Biharmonic maps between doubly warped product manifolds

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Abstract. In this paper biharmonic maps between doubly warped product manifolds are studied. We show that the inclusion maps of Riemannian manifolds B and F into the nontrivial (proper) doubly warped product manifold ${}_fB \times_b F$ can not be proper biharmonic maps. Also we analyze the conditions for the biharmonicity of projections ${}_fB \times_b F \to B$ and ${}_fB \times_b F \to F$, respectively. Some characterizations for non-harmonic biharmonic maps are given by using product of harmonic maps and warping metric. Especially, in the case of $f = 1$, the results for warped product in [4] are obtained.

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1 Introduction

The study of biharmonic maps between Riemannian manifolds, as a generalization of harmonic maps, was suggested by J. Eells and J. H. Sampson in [8]. The energy of a smooth map $\varphi : (B, g_B) \to (F, g_F)$ between two Riemannian manifolds is defined by $E(\varphi) = \frac{1}{2}\int_D |d\varphi|^2 v_{g_B}$ and φ is called harmonic if it's a critical point of energy. From the first variation formula for the energy, the Euler-Lagrange equation associated to the energy is given by $\tau(\varphi) = 0$ where $\tau(\varphi) = trace\ \nabla d\varphi$ is the tension field of φ (see also [2], [8], [9]).

The bienergy functional E_2 of a smooth map $\varphi : (B, g_B) \to (F, g_F)$ is defined by integrating the square norm of the tension field, $E_2(\varphi) = \frac{1}{2}\int_D |\tau(\varphi)|^2 v_{g_B}$. The first variation formula for the bienergy, derived in [10] and [11], shows that the Euler-Lagrange equation for E_2 is

$$\tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -\Delta\tau(\varphi) - traceR^F(d\varphi, \tau(\varphi))d\varphi = 0,$$

where J^φ is formally the Jacobi operator of φ. Since any harmonic map is biharmonic, we are interested in non-harmonic biharmonic maps which are called proper biharmonic, (see also [15], [16]).

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In [1], P. Baird and D. Kamissoko constructed new examples of proper biharmonic maps between Riemannian manifolds by firstly taking a harmonic map $\varphi : B \to F$ which is automatically biharmonic and then deforming the metric conformally on B to render φ biharmonic, (see also [3]). In [4], A. Balmuş, S. Montaldo, C. Oniciuc studied the biharmonic maps between warped product manifolds. In the same paper they investigated biharmonicity of the iclusion $i : F \to B \times_b F$ of a Riemannian manifold F into the warped product manifold $B \times_b F$ and of the projection from $B \times_b F$ into the first factor. Also in [4] the authors gave two new classes of proper biharmonic maps by using product of harmonic maps and warping the metric in the domain or codomain.

Warped products were first defined by O'Neill and Bishop in 1969, to construct Riemannian manifolds with negative sectional curvature, see [5]. Also in [14], O'Neill gave the curvature formulas of warped products in the terms of curvatures of components of warped products and studied Robertson-Walker, static, Schwarzchild and Kruskal space-times as warped products. In general doubly warped products can be considered as a generalization of singly warped products or simply warped products. A doubly warped product manifold is a product manifold $B \times F$ of two Riemannian manifolds (B, g_B) and (F, g_F) endowed with the metric $g = f^2 g_B \oplus b^2 g_F$ where $b : B \to (0, \infty)$ and $f : F \to (0, \infty)$ are smooth functions. The canonical leaves $\{x_0\} \times F$ and $B \times \{y_0\}$ of a doubly warped product manifold ${}_f B \times_b F$ are totally umbilic submanifolds, which intersect perpendicularly [17], (see also [6], [12], [13], [18]). When $f = 1$, ${}_1 B \times_b F$ becomes a warped product manifold and in this case the leaves $B \times \{y_0\}$ are totally geodesic.

This article organized as follow.

In the second and third sections we give some basic definitions on biharmonic maps and doubly warped product manifolds, respectively. In the case of warped products since $B \times \{y_0\}$ is totally geodesic so biharmonic, the authors in [4] investigated only the biharmonicity of the inclusion of the Riemannian manifold F into the warped product $B \times_b F$. In section 4, by considering the situation of doubly warped product as a generalization of warped products, we analyze the conditions for both of the leaves $\{x_0\} \times F$ and $B \times \{y_0\}$ to be biharmonic as a submanifold and we show that both of the leaves $\{x_0\} \times F$ and $B \times \{y_0\}$ can not be proper biharmonic as a submanifold of the doubly warped product manifold ${}_f B \times_b F$. The product of two harmonic maps is clearly harmonic. If the metric in the domain or codomain is deformed conformally, then the harmonicity is lost. Then it's possible to define proper biharmonic maps using products of two harmonic maps. In the next section we find some results on product maps to be proper biharmonic.

2 Biharmonic maps between Riemannian manifolds

Let (B, g_B) and (F, g_F) be Riemann manifolds and $\varphi : (B, g_B) \to (F, g_F)$ be a smooth map. The tension field of φ is given by

$$\tau(\varphi) = trace \nabla d\varphi, \tag{2.1}$$

where $\nabla d\varphi$ is the second fundamental form of φ.

Biharmonic maps $\varphi : (B, g_B) \to (F, g_F)$ between Riemannian manifolds are critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_D |\tau(\varphi)|^2 v_{g_B}, \tag{2.2}$$

for any compact domain $D \subset B$. Biharmonic maps are a natural generalization of the well-known harmonic maps, the extremal points of the energy functional defined by

$$E(\varphi) = \frac{1}{2} \int_D |d\varphi|^2 v_{g_B}. \tag{2.3}$$

The Euler-Lagrange equation for the energy is $\tau(\varphi) = 0$.

The first variation formula of $E_2(\varphi)$ is

$$\frac{\partial}{\partial t} E_2(\varphi_t)|_{t=0} = -\int_D < J_\varphi(\tau(\varphi)), w > v_{g_B}, \tag{2.4}$$

where $w = \frac{\partial \varphi}{\partial t}|_{t=0}$ is the variational vector field of the variation $\{\varphi_t\}$ of φ. The Euler-Lagrange equation corresponding to $E_2(\varphi)$ is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -\Delta\tau(\varphi) - traceR^F(d\varphi, \tau(\varphi))d\varphi, \tag{2.5}$$

where J^φ is the Jacobi operator of φ. Here Δ is the rough Laplacian on sections of the pull-back bundle $\varphi^{-1}(TF)$ defined by, for an orthonormal frame field $\{B_j\}_{j=1}^m$ on B,

$$\begin{aligned} \Delta v &= -trace_{g_b}(\nabla^\varphi)^2 v \\ &= -\sum_{j=1}^m \{\nabla^\varphi_{B_j} \nabla^\varphi_{B_j} v - \nabla^\varphi_{\nabla^B_{B_j} B_j} v\}, \;\; v \in \Gamma(\varphi^{-1}(TF)), \end{aligned} \tag{2.6}$$

with ∇^φ is representing the connection in the pull-back bundle $\varphi^{-1}(TF)$ and ∇^B is the Levi-Civita connection on M and R^F is the curvature operator

$$R^F(X,Y)Z = \nabla^F_X \nabla^F_Y Z - \nabla^F_Y \nabla^F_X Z - \nabla^F_{[X,Y]} Z. \tag{2.7}$$

Clearly any harmonic map is biharmonic. We call the non-harmonic biharmonic maps proper biharmonic maps .

3 Doubly warped product manifolds

Let (B, g_B) and (F, g_F) be Riemannian manifolds of dimensions m and n, respectively and let $b : B \to (0, \infty)$ and $f : F \to (0, \infty)$ be smooth functions. As a generalization of the warped product of two Riemannian manifolds, a doubly warped product of Riemannian manifolds (B, g_B) and (F, g_F) with warping functions b and f is a product manifold $B \times F$ with metric tensor

$$g = f^2 g_B \oplus b^2 g_F, \tag{3.1}$$

given by

$$g(X,Y) = (f\circ\sigma)^2 g_B(d\pi(X), d\pi(Y)) + (b\circ\pi)^2 g_F(d\sigma(X), d\sigma(Y)), \tag{3.2}$$

where $X, Y \in \Gamma(T(B\times F))$ and $\pi : B\times F \to B$ and $\sigma : B\times F \to F$ are the canonical projections. We denote the doubly warped product of Riemannian manifolds (B, g_B) and (F, g_F) by ${}_fB\times_b F$. If either $b = 1$ or $f = 1$, but not both, then ${}_fB\times_b F$ becomes a warped product of Riemannian manifolds B and F. If both $b = 1$ and $f = 1$, then we have a product manifold. If neither b nor f is constant, then we have a nontrivial(proper) doubly warped product manifold

Let (B, g_B) and (F, g_F) be Riemannian manifolds with Levi-Civita connections ∇^B and ∇^F, respectively and let ∇ and $\overline{\nabla}$ denote the Levi-civita connections of the product manifold $B\times F$ and doubly warped product manifold ${}_fB\times_b F$, respectively. Then we get the Levi-Civita connection of doubly warped product manifold ${}_fB\times_b F$ as follows:

$$\begin{aligned}\overline{\nabla}_X Y \quad &= \nabla_X Y + \frac{1}{2b^2}X_1(b^2)(0, Y_2) + \frac{1}{2b^2}Y_1(b^2)(0, X_2)\\ &+\frac{1}{2f^2}X_2(f^2)(Y_1, 0) + \frac{1}{2f^2}Y_2(f^2)(X_1, 0)\\ &-\frac{1}{2}g_B(X_1, Y_1)(grad\ f^2, 0) - \frac{1}{2}g_F(X_2, Y_2)(0, grad\ b^2)\end{aligned} \tag{3.3}$$

for any $X, Y \in \Gamma(T(B\times F))$, where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, $X_1, Y_1 \in \Gamma(TB)$ and $X_2, Y_2 \in \Gamma(TF)$.

If R and $\overline{R}$ denote the curvature tensors of $B\times F$ and ${}_fB\times_b F$, respectively then we have the following relation:

$$\begin{aligned}&\bar{R}(X,Y) - R(X,Y) =\\ &\frac{1}{2b^2}\{[((\nabla^B_{Y_1}\operatorname{grad} b^2 - \frac{1}{2b^2}Y_1(b^2)\operatorname{grad} b^2, 0) - \frac{1}{2f^2}(0, Y_1(b^2)\operatorname{grad} f^2)) \wedge_g (0, X_2)\\ &-((\nabla^B_{X_1}\operatorname{grad} b^2 - \frac{1}{2b^2}X_1(b^2)\operatorname{grad} b^2, 0) - \frac{1}{2f^2}(0, X_1(b^2)\operatorname{grad} f^2)) \wedge_g (0, Y_2)]\\ &+\frac{1}{2b^2}|\operatorname{grad} b^2|^2(0, X_2)\wedge_g (0, Y_2)\}\\ &+\frac{1}{2f^2}\{[((0, \nabla^F_{Y_2}\operatorname{grad} f^2 - \frac{1}{2f^2}Y_2(f^2)\operatorname{grad} f^2) - \frac{1}{2b^2}(Y_2(f^2)\operatorname{grad} b^2, 0)\wedge_g (X_1, 0)\\ &-((0, \nabla^F_{X_2}\operatorname{grad} f^2 - \frac{1}{2f^2}X_2(f^2)\operatorname{grad} f^2) - \frac{1}{2b^2}(X_2(f^2)\operatorname{grad} b^2, 0)\wedge_g (Y_1, 0)]\\ &\qquad + \frac{1}{2f^2}|\operatorname{grad} f^2|^2(X_1, 0)\wedge_g (Y_1, 0)\},\end{aligned} \tag{3.4}$$

where the wedge product $X\wedge_g Y$ denotes the linear map $Z \to g(Y,Z)X - g(X,Z)Y$ for all X, Y, $Z \in \Gamma(T(B\times F))$.

4 Biharmonicity of the inclusion maps

Let $({}_fB \times_b F, g)$ be a doubly warped product manifold. For $y_0 \in F$, let us consider the inclusion map of B

$$\begin{aligned} i_{y_0} : (B, g_B) &\to ({}_fB \times_b F, g) \\ x &\to (x, y_0) \end{aligned}$$

at the point y_0 level in ${}_fB \times_b F$ and for $x_0 \in B$ let

$$\begin{aligned} i_{x_0} : (F, g_F) &\to ({}_fB \times_b F, g) \\ y &\to (x_0, y) \end{aligned}$$

be the inclusion map of F at the point x_0 level in ${}_fB \times_b F$. In this section we obtain some non-existence results for the biharmonicity of inclusion maps i_{y_0} of B and i_{x_0} of F.

Theorem 4.1. *The inclusion map of the manifold (B, g_B) into the nontrivial (proper) doubly warped product manifold $({}_fB \times_b F, g)$ is never a proper biharmonic map.*

Proof. Let $\{B_j\}_{j=1}^m$ be an orthonormal frame on (B, g_B). By using the equation (2.1) we obtain the tension field of i_{y_0}

$$\begin{aligned} \tau(i_{y_0}) &= trace_{g_B} \nabla di_{y_0} \\ &= \sum_{j=1}^m \{\nabla_{B_j} di_{y_0}(B_j) - di_{y_0}(\nabla^B_{B_j} B_j)\} \\ &= \sum_{j=1}^m \{\overline{\nabla}_{(B_j,0)}(B_j, 0) - (\nabla^B_{B_j} B_j, 0)\} \\ &= -\frac{m}{2}(0, grad\ f^2)|_{i_{y_0}}. \end{aligned}$$

Here it's obvious from the expression of the tension field of i_{y_0} that i_{y_0} is harmonic if and only if $(grad\ f^2)|_{i_{y_0}} = 0$.

Now, to get the bitension field of $i_{y_0} : (B, g_B) \to ({}_fB \times_b F, g)$, firstly let us compute the rough Laplacian of the tension field of $\tau(i_{y_0})$. We have

$$\begin{aligned} \nabla_{B_j} \tau(i_{y_0}) &= -\frac{m}{2} \nabla_{B_j}(0, grad\ f^2)|_{i_{y_0}} \\ &= -\frac{m}{2}(\overline{\nabla}_{(B_j,0)}(0, grad\ f^2))|_{i_{y_0}} \\ &= -m(\frac{1}{4f^2}|grad\ f^2|^2(B_j, 0) + \frac{1}{4b^2}(B_j, 0)(b^2)(0, grad\ f^2))|_{i_{y_0}}. \end{aligned}$$

Then

$$\begin{aligned} \nabla_{B_j} \nabla_{B_j} \tau(i_{y_0}) &= -\frac{m}{4}\{\frac{1}{f^2}|grad\ f^2|^2(\nabla^B_{B_j} B_j, 0) \\ &\quad + \frac{1}{2b^2 f^2}(B_j, 0)(b^2)|grad\ f^2|^2(B_j, 0) \\ &\quad - \frac{1}{2}(\frac{1}{f^2}|grad\ f^2|^2 + \frac{1}{b^4}((B_j, 0)(b^2))^2)(0, grad\ f^2))\}|_{i_{y_0}}. \end{aligned} \tag{4.1}$$

Also we obtain

$$
\begin{aligned}
\nabla_{\nabla^B_{B_j}B_j}\tau(i_{y_0}) &= (-\frac{m}{4f^2}|grad\ f^2|^2(\nabla^B_{B_j}B_j,0) \\
&\quad -\frac{m}{4b^2}(\nabla^B_{B_j}B_j,0)(b^2)(0,grad\ f^2))|_{i_{y_0}}.
\end{aligned}
\tag{4.2}
$$

From the equations (4.1) and (4.2) the rough Laplacian of $\tau(i_{y_0})$ is

$$
\begin{aligned}
-\Delta\tau(i_{y_0}) &= \frac{m}{4}\sum_{j=1}^{m}\{-\frac{1}{2b^2f^2}(B_j,0)(b^2)|grad\ f^2|^2(B_j,0) \\
&\quad +(\frac{1}{2f^2}|grad\ f^2|^2-\frac{1}{2b^4}((B_j,0)(b^2))^2 \\
&\quad +\frac{1}{b^2}(\nabla^B_{B_j}B_j,0)(b^2))(0,grad\ f^2)\}|_{i_{y_0}}.
\end{aligned}
\tag{4.3}
$$

On the other hand by using (3.4) it can be seen that

$$
\begin{aligned}
trace_{g_b}\overline{R}(di_{y_0},\tau(i_{y_0}))di_{y_0} &= \{\frac{m}{4b^2}\sum_{j=1}^{m}(\nabla^B_{B_j}B_j)(b^2)(0,grad\ f^2) \\
&\quad +\frac{m^2}{8f^2}|grad\ f^2|^2(0,grad\ f^2) \\
&\quad -\frac{m}{2}\sum_{j=1}^{m}B_j(\frac{1}{2b^2}B_j(b^2))(0,grad\ f^2) \\
&\quad -\frac{m}{8b^2f^2}\sum_{j=1}^{m}((B_j,0)(b^2)|grad\ f^2|^2(B_j,0)) \\
&\quad -\frac{m}{8b^4}\sum_{j=1}^{m}(((B_j,0)(b^2))^2(0,grad\ f^2)) \\
&\quad -\frac{m^2}{4}(0,\nabla^F_{grad\ f^2}grad\ f^2) \\
&\quad +\frac{m^2}{8b^2}|grad\ f^2|^2(grad\ b^2,0)\}|_{i_{y_0}}.
\end{aligned}
$$

Subtracting the last equation from (4.3) we obtain the bitension field of $i_{y_0}:(B,g_B)\rightarrow({}_fB\times_b F,g)$ as follows:

$$
\begin{aligned}
\tau_2(i_{y_0}) &= \{-\frac{m^2}{8b^2}|grad\ f^2|^2(grad\ b^2,0) \\
&\quad +\frac{m}{2}\sum_{j=1}^{m}(B_j(\frac{1}{2b^2}B_j(b^2))(0,grad\ f^2)+\frac{m^2}{8}(0,grad(|grad\ f^2|^2)\}|_{i_{y_0}},
\end{aligned}
\tag{4.4}
$$

where $\{B_j\}_{j=1}^m$ is an orthonormal frame on (B,g_b). Therefore from (4.4) the inclusion map $i_y:(B,g_b)\rightarrow({}_fB\times_b F,g)$, $y\in F$, is a proper biharmonic map if and only if

$$
|grad\ f^2|^2 grad\ b^2=0 \tag{4.5}
$$

and

$$\sum_{j=1}^{m}(B_j(\frac{1}{2b^2}B_j(b^2))(grad\ f^2)+\frac{m}{4}grad(|grad\ f^2|^2)=0. \tag{4.6}$$

Since $grad\ f^2 \neq 0$, it can be seen from (4.5) that the warping function $b : B \to (0, \infty)$ must be a constant function. But this is a contradiction and we have the assertion of the Theorem. □

Remark 4.2. When $f = 1$, the doubly warped product manifold ${}_fB \times_b F$ becomes a warped product $B \times_b F$. Since the inclusion map $i_{y_0} : (B, g_B) \to (B \times_b F, g)$ of B at the level $y_0 \in F$ is always totally geodesic, then it's harmonic for any warping function $b \in C^\infty(B)$. So in the case of warped product biharmonicity of the inclusion map $i_{y_0} : (B, g_B) \to (B \times_b F, g)$ is trivial.

Now, let us consider the iclusion map $i_{x_0} : (F, g_F) \to ({}_fB \times_b F, g)$, $y \to (x_0, y)$, of (F, g_F) into the doubly warped product manifold $({}_fB \times_b F, g)$ where $x_0 \in B$. We have,

Theorem 4.3. *The inclusion map of the manifold (F, g_F) into the nontrivial (proper) doubly warped product manifold $({}_fB \times_b F, g)$ is never a proper biharmonic map.*

Remark 4.4. If $f = 1$, ${}_fB \times_b F$ becomes a warped product manifold and we obtain the corollaries 3.3, 3.4 and 3.5 in [4, page 454].

5 Product maps

Let $I_B : B \to B$ be the identity map on B and $\varphi : F \to F$ be a harmonic map. Obviuosly $\Psi = I_B \times \varphi : B \times F \to B \times F$ is a harmonic map. Now suppose the product manifold $B \times F$ (either as domain or codomain) with the doubly warped product metric tensor $g = f^2 g_B \oplus b^2 g_F$. In this case the product map is no longer harmonic. Therefore in this section we shall obtain some results for maps of product type to be proper biharmonic.

Firstly let us consider the product map

$$\overline{\Psi} = \overline{I_B \times \varphi} :\ {}_fB \times_b F \to B \times F.$$

Theorem 5.1. *Let (B, g_B) and (F, g_F) be Riemannian manifolds of dimensions m and n, respectively, and $b : B \to (0, \infty)$ and $f : F \to (0, \infty)$ denote smooth functions. Suppose that $\varphi : F \to F$ is a harmonic map. Then $\overline{\Psi} = \overline{I_B \times \varphi} :\ {}_fB \times_b F \to B \times F$ is a proper biharmonic map if and only if b is a non-constant solution of*

$$\frac{1}{f^2}trace_{g_b}\nabla^2 grad\ln b+\frac{1}{f^2}Ricc^B(grad\ln b)+\frac{n}{2}grad(|grad\ln b|^2)=0 \tag{5.1}$$

and f is a non-constant solution of

$$-\frac{1}{b^2}J_\varphi(d\varphi(grad\ln f))+\frac{m}{2}grad(|d\varphi(grad\ln f)|^2)=0. \tag{5.2}$$

Proof. Let $\{B_j\}_{j=1}^m$ and $\{F_r\}_{r=1}^n$ be local orthonormal frames on (B, g_B) and (F, g_F), respectively. Then $\{\frac{1}{f}(B_j, 0), \frac{1}{b}(0, F_r)\}_{j,r=1}^{m,n}$ is an local orthonormal frame on the doubly warped product manifold ${}_fB \times_b F$. In order to obtain bitension field of $\overline{\Psi}$, firstly we will compute the tension field of $\overline{\Psi}$. Since φ is harmonic,

$$\begin{aligned}
\tau(\overline{\Psi}) &= trace_g \nabla d\overline{\Psi} \\
&= \frac{1}{f^2}\sum_{j=1}^{m} \nabla d\overline{\Psi}((B_j, 0), (B_j, 0)) + \frac{1}{b^2}\sum_{r=1}^{n} \nabla d\overline{\Psi}((0, F_r), (0, F_r)) \\
&= \frac{1}{f^2}\sum_{j=1}^{m}\{\nabla^{\overline{\Psi}}_{(B_j,0)} d\overline{\Psi}(B_j, 0) - d\overline{\Psi}(\overline{\nabla}_{(B_j,0)}(B_j, 0))\} \\
&\quad + \frac{1}{b^2}\sum_{r=1}^{n}\{\nabla^{\overline{\Psi}}_{(0,F_r)} d\overline{\Psi}(0, F_r) - d\overline{\Psi}(\overline{\nabla}_{(0,F_r)}(0, F_r))\} \\
&= \frac{1}{f^2}\sum_{j=1}^{m}\{\nabla_{(B_j,0)}(B_j, 0) - d\overline{\Psi}(\overline{\nabla}_{(B_j,0)}(B_j, 0))\} \\
&\quad + \frac{1}{b^2}\sum_{r=1}^{n}\{\nabla_{d\overline{\Psi}(0,F_r)} d\overline{\Psi}(0, F_r) - d\overline{\Psi}(\overline{\nabla}_{(0,F_r)}(0, F_r))\} \\
&= n(grad \ln b, 0) + m(0, d\varphi(grad \ln f)).
\end{aligned}$$

By a straightforward calculation we have

$$\begin{aligned}
-\Delta\tau(\overline{\Psi}) &= trace_g \nabla^2 \tau(\overline{\Psi}) \\
&= \sum_{j=1}^{m}\{\nabla^{\overline{\Psi}}_{\frac{1}{f}(B_j,0)} \nabla^{\overline{\Psi}}_{\frac{1}{f}(B_j,0)} \tau(\overline{\Psi}) - \nabla^{\overline{\Psi}}_{\overline{\nabla}_{\frac{1}{f}(B_j,0)}\frac{1}{f}(B_j,0)} \tau(\overline{\Psi})\} \\
&\quad + \sum_{r=1}^{n}\{\nabla^{\overline{\Psi}}_{\frac{1}{b}(0,F_r)} \nabla^{\overline{\Psi}}_{\frac{1}{b}(0,F_r)} \tau(\overline{\Psi}) - \nabla^{\overline{\Psi}}_{\overline{\nabla}_{\frac{1}{b}(0,F_r)}\frac{1}{b}(0,F_r)} \tau(\overline{\Psi})\} \\
&= (\frac{n}{f^2} trace_{g_b} \nabla^2 grad \ln b + n^2 \nabla_{grad \ln b} grad \ln b, 0) \\
&\quad + (0, \frac{m}{b^2} trace_{g_f} \nabla^2 (d\varphi(grad \ln f)) + m^2 \nabla_{d\varphi(grad \ln f)} d\varphi(grad \ln f))
\end{aligned}$$

Also by using the usual definition of curvature tensor field on $B \times F$, one can easily see that

$$\begin{aligned}
trace_g R(d\overline{\Psi}, \tau(\overline{\Psi}))d\overline{\Psi} &= -\frac{n}{f^2}(Ricc^B(grad \ln b), 0) \\
&\quad + \frac{m}{b^2}(0, trace_{g_f} R^F(d\varphi, d\varphi(grad \ln f))d\varphi).
\end{aligned}$$

Finally bitension field of $\overline{\Psi}$ is

$$\begin{aligned}\tau_2(\overline{\Psi}) &= n(\frac{1}{f^2}trace_{g_b}\nabla^2 grad\ln b + \frac{1}{f^2}Ricc^B(grad\ln b) + \frac{n}{2}grad(|grad\ln b|^2), 0) \\ &+m(0, \frac{1}{b^2}trace_{g_f}\nabla^2(d\varphi(grad\ln f))) \\ &-m(0, \frac{1}{b^2}trace_{g_f}R^F(d\varphi, d\varphi(grad\ln f))d\varphi) \\ &+m^2(0, \frac{1}{2}grad(|d\varphi(grad\ln f)|^2)).\end{aligned}$$

and we conclude. □

Example 5.2. If $B = R$ and $\varphi : R \to R$ is an identity map of $F = R$, then Eq.(5.1) and Eq.(5.2) become

$$\frac{1}{f^2}\alpha''' + \alpha'\alpha'' = 0, \tag{5.3}$$

$$\frac{1}{b^2}\gamma''' + \gamma'\gamma'' = 0, \tag{5.4}$$

respectively, where $\alpha = \ln b$ and $\gamma = lnf$, and we obtain two non-linear partial differential equations. Considering the case when $\alpha''' = 0$ and $\gamma''' = 0$ we obtain the warping functions $b(t) = e^{a_1 t + a_2}$ and $f(s) = e^{b_1 s + b_2}$ where $a_1, a_2, b_1, b_2 \in R$. Thus we have special solutions of the non-linear partial differential equations (5.3) and (5.4) and we get two non-constant warping functions b and f that could render $\overline{\Psi} = \overline{I_B \times \varphi} : {}_fB \times_b F \to B \times F$ proper biharmonic.

We shall now investigate the biharmonicity of the projection $\bar{\pi} :_f B \times_b F \to B$ onto the first factor. By a straightforward calculation, we get $\tau(\bar{\pi}) = n(grad\ln b) \circ \bar{\pi}$, and the bitension field of $\bar{\pi}$ is

$$\begin{aligned}\tau_2(\bar{\pi}) &= \frac{n}{f^2}trace_{g_b}\nabla^2 grad\ln b \\ &\quad + \frac{n}{f^2}Ricc^B(grad\ln b) + \frac{n^2}{2}grad(|grad\ln b|^2).\end{aligned}$$

By using the bitension field of the projection $\bar{\pi}$, the biharmonic equation of the product map $\overline{\Psi} = \overline{I_B \times \varphi} : {}_fB \times_b F \to B \times F$ has the expression

$$\tau_2(\overline{\Psi}) = (\tau_2(\bar{\pi}), \Lambda(d\varphi(grad\ln f)) = 0,$$

where

$$\Lambda(d\varphi(grad\ln f)) = -\frac{m}{b^2}J_\varphi(d\varphi(grad\ln f) + \frac{m^2}{2}grad(|d\varphi(grad\ln f)|^2).$$

So we have

Corollary 5.3. *The product map* $\overline{\Psi} = \overline{I_B \times \varphi} : {}_fB \times_b F \to B \times F$ *is a proper biharmonic map if and only if* $\Lambda(d\varphi(grad\ln f)) = 0$ *and the projection* $\bar{\pi}$ *is a proper biharmonic map.*

If $f=1$ or $\varphi : F \to F$ is a non-zero constant map the biharmonicity of the product map $\overline{\Psi} = \overline{I_B \times \varphi} : {}_fB \times_b F \to B \times F$ and the biharmonicity of the projection $\overline{\pi} :_f B \times_b F \to B$ onto the first factor coincide. In the case of $f=1$, we have the same results obtained in [4] for warped product manifolds.

Now let us consider the product map $\widetilde{\Psi} = \widetilde{\varphi \times I_F} : {}_fB \times_b F \to B \times F$ of the harmonic map $\varphi : B \to B$ and the identitiy map $I_F : F \to F$.

Theorem 5.4. *Let (B, g_B) and (F, g_F) be Riemannian manifolds of dimensions m and n, respectively and let $b : B \to (0,\infty)$and $f : F \to (0,\infty)$ be smooth functions. Suppose that $\varphi : B \to B$ is a harmonic map. Then $\widetilde{\Psi} = \widetilde{\varphi \times I_F} : {}_fB \times_b F \to B \times F$ is a proper biharmonic map if and only if b is a non-constant solution of*

$$-\frac{1}{f^2} J_\varphi(d\varphi(grad \ln b)) + \frac{n}{2} grad(|d\varphi(grad \ln b)|^2) = 0$$

and f is a non-constant solution of

$$\frac{1}{b^2} trace_{g_f} \nabla^2 grad \ln f + \frac{1}{b^2} Ricc^F(grad \ln f) + \frac{m}{2} grad(|grad \ln f|^2) = 0.$$

By a similar discussion, carried out to set up the relation between the biharmonicity of product map $\overline{\Psi} = \overline{I_B \times \varphi} : {}_fB \times_b F \to B \times F$ and the biharmonicity of the projection $\overline{\pi} :_f B \times_b F \to B$ onto the first factor, the bitension field of the projection $\widetilde{\sigma} :_f B \times_b F \to F$ onto the second factor is related to the bitension field of $\widetilde{\Psi} = \widetilde{\varphi \times I_F} : {}_fB \times_b F \to B \times F$ as follows:

By computing the second fundamental form of the projection $\widetilde{\sigma}$, we get $\tau(\widetilde{\sigma}) = m(grad \ln f) \circ \widetilde{\sigma}$, and the bitension field of $\widetilde{\sigma}$ is

$$\begin{aligned} \tau_2(\widetilde{\sigma}) &= \frac{m}{b^2} trace_{g_f} \nabla^2 grad \ln f \\ &\quad + \frac{m}{b^2} Ricc^F(grad \ln f) + \frac{m^2}{2} grad(|grad \ln f|^2). \end{aligned}$$

Now by using the bitension field of the projection $\widetilde{\sigma}$, the biharmonic equation of the product map $\widetilde{\Psi} = \widetilde{\varphi \times I_F} : {}_fB \times_b F \to B \times F$ is

$$\tau_2(\widetilde{\Psi}) = (\Omega(d\varphi(grad \ln b), \tau_2(\widetilde{\sigma})) = 0,$$

where

$$\Omega(d\varphi(grad \ln b)) = -\frac{n}{f^2} J_\varphi(d\varphi(grad \ln b)) + \frac{n^2}{2} grad(|d\varphi(grad \ln b)|^2).$$

Corollary 5.5. *The product map $\widetilde{\Psi} = \widetilde{\varphi \times I_F} : {}_fB \times_b F \to B \times F$ is a proper biharmonic map if and only if $\Omega(d\varphi(grad \ln b)) = 0$ and the projection $\widetilde{\sigma}$ is a proper biharmonic map.*

Note that especially if $\varphi : B \to B$ is the identity map then the bitension field of of the product map $\widetilde{\Psi} = \widetilde{\varphi \times I_F} : {}_fB \times_b F \to B \times F$ has the expression $\tau_2(\widetilde{\Psi}) = (\tau_2(\overline{\pi}), \tau_2(\widetilde{\sigma}))$.

Consider now the case of the product map $\widehat{\Psi} = \widehat{I_B \times \varphi} : B \times F \to_f B \times_b F$, that is the case of the product metric on the codomain is doubly warped. We will see that the energy density of the harmonic map $\varphi : F \to F$ has an important role for the biharmonicity of the product map $\widehat{\Psi} = \widehat{I_B \times \varphi}$. We have

Theorem 5.6. *Let (B, g_B) and (F, g_F) be Riemannian manifolds of dimensions m and n, respectively and let $b : B \to (0, \infty)$ and $f : F \to (0, \infty)$ be smooth functions. Suppose $I_B : B \to B$ is the identity map and $\varphi : F \to F$ is a harmonic map. Then the product map $\widehat{\Psi} = \widehat{I_B \times \varphi} : B \times F \to_f B \times_b F$ is a proper biharmonic map if and only if*

$$\begin{aligned} 0 \;=\;& e(\varphi)\{-trace_{g_b}\nabla^2(\operatorname{grad} b^2) - Ricc^B(\operatorname{grad} b^2) + \frac{e(\varphi)}{2}\operatorname{grad}(|\operatorname{grad} b^2|^2)\} \\ &+\frac{m}{2}\{\frac{e(\varphi)}{f^2} - \frac{m}{2b^2}\}|\operatorname{grad} f^2|^2\operatorname{grad} b^2 \\ &+\{\frac{m}{4b^2}\Delta^\varphi(f^2) + \sum_{r=1}^{n}\zeta^F_{f^2}(F_r)\}\operatorname{grad} b^2 \end{aligned}$$

and

$$\begin{aligned} 0 \;=\;& \frac{m}{2}\{-J_\varphi(\operatorname{grad} f^2) + \frac{m}{4}\operatorname{grad}(|\operatorname{grad} f^2|^2)\} \\ &-\frac{e(\varphi)}{2}\{\frac{e(\varphi)}{f^2} - \frac{m}{b^2}\}|\operatorname{grad} b^2|^2\operatorname{grad} f^2 - \frac{|\operatorname{grad} b^2|^2}{2b^2}d\varphi(\operatorname{grad} e(\varphi)) \\ &+\{\frac{e(\varphi)}{2f^2}\Delta(b^2) + \sum_{j=1}^{m}\eta^B_{b^2}(B_j)\}\operatorname{grad} f^2, \end{aligned}$$

where

$$\eta^B_{b^2}(X) = -\frac{e(\varphi)}{2f^2}(\nabla^B_X X)(b^2) + \frac{m}{4}X(\frac{1}{b^2}X(b^2))$$

and

$$\zeta^F_{f^2}(Y) = -\frac{m}{4b^2}(\nabla^F_{d\varphi(Y)}d\varphi(Y))(f^2) + \frac{e(\varphi)}{2}d\varphi(Y)(\frac{1}{f^2}d\varphi(Y)(f^2)).$$

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