Conformal change of special Finsler spaces

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Abstract. The present paper is a continuation of the foregoing paper [16]. The main aim is to establish *an intrinsic investigation* of the conformal change of the most important special Finsler spaces. Necessary and sufficient conditions for such special Finsler manifolds to be invariant under a conformal change are obtained. Moreover, the conformal change of Chern and Hashiguchi connections, as well as their curvature tensors, are given.

1 Introduction

Studying Finsler geometry one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

The infinitesimal transformations in Riemannian and Finsler geometries are important, not only in differential geometry, but also in application to other branches of science, especially in the process of geometrization of physical theories.

The theory of conformal changes in Riemannian geometry has been deeply studied (*locally and intrinsically*) by many authors. As regards to Finsler geometry, an almost complete *local* theory of conformal changes has been established ([1], [6], [7], [8], [9], [11], [12], , $\cdots$, etc.).

In [16], we investigated intrinsically conformal changes in Finsler geometry, where we got, among other results, a characterization of conformal changes. Also the conformal change of Barthel connection and its curvature tensor were studied. Moreover, the conformal changes of Cartan and Berwald connections as well as their curvature tensors, were obtained.

The present paper is a continuation of [16] where we present an *intrinsic* theory of conformal changes of special Finsler spaces. Moreover, we study the conformal change of Chern and Hashiguchi connections

The paper consists of two parts preceded by an introductory section (§1), which provides a brief account of the basic definitions and concepts necessary for this work.

In the first part (§2), the conformal change of Chern and Hashiguchi connections, as well as their curvature tensors, are given.

In the second part (§3), we provide an intrinsic investigation of the conformal change of the most important special Finsler spaces. Moreover, we obtain necessary and sufficient conditions for such special Finsler manifolds to be invariant under a conformal change.

Finally, it should be noted that the present work is formulated in a prospective modern coordinate-free form, without being trapped into the complications of indices. However, some important results of [6], [8], [9] and others (obtained in local coordinates) are immediately derived from the obtained global results (when localized).

2   Notation and preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to global Finsler geometry necessary for this work. For more detail, we refer to [2], [3] and [13]. We assume that all geometric objects treated are of class C^∞. The following notations will be used throughout this paper:

M: a real paracompact differentiable manifold of finite dimension n and of class C^∞,

$\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on M,

$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on M,

$\pi_M : TM \longrightarrow M$: the tangent bundle of M,

$\pi_M^* : T^*M \longrightarrow M$: the cotangent bundle of M,

$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M,

$V(TM)$: the vertical subbundle of the bundle TTM,

$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π,

$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(TM)$-module of differentiable sections of $\pi^{-1}(TM)$,

i_X: the interior product with respect to $X \in \mathfrak{X}(M)$,

df: the exterior derivative of $f \in \mathfrak{F}(M)$,

$d_L := [i_L, d]$, i_L being the interior derivative with respect to a vector form L.

The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles

$$0 \longrightarrow \pi^{-1}(TM) \stackrel{\gamma}{\longrightarrow} T(\mathcal{T}M) \stackrel{\rho}{\longrightarrow} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) := j_u(v)$, j_u being the natural isomorphism $j_u : T_{\pi_M(u)}M \longrightarrow T_u(T_{\pi_M(u)}M)$. The vector 1-form J on $\mathcal{T}M$ defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM. The vertical vector field $\mathcal{C}$ on TM defined by $\mathcal{C} := \gamma \circ \overline{\eta}$ is called the canonical or Liouville vector field.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. The map $K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$ is called the connection

map or the deflection map associated with D. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M)$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space of M at u. The connection D is said to be regular if $T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M)$ for all $u \in \mathcal{T}M$.

If M is endowed with a regular connection D, then the maps $\rho|_{H(\mathcal{T}M)}$, $K|_{V(\mathcal{T}M)}$ and $\gamma : \pi^{-1}(TM) \longrightarrow V(\mathcal{T}M)$ are vector bundle isomorphisms. Let $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, called the horizontal map associated with D, then $\beta \circ \rho = id_{H(\mathcal{T}M)}$ on $H(\mathcal{T}M)$.

The (classical) torsion tensor $\mathbf{T}$ of the connection D is given by

$$\mathbf{T}(X, Y) = D_X \rho Y - D_Y \rho X - \rho[X, Y] \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}M),$$

from which the horizontal or (h)h-torsion tensor Q and the mixed or (h)hv-torsion tensor T are defined respectively by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta \overline{X} \beta \overline{Y}), \; T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma \overline{X}, \beta \overline{Y}) \quad \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

The (classical) curvature tensor $\mathbf{K}$ of the connection D is given by

$$\mathbf{K}(X, Y)\rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X,Y]} \rho Z \quad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M),$$

from which the horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by R, P and S respectively, are defined by

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta \overline{X} \beta \overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta \overline{X}, \gamma \overline{Y})\overline{Z}, \quad S(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\gamma \overline{X}, \gamma \overline{Y})\overline{Z}.$$

The contracted curvature tensors $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$, also known as the (v)h-, (v)hv- and (v)v-torsion tensors, are defined by

$$\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y})\overline{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

(2.1) $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(R(\overline{X}, \overline{Y})\overline{Z}, \overline{W}), \cdots, S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(S(\overline{X}, \overline{Y})\overline{Z}, \overline{W}).$

On a Finsler manifold (M, L), there are *canonically* associated four linear connections on $\pi^{-1}(TM)$ [14]: the Cartan connection ∇, the Chern (Rund) connection D^c, the Hashiguchi connection D^* and the Berwald connection D°. Each of these connections is regular with (h)hv-torsion T satisfying $T(\overline{X}, \overline{\eta}) = 0$. Moreover, the Chern connection D^c and Hashiguchi connection D^* are given, in terms of Cartan connection, by

(2.2) $D^c_X \overline{Y} \;=\; \nabla_X \overline{Y} - T(KX, \overline{Y}) = D^\circ_X \overline{Y} - \widehat{P}(\rho X, \overline{Y}).$

(2.3) $D^*_X \overline{Y} \;=\; \nabla_X \overline{Y} + \widehat{P}(\rho X, \overline{Y}) = D^\circ_X \overline{Y} + T(KX, \overline{Y}).$

Now, we give some concepts and results concerning the Klein-Grifone approach to intrinsic Finsler geometry. For more details, we refer to [4], [5] and [10].

Proposition 2.1. *Let (M, L) be a Finsler manifold. The vector field G on TM defined by $i_G \Omega = -dE$ is a spray, where $E := \frac{1}{2}L^2$ is the energy function and $\Omega := dd_J E$. Such a spray is called the canonical spray.*

Theorem 2.2. *On a Finsler manifold* (M, L), *there exists a unique conservative homogenous nonlinear connection with zero torsion. It is given by*:

$$\Gamma = [J, G],$$

where G *is the canonical spray. Such a nonlinear connection is called the canonical connection, the Cartan nonlinear connection or the Barthel connection associated with* (M, L).

3 Conformal change of the fundamental regular connections and their curvature tensors

In this section, we first review some concepts and results concerning the conformal changes of the Cartan and Berwald connections [16]. Then, using these results, the conformal changes of Chern and Hashiguchi connections, as well as their curvature tensors, are investigated.

Definition 3.1. Let (M, L) and $(M, \widetilde{L})$ be two Finsler manifolds. The two associated metrics g and $\widetilde{g}$ are said to be conformal if there exists a positive differentiable function $\sigma(x)$ such that $\widetilde{g}(\overline{X}, \overline{Y}) = e^{2\sigma(x)} g(\overline{X}, \overline{Y})$. Equivalently, g and $\widetilde{g}$ are conformal iff $\widetilde{L}^2 = e^{2\sigma(x)} L^2$. In this case, the transformation $L \longrightarrow \widetilde{L}$ is said to be a conformal transformation and the two Finsler manifold (M, L) and $(M, \widetilde{L})$ are said to be conformal or conformally related.

Lemma 3.2. *[16] Let* (M, L) *and* $(M, \widetilde{L})$ *be conformally related Finsler manifolds with* $\widetilde{g} = e^{2\sigma(x)} g$. *The associated Barthel connections* Γ *and* $\widetilde{\Gamma}$ *are related by*

$$(3.1) \qquad \begin{aligned} \widetilde{\Gamma} &= \Gamma - 2\text{Ł}, \\ \text{where} \qquad \text{Ł} &:= d\sigma \otimes \mathcal{C} + \sigma_1 J - d_J E \otimes grad_v \sigma - EF = \gamma o N o \rho, \end{aligned}$$

$\sigma_1 := d_G \sigma$ *and* $F := [J, grad_v \sigma]$.
Consequently, $\widetilde{h} = h - \text{Ł}, \quad \widetilde{v} = v + \text{Ł}$ *or, equivalently,* $\widetilde{\beta} = \beta - \text{Ł}o\beta, \quad \widetilde{K} = K + Ko\text{Ł}$.

Concerning the conformal change of the Cartan and Berwald connections and their curvature tensors, we have the following two results [16].

Theorem 3.3. *If* (M, L) *and* $(M, \widetilde{L})$ *are conformally related Finsler manifolds, then the associated Cartan connections* ∇ *and* $\widetilde{\nabla}$ *are related by*:

$$\widetilde{\nabla}_X \overline{Y} = \nabla_X \overline{Y} + \omega(X, \overline{Y}),$$

where

$$\begin{aligned} \omega(X, \overline{Y}) :=& (hX \cdot \sigma(x)) \overline{Y} + (\beta \overline{Y} \cdot \sigma(x)) \rho X - g(\rho X, \overline{Y}) \overline{P} \\ & - T(N\overline{Y}, \rho X) + T'(\text{Ł}X, \beta \overline{Y}), \end{aligned}$$

$\overline{P}$ *being a* π-*vector field defined by* $g(\overline{P}, \rho Z) = hZ \cdot \sigma(x)$ *and* T' *being a 2-form on* TM, *with values in* $\pi^{-1}(TM)$, *defined by*

$$g(T'(\text{Ł}X, hY), \rho Z) = g(T(N\rho Z, \rho Y), \rho X).$$

The associated curvature tensors are related by:

(a) $\widetilde{S}(\overline{X}, \overline{Y})\overline{Z} = S(\overline{X}, \overline{Y})\overline{Z}.$

(b) $\widetilde{P}(\overline{X}, \overline{Y})\overline{Z} = P(\overline{X}, \overline{Y})\overline{Z} + V(\overline{X}, \overline{Y})\overline{Z},$

(c) $\widetilde{R}(\overline{X}, \overline{Y})\overline{Z} = R(\overline{X}, \overline{Y})\overline{Z} + H(\overline{X}, \overline{Y})\overline{Z},$

where H and V are the π-tensor fields defined by

$$\begin{aligned}
V(\overline{X}, \overline{Y})\overline{Z} &= (\nabla_{\gamma\overline{Y}}B)(\overline{X}, \overline{Z}) + B(T(\overline{Y}, \overline{X}), \overline{Z}) - S(N\overline{X}, \overline{Y})\overline{Z}, \\
H(\overline{X}, \overline{Y})\overline{Z} &= S(N\overline{X}, N\overline{Y})\overline{Z} - \mathfrak{U}_{\overline{X}, \overline{Y}}\{(\nabla_{\beta\overline{X}}B)(\overline{Y}, \overline{Z}) - (\nabla_{\mathit{L}\beta\overline{X}}B)(\overline{Y}, \overline{Z}) \\
&\quad + P(\overline{X}, N\overline{Y})\overline{Z} + B(\overline{X}, B(\overline{Y}, \overline{Z})) - B(T(N\overline{X}, \overline{Y}), \overline{Z})\};
\end{aligned}$$

B being defined by $B(\overline{X}, \overline{Y}) := \omega(\beta\overline{X}, \overline{Y})$ and L by (3.1).

Theorem 3.4. *If (M, L) and $(M, \widetilde{L})$ are conformally related Finsler manifolds, then the associated Berwald connections D° and $\widetilde{D^\circ}$ are related by:*

$$\widetilde{D^\circ}_X\overline{Y} = D^\circ_X\overline{Y} + \omega^\circ(X, \overline{Y}),$$

where $\omega^\circ(X, \overline{Y}) = K([\gamma\overline{Y}, \mathit{L}]X) + D^\circ_{\mathit{L}X}\overline{Y}.$ *The associated curvature tensors are related by:*

(a) $\widetilde{S^\circ}(\overline{X}, \overline{Y})\overline{Z} = S^\circ(\overline{X}, \overline{Y})\overline{Z} = 0.$

(b) $\widetilde{P^\circ}(\overline{X}, \overline{Y})\overline{Z} = P^\circ(\overline{X}, \overline{Y})\overline{Z} + (D^\circ_{\gamma\overline{Y}}B^\circ)(\overline{X}, \overline{Z}).$

(c) $\widetilde{R^\circ}(\overline{X}, \overline{Y})\overline{Z} = R^\circ(\overline{X}, \overline{Y})\overline{Z} + \mathfrak{U}_{\overline{X}, \overline{Y}}\{(D^\circ_{\gamma N\overline{X}}B^\circ)(\overline{Y}, \overline{Z}) - (D^\circ_{\beta\overline{X}}B^\circ)(\overline{Y}, \overline{Z})$
$\qquad\qquad\qquad + P^\circ(\overline{Y}, N\overline{X})\overline{Z} - B^\circ(\overline{X}, B^\circ(\overline{Y}, \overline{Z}))\},$

where $B^\circ(\overline{X}, \overline{Y}) := \omega^\circ(\beta\overline{X}, \overline{Y}).$

Now, we turn our attention to the Chern and Hashiguchi connections.

Theorem 3.5. *Let (M, L) and $(M, \widetilde{L})$ be conformally related Finsler manifolds with $\widetilde{g} = e^{2\sigma(x)}g$. The associated Chern connections D^c and $\widetilde{D^c}$ are related by*

(3.2) $$\widetilde{D^c}_X\overline{Y} = D^c_X\overline{Y} + \omega^c(X, \overline{Y}),$$

where

$$\begin{aligned}
\omega^c(X, \overline{Y}) :=&(hX \cdot \sigma(x))\overline{Y} + (\beta\overline{Y} \cdot \sigma(x))\rho X - g(\rho X, \overline{Y})\overline{P} \\
&- T(N\overline{Y}, \rho X) + T'(\mathit{L}X, \beta\overline{Y}) - T(N\rho X, \overline{Y}).
\end{aligned}$$

Proof. The proof follows from (2.2), Theorem 3.3 and Lemma 3.2, taking into account the fact that the (h)hv-torsion tensor T is conformally invariant [16]. □

In view of the above theorem, we have

Theorem 3.6. *Under a Finsler conformal change $\widetilde{g} = e^{2\sigma(x)}g$, we have*

(a) $\widetilde{S^c}(\overline{X}, \overline{Y})\overline{Z} = S^c(\overline{X}, \overline{Y})\overline{Z} = 0,$

(b) $\widetilde{P^c}(\overline{X},\overline{Y})\overline{Z} = P^c(\overline{X},\overline{Y})\overline{Z} + (D^c_{\gamma\overline{Y}}B^c)(\overline{X},\overline{Z}),$

(c) $\widetilde{R^c}(\overline{X},\overline{Y})\overline{Z} = R^c(\overline{X},\overline{Y})\overline{Z} + \mathfrak{U}_{\overline{X},\overline{Y}}\{(D^c_{\gamma N\overline{X}}B^c)(\overline{Y},\overline{Z}) - (D^c_{\beta\overline{X}}B^c)(\overline{Y},\overline{Z})$
$$+ P^c(\overline{Y}, N\overline{X})\overline{Z} - B^c(\overline{X}, B^c(\overline{Y},\overline{Z}))\},$$

where $B^c(\overline{X},\overline{Y}) := \omega^c(\beta\overline{X},\overline{Y}).$

Theorem 3.7. *Let* (M,L) *and* $(M,\widetilde{L})$ *be conformally related Finsler manifolds with* $\widetilde{g} = e^{2\sigma(x)}g.$ *The associated Hashiguchi connections* D^* *and* $\widetilde{D}^*$ *are related by*

(3.3) $$\widetilde{D^*}_X\overline{Y} = D^*_X\overline{Y} + \omega^*(X,\overline{Y}),$$

where $\omega^*(X,\overline{Y}) = (D^*_{\gamma\overline{Y}}N)(\rho X) + NT(\overline{Y},\rho X).$

Proof. The proof follows from (2.3) and Theorem 3.3(**b**). $\qquad\square$

Theorem 3.8. *Under a Finsler conformal change* $\widetilde{g} = e^{2\sigma(x)}g,$ *we have*

(a) $\widetilde{S^*}(\overline{X},\overline{Y})\overline{Z} = S^*(\overline{X},\overline{Y})\overline{Z},$

(b) $\widetilde{P^*}(\overline{X},\overline{Y})\overline{Z} = P^*(\overline{X},\overline{Y})\overline{Z} - S^*(N\overline{X},\overline{Y})\overline{Z} + (D^*_{\gamma\overline{Y}}B^*)(\overline{X},\overline{Z}) + B^*(T(\overline{Y},\overline{X}),\overline{Z}),$

(c) $\widetilde{R^*}(\overline{X},\overline{Y})\overline{Z} = R^*(\overline{X},\overline{Y})\overline{Z} + S^*(N\overline{X}, N\overline{Y})\overline{Z} - \mathfrak{U}_{\overline{X},\overline{Y}}\{P^*(\overline{X}, N\overline{Y})\overline{Z} + (D^*_{\beta\overline{X}}B^*)(\overline{Y},\overline{Z})$
$$- (D^*_{\gamma N\overline{X}}B^*)(\overline{Y},\overline{Z}) + B^*(\overline{X}, B^*(\overline{Y},\overline{Z})) - B^*(T(N\overline{X},\overline{Y}),\overline{Z})\},$$

where $B^*(\overline{X},\overline{Y}) := \omega^*(\beta\overline{X},\overline{Y}).$

4 Conformal change of special Finsler spaces

In this section, we establish *an intrinsic investigation* of the conformal change of the most important special Finsler spaces. Moreover, we obtain necessary and sufficient conditions for such special Finsler spaces to be conformally invariant.

Throughout this section, g, $\hbar$, $\widehat{g}$, ∇ and D° denote respectively the Finsler metric on $\pi^{-1}(TM)$, the angular metric tensor, the induced metric on $\pi^{-1}(T^*M)$, the Cartan connection and the Berwald connection associated with a Finsler manifold (M,L). Also, R, P and S denote respectively the h-, hv- and v-curvature tensors of Cartan connection, whereas R°, P° and S° denote respectively the h-, hv- and v-curvature tensors of Berwald connection. Finally, $T(\overline{X},\overline{Y},\overline{Z}) := g(T(\overline{X},\overline{Y}),\overline{Z})$ denotes the Cartan tensor, where T is the (h)hv-torsion tensor of the Cartan connection.

We first set the intrinsic definitions of the special Finsler spaces that will be treated. These definitions are quoted from [15].

Definition 4.1. A Finsler manifold (M,L) is:
(a) Riemannian if the metric tensor $g(x,y)$ is independent of y or, equivalently, if

$$T = 0.$$

(b) locally Minkowskian if the metric tensor $g(x,y)$ is independent of x or, equivalently, if $\nabla_{\beta\overline{X}}T = 0$ and $R = 0.$
The above conditions are also equivalent to $\widehat{R} = 0$ and $P^\circ = 0.$

Definition 4.2. A Finsler manifold (M, L) is said to be:

(a) Berwald if the torsion tensor T is horizontally parallel. That is, $\nabla_{\beta\overline{X}} T = 0$.

(b) C^h-recurrent if the torsion tensor T satisfies the condition $\nabla_{\beta\overline{X}} T = \lambda_o(\overline{X})\,T$, where λ_o is a π-form of order one.

(c) P^*-Finsler manifold if the π-tensor field $\nabla_{\beta\overline{\eta}}T$ is has the form $\nabla_{\beta\overline{\eta}} T = \lambda(x, y)\,T$, where $\lambda(x, y) = \frac{\widehat{g}(\nabla_{\beta\overline{\eta}} C, C)}{C^2}$, $C^2 := \widehat{g}(C, C) \neq 0$ and C is the contracted torsion.

Definition 4.3. A Finsler manifold (M, L) is said to be:

(a) semi-C-reducible if $dim\, M \geq 3$ and the Cartan tensor T has the form[1]

$$(4.1) \quad T(\overline{X}, \overline{Y}, \overline{Z}) = \frac{\mu}{n+1}\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{\hbar(\overline{X}, \overline{Y})C(\overline{Z})\} + \frac{\tau}{C^2}C(\overline{X})C(\overline{Y})C(\overline{Z}),$$

where μ and τ are scalar functions satisfying $\mu + \tau = 1$.

(b) C-reducible if $dim\, M \geq 3$ and the scaler function τ in **(a)** is zero.

(c) C_2-like if $dim\, M \geq 2$ and the scaler function μ in **(a)** is zero.

Definition 4.4. A Finsler manifold (M, L), where $dim\, M \geq 3$, is said to be quasi-C-reducible if the Cartan tensor T is written as:

$$T(\overline{X}, \overline{Y}, \overline{Z}) = A(\overline{X}, \overline{Y})C(\overline{Z}) + A(\overline{Y}, \overline{Z})C(\overline{X}) + A(\overline{Z}, \overline{X})C(\overline{Y}),$$

where A is a symmetric indicatory (2) π-form $(A(\overline{X}, \overline{\eta}) = 0$ *for all* $\overline{X})$.

Definition 4.5. A Finsler manifold (M, L) is said to be:

(a) S_3-like if $dim\, M \geq 4$ and the vertical curvature tensor S has the form:

$$S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \frac{Sc^v}{(n-1)(n-2)}\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W}) - \hbar(\overline{X}, \overline{W})\hbar(\overline{Y}, \overline{Z})\}.$$

(b) S_4-like if $dim\, M \geq 5$ and the vertical curvature tensor S has the form:

$$(4.2) \quad \begin{aligned} S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) =&\, \hbar(\overline{X}, \overline{Z})\mathbf{F}(\overline{Y}, \overline{W}) - \hbar(\overline{Y}, \overline{Z})\mathbf{F}(\overline{X}, \overline{W}) \\ &+ \hbar(\overline{Y}, \overline{W})\mathbf{F}(\overline{X}, \overline{Z}) - \hbar(\overline{X}, \overline{W})\mathbf{F}(\overline{Y}, \overline{Z}), \end{aligned}$$

where $\mathbf{F} = \frac{1}{n-3}\{Ric^v - \frac{Sc^v\, \hbar}{2(n-2)}\}$, Ric^v is the vertical Ricci tensor and Sc^v is the vertical scalar curvature.

Definition 4.6. A Finsler manifold (M, L) is said to be S^v-recurrent if the v-curvature tensor S satisfies the condition $(\nabla_{\gamma\overline{X}}S)(\overline{Y}, \overline{Z}, \overline{W}) = \lambda(\overline{X})S(\overline{Y}, \overline{Z})\overline{W}$, where λ is a π-form of order one.

Definition 4.7. A Finsler manifold (M, L) is said to be:

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[1] the symbol $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}$ denotes cyclic sum over $\overline{X}, \overline{Y}$ and $\overline{Z}$.

(a) a Landsberg manifold if $\widehat{P} = 0$, or equivalently $\nabla_{\beta\overline{\eta}} T = 0$.

(b) a general Landsberg manifold if $Tr\{\overline{Y} \longrightarrow \widehat{P}(\overline{X}, \overline{Y})\} = \nabla_{\beta\overline{\eta}} C = 0$.

Definition 4.8. A Finsler manifold (M, L) is said to be P-symmetric if the mixed curvature tensor P satisfies $P(\overline{X}, \overline{Y})\overline{Z} = P(\overline{Y}, \overline{X})\overline{Z}$.

Definition 4.9. A Finsler manifold (M, L), where $dim\, M \geq 3$, is said to be P_2-like if the mixed curvature tensor P has the form:

$$P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \alpha(\overline{Z})T(\overline{X}, \overline{Y}, \overline{W}) - \alpha(\overline{W})\, T(\overline{X}, \overline{Y}, \overline{Z}),$$

where α is a $(1)\pi$-form, positively homogeneous of degree 0.

Definition 4.10. A Finsler manifold (M, L), where $dim\, M \geq 3$, is said to be P-reducible if the π-tensor field $P(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$ is expressed in the form:

$$P(\overline{X}, \overline{Y}, \overline{Z}) = \delta(\overline{X})\hbar(\overline{Y}, \overline{Z}) + \delta(\overline{Y})\hbar(\overline{Z}, \overline{X}) + \delta(\overline{Z})\hbar(\overline{X}, \overline{Y}),$$

where δ is the π-form defined by $\delta = \frac{1}{n+1}\nabla_{\beta\overline{\eta}} C$.

Definition 4.11. A Finsler manifold (M, L), where $dim\, M \geq 3$, is said to be h-isotropic if there exists a scalar k_o such that the horizontal curvature tensor R has the form $R(\overline{X}, \overline{Y})\overline{Z} = k_o\{g(\overline{X}, \overline{Z})\overline{Y} - g(\overline{Y}, \overline{Z})\overline{X}\}$.

Definition 4.12. A Finsler manifold (M, L), where $dim\, M \geq 3$, is said to be:

(a) of scalar curvature if there exists a scalar function $k : \mathcal{T}M \longrightarrow \mathbb{R}$ such that the horizontal curvature tensor R satisfies the relation $R(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = kL^2\hbar(\overline{X}, \overline{Y})$.

(b) of constant curvature if the function k in **(a)** is constant.

Definition 4.13. A Finsler manifold (M, L) is said to be R_3-like if $dim\, M \geq 4$ and the horizontal curvature tensor R is expressed in the form

$$(4.3) \qquad \begin{aligned} R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) =& g(\overline{X}, \overline{Z})F(\overline{Y}, \overline{W}) - g(\overline{Y}, \overline{Z})F(\overline{X}, \overline{W}) \\ &+ g(\overline{Y}, \overline{W})F(\overline{X}, \overline{Z}) - g(\overline{X}, \overline{W})F(\overline{Y}, \overline{Z}), \end{aligned}$$

where F is the $(2)\pi$-form defined by $F = \frac{1}{n-2}\{Ric^h - \frac{Sc^h\, g}{2(n-1)}\}$, Ric^h is the horizontal Ricci tensor and Sc^h is the horizontal scalar curvature.

Definition 4.14. A Finsler manifold (M, L) is called of perpendicular scalar *(simply, p-scaler)* curvature if the h-curvature tensor R satisfies the condition

$$R(\phi(\overline{X}), \phi(\overline{Y}), \phi(\overline{Z}), \phi(\overline{W})) = R_o\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W}) - \hbar(\overline{X}, \overline{W})\hbar(\overline{Y}, \overline{Z})\},$$

where R_o is a function on TM, called perpendicular scaler curvature, and ϕ is the π-tensor field defined by $\phi(\overline{X}) := \overline{X} - L^{-1}\ell(\overline{X})\overline{\eta}$.

Definition 4.15. A Finsler manifold (M, L) is called of *s-ps* curvature if (M, L) is both of scalar curvature and of p-scalar curvature.

Definition 4.16. A Finsler manifold (M, L) is said to be symmetric if the h-curvature tensor R° of the Berwald connection D° is horizontally parallel: $D^\circ_{\beta\overline{X}} R^\circ = 0$.

Now, we focus our attention to the change of the above mentioned special Finsler manifolds under a conformal transformation $g \longrightarrow \widetilde{g} = e^{2\sigma(x)} g$. *In what follows we assume that the Finsler manifolds (M, L) and $(M, \widetilde{L})$ are conformally related.*

Proposition 4.17.

(a) *(M, L) is Riemaniann if, and only if, $(M, \widetilde{L})$ is Riemaniann.*

(b) *Assume that $D^\circ_{\gamma\overline{X}} B^\circ = 0$ and $H(\overline{X}, \overline{Y})\overline{\eta} = 0$. Then, (M, L) is Locally Minkowskian if, and only if, $(M, \widetilde{L})$ is Locally Minkowskian.*

Proof. **(a)** Follows from Definition 4.1 together with the fact that the $(h)hv$-torsion tensor T is conformally invariant.

(b) By Theorem 3.3(**c**) and Theorem 3.4(**b**), we get

$$\widetilde{R}(\overline{X}, \overline{Y})\overline{\eta} = R(\overline{X}, \overline{Y})\overline{\eta}, \quad \text{and} \quad \widetilde{P^\circ}(\overline{X}, \overline{Y})\overline{Z} = P^\circ(\overline{X}, \overline{Y})\overline{Z}.$$

The result follows then from Definition 4.1.   □

Let us introduce the π-tensor field

$$(4.4) \qquad \mathcal{A}(\overline{X}, \overline{Y}, \overline{Z}) := T(U(\beta\overline{X}, \overline{Y}), \overline{Z}) + T(U(\beta\overline{X}, \overline{Z}), \overline{Y}) - U(\beta\overline{X}, T(\overline{Y}, \overline{Z})),$$

where $U(\beta\overline{X}, \overline{Y}) := B(\overline{X}, \overline{Y}) - \nabla_{\mathbf{L}\beta\overline{X}} \overline{Y}$.

One can show that the π-tensor field $\mathcal{A}$ has the property that $\mathcal{A}(\overline{X}, \overline{Y}, \overline{\eta}) = 0$.

Proposition 4.18. *Assume that the π-tensor field $\mathcal{A}$ vanishes. Then, (M, L) is a Berwald (resp. C^h-recurrent) manifold if, and only if, $(M, \widetilde{L})$ is a Berwald (resp. C^h-recurrent) manifold.*

Proof. Using Theorem 3.3, taking into account the fact that T is conformally invariant, we get

$$
\begin{aligned}
(\widetilde{\nabla}_{\widetilde{\beta}\overline{X}} \widetilde{T})(\overline{Y}, \overline{Z}) \;=\; & \nabla_{\beta\overline{X}} T(\overline{Y}, \overline{Z}) - T(\nabla_{\beta\overline{X}} \overline{Y}, \overline{Z}) - T(\overline{Y}, \nabla_{\beta\overline{X}} \overline{Z}) \\
& - \{ T(U(\beta\overline{X}, \overline{Y}), \overline{Z}) + T(\overline{Y}, U(\beta\overline{X}, \overline{Z})) - U(\beta\overline{X}, T(\overline{Y}, \overline{Z})) \}.
\end{aligned}
$$

Consequently,

$$(4.5) \qquad\qquad (\widetilde{\nabla}_{\widetilde{\beta}\overline{X}} \widetilde{T})(\overline{Y}, \overline{Z}) = (\nabla_{\beta\overline{X}} T)(\overline{Y}, \overline{Z}) - \mathcal{A}(\overline{X}, \overline{Y}, \overline{Z}).$$

Hence, under the given assumption, we have

$$(4.6) \qquad\qquad\qquad \widetilde{\nabla}_{\widetilde{\beta}\overline{X}} \widetilde{T} = \nabla_{\beta\overline{X}} T.$$

Therefore, (M, L) is Berwald iff $(M, \widetilde{L})$ is Berwald.

The same argument can be applied to the C^h-recurrence property.   □

Proposition 4.19. *Assume that the π-tensor field $\mathcal{A}$ has the property that $i_{\overline{\eta}}\mathcal{A} = 0$. Then, (M, L) is a P^*-Finsler manifold if, and only if, $(M, \widetilde{L})$ is a P^*-Finsler manifold.*

Proof. From relation (4.5), we have $\nabla_{\beta\overline{\eta}}T = \widetilde{\nabla}_{\widetilde{\beta}\overline{\eta}}\widetilde{T}$. Hence, the π-tensor field $\nabla_{\beta\overline{\eta}}C$ is conformally invariant. This, together with the fact that $\widetilde{C} = C$, imply that the scalar function $\lambda(x, y)$ defined by $\lambda(x, y) := \frac{\widehat{g}(\nabla_{\beta\overline{\eta}}C, C)}{\widehat{g}(C, C)}$ is also conformally invariant. Hence the result. $\qquad\square$

Proposition 4.20. *A Finsler manifold (M, L) is semi-C-reducible if, and only if, $(M, \widetilde{L})$ is semi-C-reducible. Consequently, (M, L) is C-reducible (resp. C_2-like) if, and only if, $(M, \widetilde{L})$ is C-reducible (resp. C_2-like).*

Proof. We have $\widetilde{C}^2 := \widetilde{\widehat{g}}(\widetilde{C}, \widetilde{C}) = e^{-2\sigma}\widehat{g}(C, C) = e^{-2\sigma}C^2$, $\widetilde{T}(\overline{X}, \overline{Y}, \overline{Z}) = e^{2\sigma(x)}T(\overline{X}, \overline{Y}, \overline{Z})$ and the angular metric tensor $\hbar$ is conformally σ-invariant [16].
Thus (4.1) is equivalent to

$$\widetilde{T}(\overline{X}, \overline{Y}, \overline{Z}) = \frac{\mu}{n+1}\mathfrak{U}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\widetilde{C}(\overline{Z})\} + \frac{\tau}{\widetilde{C}^2}\widetilde{C}(\overline{X})\widetilde{C}(\overline{Y})\widetilde{C}(\overline{Z}).$$

Hence, the semi-C-reducibility property is preserved.
Finally, the proof of the cases of C-reducibility and C_2-likeness is similar. $\qquad\square$

Proposition 4.21. *A Finsler manifold (M, L) is quasi-C-reducible if, and only if, $(M, \widetilde{L})$ is quasi-C-reducible.*

Theorem 4.22. *A necessary and sufficient condition for a Finsler manifold to be conformal to a Landsberg manifold is that*

$$\widehat{P} = i_{\overline{\eta}}\mathcal{A}.$$

Proof. We have $\widehat{P} = \nabla_{\beta\overline{\eta}}T$ ([17]). From which, together with (4.5), we obtain

$$(4.7) \qquad \widetilde{\widehat{P}} - \widehat{P} = \widetilde{\nabla}_{\widetilde{\beta}\overline{\eta}}\widetilde{T} - \nabla_{\beta\overline{\eta}}T = -i_{\overline{\eta}}\mathcal{A}.$$

Hence, the result follows. $\qquad\square$

Let us define the π-tensor field

$$(4.8) \qquad \mathcal{A}_o(\overline{X}) := Tr\{\overline{Y} \longmapsto (i_{\overline{\eta}}\mathcal{A})(\overline{X}, \overline{Y})\},$$

where $\mathcal{A}$ is the π-tensor field defined by (4.4).

Proposition 4.23.

(a) *Assume that $i_{\overline{\eta}}\mathcal{A} = 0$. Then, (M, L) is Landsberg if, and only if, $(M, \widetilde{L})$ is Landsberg.*

(b) *Assume that $\mathcal{A}_o = 0$. Then, (M, L) is general Landsberg if, and only if, $(M, \widetilde{L})$ is general Landsberg.*

Proposition 4.24. *Assume that $i_{\overline{\eta}}\mathcal{A} = 0$. Then, (M, L) is P-reducible if, and only if, $(M, \widetilde{L})$ is P-reducible.*

Proof. Under a conformal change, the angular metric tensor $\hbar$ is conformally σ-invariant. On the other hand, $\widehat{P}$ is conformally invariant by our assumption together with (4.7). Consequently, $\nabla_{\beta\overline{\eta}}T$ is conformally invariant, which implies that $\nabla_{\beta\overline{\eta}}C$ (or δ of Definition 4.10) is also conformally invariant.

Now, since $P(\overline{X}, \overline{Y}, \overline{Z}) = g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$ is conformally σ-invariant, then, the tensor field

$$\mathbf{U}_1(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z}) - \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{\delta(\overline{X})\hbar(\overline{Y}, \overline{Z})\}$$

is conformally σ-invariant. From which, the result follows (provided that $\sigma \neq 0$). $\square$

Proposition 4.25. *(M, L) is S_3-like (resp. S_4-like) if, and only if, $(M, \widetilde{L})$ is S_3-like (resp. S_4-like).*

Proof. The proof is clear and we omit it. $\square$

Proposition 4.26. *(M, L) is S^v-recurrent if, and only if, $(M, \widetilde{L})$ is S^v-recurrent.*

Proposition 4.27. *Assume that the π-tensor field H defined in Theorem 3.3 has the property that $H(\overline{\eta}, \overline{X})\overline{\eta} = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$. Then, (M, L) is of scalar curvature if, and only if, $(M, \widetilde{L})$ is of scalar curvature.*

Proof. By Theorem 3.3(**c**), we have

$$\widetilde{R}(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = e^{2\sigma(x)}R(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) + e^{2\sigma(x)}g(H(\overline{\eta}, \overline{X})\overline{\eta}, \overline{Y}),$$

which implies, by hypothesis, that

$$(4.9) \qquad\qquad \widetilde{R}(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = e^{2\sigma(x)}R(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}).$$

Now, let (M, L) be of scalar curvature, then the h-curvature tensor R has the form $R(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = kL^2\hbar(\overline{X}, \overline{Y})$. This, together with (4.9), imply that $\widetilde{R}(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = e^{2\sigma(x)}kL^2\hbar(\overline{X}, \overline{Y}) = e^{-2\sigma(x)}k\widetilde{L}^2\widetilde{\hbar}(\overline{X}, \overline{Y})$, where we have used the fact that both L^2 and $\hbar$ are conformally σ-invariant [16].
Hence $\widetilde{R}(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = k_o\widetilde{L}^2\widetilde{\hbar}(\overline{X}, \overline{Y})$, where $k_o = e^{-2\sigma(x)}k$. $\square$

Proposition 4.28. *Assume that the given conformal change is homothetic. Then, the following properties are conformally invariant*:

– *being P-symmetric,*   – *being P_2-like,*
– *being symmetric,*   – *being of s-ps curvature,*
– *being of scalar curvature,*   – *being of constant curvature,*
– *being R_3-like,*   – *being of p-scaler curvature,*

Proof. The proof follows from the fact that:
$$\sigma(x) \text{ is constant} \Longleftrightarrow \widetilde{\nabla}_X\overline{Y} = \nabla_X\overline{Y} \text{ [16].} \qquad \square$$

Summing up, the results of this section can be gathered in the following

Theorem 4.29. *The following properties are conformally invariant*:

– *being Riemannian,*   – *being semi-C-reducible,*
– *being C-reducible,*   – *being C_2-like,*
– *being quasi-C-reducible,*   – *being S^v-recurrent,*
– *being S_3-like,*   – *being S_4-like.*

The following properties are conformally invariant under certain conditions:

– *being locally Minkowskian,*   – *being Berwald,*
– *being C^h-recurrent,*   – *being P^*-manifold,*
– *being Landsberg,*   – *being general Landsberg,*
– *being P-symmetric,*   – *being P_2-like,*
– *being P-reducible,*   – *being h-isotropic,*
– *being of scalar curvature,*   – *being of constant curvature,*
– *being R_3-like,*   – *being of p-scaler curvature,*
– *being of s-ps curvature,*   – *being symmetric.*

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