$$u(x,0) = h(x),\ in\ M,$$

where $c \geq 0$ is a constant, $h \in C^1(M)$, and $\lambda(t)$ is chosen such that the flow preserves the L^2-norm of the solution. By the fact that $\partial_t dx = -Rdx$ [6], a direct computation shows

$$\frac{1}{2}\frac{d}{dt}\int_M u^2\ dx = -\int_M |\nabla u|^2\ dx + \lambda \int_M u^2\ dx - (\frac{1}{2} - c)\int_M u^2 R\ dx,$$

hence

$$\lambda(t) = \frac{\int_M |\nabla u|^2\ dx + (\frac{1}{2} - c)\int_M u^2 R\ dx}{\int_M h^2\ dx}$$

can preserve the L^2-norm. Without loss of generality, we may assume that $\int_M |h|^2\ dx = 1$ and $h(x) \geq 0$. Thus we will consider the following L^2-preserving Schrödinger heat flow on a closed manifold M with the metric evolving under Ricci flow (1.1):

$$\partial_t u = \triangle u + \lambda(t)u + cRu,\ in\ M \times [0,T), \tag{1.2}$$
$$u(x,0) = h(x),\ in\ M, \tag{1.3}$$

where

$$\lambda(t) = \int_M |\nabla u|^2\ dx + (\frac{1}{2} - c)\int_M u^2 R\ dx, \tag{1.4}$$

$h(x) \geq 0$, $\int_M h^2\ dx = 1$, $h \in C^1(M)$ and T may be $+\infty$. We establish a global existence result about L^2-preserving Schrödinger heat flow.

Theorem 1.1. *We assume that the Ricci flow (1.1) has a smooth solution on* $[0,T)$*(*T *may be* $+\infty$*) and the scalar curvature of* $g(0)$ *is nonnegative. Then the equation (1.2) with initial value* $h(x) \geq 0$, $\int_M h^2(x)\ dx = 1$ *and* $h \in C^1(M)$ *has a global solution*

$$u \in L^\infty([0,T), H_t^1(M)) \cap L^2_{loc}([0,T), H_t^2(M)),$$

where $H_t^1(M), H_t^2(M)$ *denote* $H^1(M), H^2(M)$ *about the metric* $g(t)$ *respectively.*

Theorem 1.2. *We assume that the Ricci flow (1.1) has a smooth solution on* $[0,T)$*(*T *may be* $+\infty$*) and the scalar curvature of* $g(0)$ *is nonnegative. Then the solution to equation (1.2) with initial value* $h(x) \geq 0$, $\int_M h^2(x)\ dx = 1$ *and* $h \in C^1(M)$ *is unique.*

The Schrödinger heat equation

$$u_t = \triangle u + qu$$

was discussed in [12], in this paper, by some parabolic type gradient estimates, the authors got Harnack inequalities as the following type:

$$u(x_1,t_1) \leq u(x_2,t_2)(\frac{t_2}{t_1})^{\frac{n\alpha}{2}} \exp\{\frac{\alpha r^2(x_1,x_2)}{4(t_2-t_1)} + \frac{n\alpha K(t_2-t_1)}{\sqrt{2}(\alpha-1)}\}, \tag{1.5}$$

where $\alpha > 1$ is a given constant, $r(x_1, x_2)$ is the geodesic distance between x_1, x_2 and $0 < t_1 < t_2 < +\infty$.

The parabolic type gradient estimate for the positive solutions to the non-local heat flow (1.2) when $c = 0$ was studied in [8], the Harnack inequality has the same type as (1.5). But we can only compare the solutions at different times from the Harnack inequalities with the type similar to (1.5). In order to overcome this blemish, we can consider the elliptic type gradient estimate, the Harnack inequality deduced from this estimate can be used to compare the solutions at the same time. Hamilton firstly got this gradient estimate on a closed manifold in [9]; then in [13], this elliptic type gradient estimate was proved on a complete noncompact Riemannian manifold, the Harnack inequality with the following type was deduced by using this estimate:

$$u(x_1, t) \leq u(x_2, t)^{\eta(t, r(x_1, x_2))} e^{1-\eta(t, r(x_1, x_2))},$$

where $\eta(t, r(x_1, x_2))$ depends on $t, r(x_1, x_2)$.

In this paper, we also prove an elliptic type gradient estimate for solutions to equation (1.2), as an application, we get a Harnack inequality.

Theorem 1.3. *We assume that the Ricci flow (1.1) has a smooth solution on* $[0, T_0]$ *and the scalar curvature of* $g(0)$ *is nonnegative, for a constant* $G > 0$, $|\nabla\sqrt{R}| \leq G, t \in [0, T_0]$, $u(x, t)$ *is a positive smooth solution to (1.2) with* $u(x, t) \leq e^{-1}$ *for all* $(x, t) \in M \times (0, T)$. *Then*

$$|\frac{\nabla u}{u}| \leq \sqrt{\frac{1}{t} + 2\lambda(t) + \sqrt{2c}G}(1 - \log u). \tag{1.6}$$

Theorem 1.4. *We assume that the Ricci flow (1.1) has a smooth solution on* $[0, T_0]$ *and the scalar curvature of* $g(0)$ *is nonnegative, for a constant* $G > 0$, $|\nabla\sqrt{R}| \leq G, t \in [0, T_0]$, $u(x, t)$ *is a positive smooth solution to (1.2) with* $u(x, t) \leq e^{-1}$ *for all* $(x, t) \in M \times (0, T)$. *Then for* $x_1, x_2 \in M, t \in (0, T)$,

$$u(x_2, t) \leq e^{1-\eta(t, r(x_1, x_2))} u(x_1, t)^{\eta(t, r(x_1, x_2))} \tag{1.7}$$

where

$$\eta(t, r(x_1, x_2)) = e^{-\sqrt{\frac{1}{t} + 2\lambda(t) + \sqrt{2c}G}\, r(x_1, x_2)} \tag{1.8}$$

and $r(x_1, x_2)$ *denotes the geodesic distance between* x_1, x_2.

Remark We always need some conditions about the Ricci curvature when we consider the elliptic or parabolic type gradient estimate(see [8, 12, 9, 13]). But in Theorem 1.3 and Theorem 1.4, we give the elliptic type gradient estimate and the Harnack inequality without any assumption about the Ricci curvature.

2 Some estimates

We assume that the Ricci flow (1.1) has a smooth solution on $[0, T)$ with $R(x, 0) \geq 0$. In [10], Hamilton get the evolution of the scalar curvature

$$\partial_t R = \triangle R + 2|Ric|^2.$$

By the maximum principle, we deduce that

$$R(x,t) \geq 0, (x,t) \in M \times [0,T).$$

We define a series $u^{(k)}$ as

$$\begin{aligned} & u^{(0)} = h, \lambda^{(k)}(t) = \int_M (|\nabla u^{(k)}|^2 + (\frac{1}{2} - c)(u^{(k)})^2 R)\ dx, \\ & \partial_t u^{(k+1)} = \triangle u^{(k+1)} + \lambda^{(k)}(t) u^{(k+1)} + cRu^{(k+1)}, \\ & u^{(k+1)}(x,0) = h(x). \end{aligned} \tag{2.1}$$

For $k \geq 0$,

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_M (u^{(k+1)})^2\ dx + \int_M |\nabla u^{(k+1)}|^2\ dx \\ = \ & \lambda^{(k)}(t)\int_M (u^{(k+1)})^2\ dx + (c - \frac{1}{2})\int_M (u^{(k+1)})^2 R\ dx. \end{aligned} \tag{2.2}$$

By $\partial_t g^{ij} = 2g^{ik}g^{jl}R_{kl}$, for a smooth function $f(x,t)$,

$$\frac{d}{dt}\int_M |\nabla f|^2\ dx = \int_M (2\nabla f \cdot \nabla f_t + 2Ric(\nabla f, \nabla f) - |\nabla f|^2 R)\ dx,$$

so

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_M |\nabla u^{(k+1)}|^2\ dx + (\frac{1}{2} - c)\int_M |\nabla u^{(k+1)}|^2 R\ dx \\ = \ & \lambda^{(k)}\int_M |\nabla u^{(k+1)}|^2\ dx + \int_M Ric(\nabla u^{(k+1)}, \nabla u^{(k+1)})\ dx \\ & + c\int_M u^{(k+1)}\nabla R \cdot \nabla u^{(k+1)}\ dx - \int_M (\triangle u^{(k+1)})^2\ dx. \end{aligned} \tag{2.3}$$

From (2.1),

$$\begin{aligned} & \int_M (u_t^{(k+1)})^2\ dx + \frac{1}{2}\frac{d}{dt}\int_M |\nabla u^{(k+1)}|^2\ dx \\ = \ & \frac{\lambda^{(k)}}{2}\frac{d}{dt}\int_M |u^{(k+1)}|^2\ dx + \frac{\lambda^{(k)}}{2}\int_M (u^{(k+1)})^2 R\ dx \\ & -\frac{1}{2}\int_M |\nabla u^{(k+1)}|^2 R\ dx + \int_M Ric(\nabla u^{(k+1)}, \nabla u^{(k+1)})\ dx \\ & +\frac{c}{2}[\frac{d}{dt}\int_M R(u^{(k+1)})^2\ dx + \int_M R^2(u^{(k+1)})^2\ dx] \\ & -\frac{c}{2}\int_M \partial_t R(u^{(k+1)})^2\ dx. \end{aligned} \tag{2.4}$$

By the Ricci identity

$$v_{iij} = v_{iji} - R_{ij}v_i, \tag{2.5}$$

we have

$$\begin{aligned}\int_M (\triangle f)^2\ dx &= -\int_M f_{iij}f_j\ dx\\ &= -\int_M (v_{iji}v_j - R_{ij}v_iv_j)\ dx\\ &= \int_M v_{ij}^2\ dx + \int_M R_{ij}v_iv_j\ dx. \qquad (2.6)\end{aligned}$$

Note that

$$\int_M f\nabla R\cdot\nabla f\ dx = -\frac{1}{2}\int_M f^2\triangle R\ dx,$$

hence (2.3) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_M |\nabla u^{(k+1)}|^2\ dx + (\frac{1}{2}-c)\int_M |\nabla u^{(k+1)}|^2 R\ dx + \int_M |\nabla^2 u^{(k+1)}|^2\ dx\\ (2.7) = \ &\lambda^{(k)}\int_M |\nabla u^{(k+1)}|^2\ dx - \frac{c}{2}\int_M \triangle R(u^{(k+1)})^2\ dx,\end{aligned}$$

which means that

$$\begin{aligned}&\frac{d}{dt}\int_M |\nabla u^{(k+1)}|^2\ dx + c\int_M \triangle R(u^{(k+1)})^2\ dx\\ \le\ & 2\lambda^{(k)}\int_M |\nabla u^{(k+1)}|^2\ dx + 2c\int_M |\nabla u^{(k+1)}|^2 R\ dx. \qquad (2.8)\end{aligned}$$

(2.2) tells us that

$$\frac{d}{dt}\int_M (u^{(k+1)})^2\ dx \le 2\lambda^{(k)}(t)\int_M (u^{(k+1)})^2\ dx + 2c\int_M R(u^{(k+1)})^2\ dx. \qquad (2.9)$$

By (2.6) and the fact that

$$|\nabla^2 f|^2 \ge \frac{1}{n}(\triangle f)^2,$$

we get

$$\int_M Ric(\nabla f,\nabla f)\ dx \le (n-1)\int_M |\nabla^2 f|^2\ dx. \qquad (2.10)$$

By (2.4) and (2.10),

$$\begin{aligned}&\int_M (u_t^{(k+1)})^2\ dx + \frac{1}{2}\frac{d}{dt}\int_M |\nabla u^{(k+1)}|^2\ dx\\ \le\ & \frac{\lambda^{(k)}}{2}\int_M (u^{(k+1)})^2 R\ dx + (n-1)\int_M |\nabla^2 u^{(k+1)}|^2\ dx\\ &+\frac{c}{2}[\frac{d}{dt}\int_M R(u^{(k+1)})^2\ dx + \int_M R^2(u^{(k+1)})^2\ dx]\\ &-\frac{c}{2}\int_M \partial_t R(u^{(k+1)})^2\ dx + \frac{\lambda^{(k)}}{2}\frac{d}{dt}\int_M |u^{(k+1)}|^2\ dx. \qquad (2.11)\end{aligned}$$

Lemma 2.1. *We can choose a positive number* $\delta < \frac{T}{2}$, *such that for some constant* $C > 0$ *depending on* $M = \max\{R(x,t) + |\triangle R(x,t)| | (x,t) \in M \times [0, \frac{T}{2}]\}$ *and* $\int_M |\nabla h|^2 \ dx(g(0))$,

$$\int_M |u^{(k+1)}|^2 \ dx(t) \leq C, t \in [0,\delta], \tag{2.12}$$

$$\int_M |\nabla u^{(k+1)}|^2 \ dx(t) \leq C, t \in [0,\delta], \tag{2.13}$$

$$\int_0^\delta \int_M |\nabla^2 u^{(k+1)}|^2 \ dx \ dt \leq C, t \in [0,\delta], \tag{2.14}$$

$$\int_0^\delta \int_M (u_t^{(k+1)})^2 \ dx(t) \ dt \leq C, t \in [0,\delta]. \tag{2.15}$$

Proof. We firstly show that (2.14) and (2.15) can be deduced by (2.12) and (2.13). Integrate (2.7) with t on $[0,\delta]$,

$$\begin{aligned} & 2\int_0^\delta \int_M |\nabla^2 u^{(k+1)}|^2 \ dx \ dt \\ \leq \ & \int_M |\nabla h|^2 \ dx(0) + 2\int_0^\delta \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 \ dx(t) \ dt \\ & +cM \int_0^\delta \int_M (u^{(k+1)})^2 \ dx \ dt + 2cM \int_0^\delta \int_M |\nabla u^{(k+1)}|^2 \ dx(t) \ dt. \end{aligned} \tag{2.16}$$

Note the evolution of R, integrate (2.11) with t on $[0,\delta]$,

$$\begin{aligned} & 2\int_0^\delta \int_M (u_t^{(k+1)})^2 \ dx(t) \ dt \\ \leq \ & \int_M |\nabla h|^2 \ dx(0) + \int_0^\delta \lambda^{(k)}(t) \frac{d}{dt} \int_M |u^{(k+1)}|^2 \ dx(t) \ dt \\ & +M \int_0^\delta \lambda^{(k)}(t) \int_M (u^{(k+1)})^2 \ dx(t) \ dt + (n-1) \int_0^\delta \int_M |\nabla^2 u^{(k+1)}|^2 \ dx(t) \ dt \\ & +c\int_M R(x,\delta)(u^{(k+1)})^2 \ dx(\delta) + cM(M+1) \int_0^\delta \int_M (u^{(k+1)})^2 \ dx \ dt. \end{aligned} \tag{2.17}$$

By (2.12) and (2.13), after a suitable adjustment of C, we know

$$\lambda^{(k)}(t) \leq C, t \in [0,\delta]. \tag{2.18}$$

By (2.12), (2.13), (2.16), (2.17) and (2.18), after a suitable adjustment of C, we get (2.14) and (2.15).

Now we prove (2.12) and (2.13) by induction, we assume that for suitable $\delta > 0$ and C picked later, (2.12) and (2.13) are right for k. Integrate (2.9) with t, we get

$$\begin{aligned} \int_M |u^{(k+1)}|^2 \ dx \ & \leq \ \int_M |h|^2 \ dx(0) \exp\left(\int_0^t (2\lambda^{(k)}(t) + 2cM) \ dt\right) \\ & \leq \ \int_M |h|^2 \ dx(0) e^{(2C+CM+2cM)t}, t \in [0,\delta]. \end{aligned} \tag{2.19}$$

By (2.8) and (2.19), for $t \in [0,\delta]$,

$$\begin{aligned} & \frac{d}{dt}\int_M |\nabla u^{(k+1)}|^2 \; dx \\ \leq \;& M\int_M |h|^2 \; dx(0) e^{(2C+CM+2cM)t} + (2C+CM+2cM)\int_M |\nabla u^{(k+1)}|^2 \; dx, \end{aligned}$$

or

$$\frac{d}{dt}[\int_M |\nabla u^{(k+1)}|^2 \; dx e^{-(2C+CM+2cM)t} - M\int_M |h|^2 \; dx(0) t] \leq 0,$$

which means that for $t \in [0,\delta]$,

$$\int_M |\nabla u^{(k+1)}|^2 \; dx \leq \int_M (h^2 + M\delta|\nabla h|^2) \; dx(0) e^{(2C+CM+2cM)\delta}. \tag{2.20}$$

We choose $\delta > 0$ small so that for suitable constant $C > 0$,

$$\int_M (h^2 + M\delta|\nabla h|^2) \; dx(0) e^{(2C+CM+2cM)\delta} \leq C.$$

By (2.19) and (2.20), we can see (2.12) and (2.13) are right for $k+1$. □

3 Global existence

Let Γ be the maximal subset of $[0,T)$ such that (1.2) has a global solution

$$u \in L^\infty(\Gamma, H^1_t(M)) \cap L^2_{loc}(\Gamma, H^2_t(M)).$$

In this section, we shall prove that $\Gamma = [0,T)$. The proof consists of three steps, the main idea comes from [2, 8].

Step1 We will show that there exists a positive number δ depending on $\int_M |\nabla h|^2 \; dx(0)$ and M, such that $[0,\delta] \subset \Gamma$.

We use $H^1_t(M), H^2_t(M), L^2_t(M)$ to denote $H^1(M), H^2(M), L^2(M)$ of $g(t)$. By Lemma 2.1, there is a subsequence of $\{u^{(k)}\}$ (still denoted by $\{u^{(k)}\}$) and a function

$$u(x,t) \in L^\infty([0,\delta], H^1_t(M)) \cap L^2([0,\delta], H^2_t(M))$$

with $\partial_t u(x,t) \in L^2([0,\delta], L^2_t(M))$, such that $u^{(k)} \rightharpoonup u$ weak* in $L^\infty([0,\delta], H^1_t(M))$ and weakly in $L^2([0,\delta], H^2_t(M))$. Then we have $u^{(k)} \to u$ strongly in $L^2([0,\delta], H^1_t(M))$ and $u(x,t) \in C([0,\delta], L^2_t(M))$. Hence $\lambda^{(k)}(t) \to \lambda(t)$ strongly in $L^2([0,\delta])$. Thus, we get a local strong solution to equation (1.2) under the Ricci flow (1.1). So Γ is not empty.

Step2 Let $u(x,t)$ be a solution to (1.2) on $[0,t_0)$, where $t_0 \in (0,T)$, we assume that

$$Ric(x,t) \leq N(t_0) g(x,t), \quad (x,t) \in M \times [0,t_0],$$

and

$$R(x,t) \leq M(t_0), \quad (x,t) \in M \times [0,t_0],$$

for some constants $N(t_0), M(t_0)$ depending on t_0.

We can compute as

$$\begin{aligned}&\frac{d}{dt}\int_M |\nabla u|^2\ dx + 2\int_M (\triangle u)^2\ dx - 2\lambda(t)\int_M |\nabla u|^2\ dx\\ = \ & 2\int_M Ric(\nabla u, \nabla u)\ dx + (2c-1)\int_M R|\nabla u|^2\ dx - c\int_M \triangle R u^2\ dx.\end{aligned} \tag{3.1}$$

Note that

$$(\int_M |\nabla u|^2\ dx)^2 = (-\int_M u\triangle u\ dx)^2 \leq \int_M (\triangle u)^2\ dx. \tag{3.2}$$

For $t \in [0, t_0)$,

$$\frac{d}{dt}\int_M |\nabla u|^2\ dx \leq D\int_M |\nabla u|^2\ dx + cM(t_0),$$

where

$$D = (1+2c)M(t_0) + 2N(t_0),$$

or

$$\frac{d}{dt}[(\int_M |\nabla u|^2\ dx + \frac{cM(t_0)}{D})e^{-Dt}] \leq 0.$$

So for $t \in [0, t_0)$,

$$\int_M |\nabla u|^2\ dx \leq [\int_M |\nabla h|^2\ dx(g(0)) + \frac{cM(t_0)}{D}]e^{Dt} - \frac{cM(t_0)}{D}. \tag{3.3}$$

Similar to (2.7) and (2.11), we have

$$\begin{aligned}&\partial_t\int_M |\nabla u|^2\ dx + 2\int_M |\nabla^2 u|^2\ dx\\ \leq\ & 2\lambda(t)\int_M |\nabla u|^2\ dx + 2c\int_M R|\nabla u|^2\ dx - c\int_M \triangle R u^2\ dx,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\int_M (u_t)^2\ dx + \frac{1}{2}\partial_t\int_M |\nabla u|^2\ dx + \frac{c}{2}\int_M \partial_t R u^2\ dx\\ \leq\ & \frac{\lambda(t)}{2}\int_M R u^2\ dx + (n-1)\int_M |\nabla^2 u|^2\ dx\\ & + \frac{c}{2}[\frac{d}{dt}\int_M R u^2\ dx + \int_M R^2 u^2\ dx].\end{aligned} \tag{3.5}$$

By (3.3), (3.4) and (3.5), we conclude that there exists a constant C depending on

$\int_M |\nabla h|^2 \; dx(g(0))$ and t_0, such that for $t \in [0, t_0)$,

$$\int_M |\nabla u|^2 \; dx \le C, \tag{3.6}$$

$$\partial_t \int_M |\nabla u|^2 \; dx \le C, \tag{3.7}$$

$$\int_M |\nabla^2 u|^2 \; dx \le C, \tag{3.8}$$

$$\int_M (u_t)^2 \; dx \le C. \tag{3.9}$$

So for $t_0 \in (0, T)$, if $[0, t_0) \subset \Gamma$, then $[0, t_0] \subset \Gamma$.

Step3 Assume that $[0, t_0] \subset \Gamma$, from Step 2, we know that $\int_M |\nabla u|^2 \; dx(g(t_0))$ is bounded. Now follow the procedure in Step 1, we will get a local strong solution to (1.2) under the Ricci flow (1.1) on $[t_0, t_0 + \delta)$ for $\delta > 0$ small enough depending on $\int_M |\nabla u|^2 \; dx(g(t_0))$ and $\max\{R(x,t)|(x,t) \in M \times [t_0, \frac{t_0+T}{2}]\}$. Which means that $[t_0, t_0 + \delta) \subset \Gamma$.

As a subset of $[0, T)$, Γ is not empty, moreover, it is both open and closed, so we conclude that $\Gamma = [0, T)$ and Theorem 1.1 follows.

4 Uniqueness

We assume that u, v are two solutions to (1.2) with initial value h_u, h_v under the Ricci flow (1.1). Then,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_M (u-v)^2 \; dx &= \int_M [(u-v)(u_t - v_t) - \frac{1}{2}(u-v)^2 R] \; dx \\ \le \quad & -\int_M |\nabla(u-v)|^2 \; dx + \int_M (u-v)(\lambda_u(t)u - \lambda_v(t)v) \; dx \\ & +c\int_M R(u-v)^2 \; dx, \end{aligned} \tag{4.1}$$

where

$$\lambda_u(t) = \int_M |\nabla u|^2 \; dx + (\frac{1}{2} - c)\int_M u^2 R \; dx,$$

and

$$\lambda_v(t) = \int_M |\nabla v|^2 \; dx + (\frac{1}{2} - c)\int_M v^2 R \; dx.$$

Note that

$$\begin{aligned} & \int_M (u-v)(\lambda_u(t)u - \lambda_v(t)v) \; dx \\ = \quad & (\lambda_u(t) - \lambda_v(t))\int_M (u-v)u \; dx + \lambda_v(t)\int_M (u-v)^2 \; dx \\ \le \quad & |\lambda_u(t) - \lambda_v(t)|(\int_M (u-v)^2 \; dx)^{1/2} + |\lambda_v(t)|\int_M (u-v)^2 \; dx, \end{aligned} \tag{4.2}$$

and

$$
\begin{aligned}
|\lambda_u(t)-\lambda_v(t)| &= |\int_M (|\nabla u|^2-|\nabla v|^2)\ dx + (\frac{1}{2}-c)\int_M (u^2-v^2)R\ dx| \\
&\le \int_M |\nabla(u-v)|(|\nabla u|+|\nabla v|)\ dx + |\frac{1}{2}-c|\int_M (u^2-v^2)R\ dx \\
&\le (\int_M |\nabla(u-v)|^2\ dx)^{1/2}(\int_M (|\nabla u|+|\nabla v|)^2\ dx)^{1/2} \\
&\quad +|\frac{1}{2}-c|(\int_M (u-v)^2\ dx)^{1/2}(\int_M ((u+v)R)^2\ dx)^{1/2}. \qquad (4.3)
\end{aligned}
$$

From Section 2, we know that for every $t_0 \in (0,T)$, there exists a constant C depending on $\int_M |\nabla h_u|^2\ dx(g(0)), \int_M |\nabla h_v|^2\ dx(g(0))$ and t_0 such that

$$
|R(x,t)| + |\triangle R(x,t)| \le C, (x,t) \in M \times [0,t_0],
$$

$$
\int_M |\nabla u|^2\ dx \le C, \int_M |\nabla v|^2\ dx \le C, t \in [0,t_0],
$$

and

$$
|\lambda_u(t)| \le C, |\lambda_v(t)| \le C, t \in [0,t_0].
$$

Together with (4.1), (4.2) and (4.3), we get that on $[0,t_0]$,

$$
\frac{1}{2}\frac{d}{dt}\int_M (u-v)^2\ dx \le (3c+3)C\int_M (u-v)^2\ dx,
$$

and we conclude that

$$
\int_M (u-v)^2\ dx \le \int_M (h_u-h_v)^2\ dx(g(0))e^{(6c+6)Ct}, t \in [0,t_0]. \qquad (4.4)
$$

Further more,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_M |\nabla(u-v)|^2\ dx \\
\le\ & -\int_M |\nabla^2(u-v)|^2\ dx + \int_M \nabla(u-v)\cdot\nabla(\lambda_u(t)u-\lambda_v(t)v)\ dx \\
& +c\int_M R|\nabla(u-v)|^2\ dx - \frac{c}{2}\int_M \triangle R(u-v)^2\ dx, \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_M \nabla(u-v)\cdot\nabla(\lambda_u(t)u-\lambda_v(t)v)\ dx \\
=\ & (\lambda_u(t)-\lambda_v(t))\int_M \nabla(u-v)\cdot\nabla u\ dx + \lambda_v(t)\int_M |\nabla(u-v)|^2\ dx \\
\le\ & |\lambda_u(t)-\lambda_v(t)|(\int_M |\nabla u|^2\ dx)^{1/2}(\int_M \nabla(u-v)\ dx)^{1/2} \\
& +|\lambda_v(t)|\int_M |\nabla(u-v)|^2\ dx. \qquad (4.6)
\end{aligned}
$$

By (4.3) and (4.4), we know that for $t \in [0, t_0]$,

$$|\lambda_u(t) - \lambda_v(t)| \leq 2\sqrt{C}(\int_M |\nabla(u-v)|^2 \, dx)^{1/2} + (1+2c)C\sqrt{D}e^{(3c+3)Ct},$$

where

$$D = \int_M (h_u - h_v)^2 \, dx(g(0)).$$

Hence we get the following differential inequality on $[0, t_0]$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}X(t) &\leq (3+c)CX(t) + (1+2c)C\sqrt{CD}e^{(3c+3)Ct}\sqrt{X(t)} + \frac{cCD}{2}e^{(6c+6)Ct} \\ &\leq (4+c)CX(t) + [\frac{1+2c}{4}C + \frac{c}{2}]CDe^{(6c+6)Ct}, \qquad (4.7)\end{aligned}$$

where

$$X(t) = \int_M |\nabla(u-v)|^2 \, dx, t \in [0, t_0].$$

Now we begin to prove Theorem 1.2.

Proof. when $h_u = h_v$, by (4.4),

$$\int_M (u-v)^2 \, dx = 0.$$

By (4.7),

$$X(t) \leq X(0)e^{(8+2c)Ct} = 0, \quad t \in [0, \delta].$$

So $u = v, \quad t \in [0, t_0]$, Theorem 1.2 follows for the arbitrariness of $t_0 \in [0, T)$. □

5 Elliptic type gradient estimate

Maximum principle is an important tool in geometric analysis, we can refer to [12, 13, 16] and the references therein. In this section, by using maximum principle, we will prove elliptic type gradient estimate for positive solutions to (1.2).

Let u be a positive smooth solution to (1.2), $f = \log u$, then f satisfies

$$f_t = \triangle f + |\nabla f|^2 + \lambda + cR. \qquad (5.1)$$

Let

$$w = |\nabla f|^2(1-f)^{-2}. \qquad (5.2)$$

By the Ricci identity (2.5), we can compute as follows,

$$\begin{aligned}\partial_t w &= [2\nabla f \cdot \nabla\triangle f + 2\nabla f \cdot \nabla|\nabla f|^2 + 2c\nabla f \cdot \nabla R + 2Ric(\nabla f, \nabla f)](1-f)^{-2} \\ &\quad +2|\nabla f|^2(\triangle f + |\nabla f|^2 + \lambda + cR)(1-f)^{-3}, \qquad (5.3)\end{aligned}$$

$$\nabla w = \nabla|\nabla f|^2(1-f)^{-2} + 2|\nabla f|^2(1-f)^{-3}\nabla f, \qquad (5.4)$$

and

$$\begin{aligned}\triangle w &= 4\nabla f\cdot\nabla|\nabla f|^2(1-f)^{-3}+6|\nabla f|^4(1-f)^{-4}+2\triangle f|\nabla f|^2(1-f)^{-3}\\ &\quad+[2|\nabla^2 f|^2+2\nabla f\cdot\nabla\triangle f+2Ric(\nabla f,\nabla f)](1-f)^{-2}.\end{aligned}\tag{5.5}$$

By (5.2), (5.3), (5.4) and (5.5),

$$\begin{aligned}(\partial_t-\triangle)w &= 2fw^2+(2-4(1-f)^{-1})\nabla f\cdot\nabla w+2c\nabla f\cdot\nabla R(1-f)^{-2}\\ &\quad+2\lambda(1-f)^{-1}w-2|\nabla^2 f|^2(1-f)^{-2}+2cRw(1-f)^{-1},\end{aligned}$$

or

$$\begin{aligned}&(\partial_t-\triangle)(tw)\\ =\ &2tfw^2+(2-4(1-f)^{-1})t\nabla f\cdot\nabla w+2ct\nabla f\cdot\nabla R(1-f)^{-2}\\ &+2t\lambda(1-f)^{-1}w-2t|\nabla^2 f|^2(1-f)^{-2}+w+2ctRw(1-f)^{-1}.\end{aligned}\tag{5.6}$$

For $T_0, 0<T_0<T$, let (x_0,t_0) be the maximal point of wt on $M\times[0,T_0]$, we assume the maximum value is positive and so $t_0>0$, then at (x_0,t_0), we have $(\partial_t-\triangle)(tw)\geq 0$, and $\nabla(tw)=0$. Since $f\leq -1$, by (5.6), we get that at (x_0,t_0),

$$\begin{aligned}&t(1-f)w^2\leq -2ftw^2\\ \leq\ &w+2\lambda t(1-f)^{-1}w+2ct(1-f)^{-2}\nabla f\cdot\nabla R+2ctRw(1-f)^{-1}.\end{aligned}$$

Note that

$$(1-f)^{-2}\nabla f\cdot\nabla R+(1-f)^{-1}wR\leq(1-f)^{-1}|\nabla\sqrt{R}|^2,$$

so at (x_0,t_0),

$$(tw)^2\leq(1+2\lambda T_0)tw+2cGT_0^2,$$

in particular, for all $x\in M$,

$$T_0 w(x,T_0)\leq t_0 w(x_0,t_0)\leq 1+2\lambda T_0+\sqrt{2c}GT_0,$$

so

$$w(x,T_0)\leq\frac{1}{T_0}+2\lambda+\sqrt{2c}G.\tag{5.7}$$

For the reason that $T_0\in(0,T)$ is arbitrary, we get (1.6).

In order to show the Harnack inequality, let's consider the minimal geodesic $\gamma(s):[0,1]\to M$, so that $\gamma(0)=x_2,\gamma(1)=x_1$, by using (1.6), we have

$$\begin{aligned}\ln\frac{1-f(x_1,t)}{1-f(x_2,t)} &= \int_0^1\frac{d\ln(1-f(\gamma(s),t))}{ds}ds=\int_0^1\frac{-\gamma'\cdot\nabla f}{1-f(\gamma(s),t)}ds\\ &\leq\int_0^1|\gamma'|\frac{|\nabla u|}{u(1-\ln u)}ds\leq r(x_1,x_2)\sqrt{\frac{1}{t}+2\lambda+\sqrt{2c}G}.\end{aligned}$$

This inequality means (1.7) is right.

Acknowledgement. The present work was supported by the Doctoral Foundation of Nantong University (08B04), by NSF of Jiangsu Educational Ministry (08KJD110015) and by NSF of China (10871070, 10971066).

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