Linear Weingarten hypersurfaces in locally symmetric manifolds

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Abstract. In this paper, we discuss about the complete linear Weingarten hypersurfaces in locally symmetric manifold and obtain a rigidity theorem. More precisely, under a suitable restriction on the square norm of the second fundamental form, we prove that such a hypersurface must be either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.

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1 Introduction

Recently, many researchers studied the minimal hypersurfaces or hypersurfaces with constant mean (or scalar) curvature in the locally symmetric manifolds and the δ-pinched manifolds, and obtained many rigidity results about these hypersurfaces ([4, 8, 9] and the references therein). As a natural generalization of hypersurface with constant scalar curvature or with constant mean curvature, linear Weingarten hypersurface has been studied in many places ([1, 2, 5]). Recall that a hypersurface in a Riemannian manifold is said to be linear Weingarten if its (normalized) scalar curvature r and its mean curvature H are related by $r = aH + b$ for some constants $a, b \in \mathbb{R}$. In this paper, we modify Cheng-Yau's technique to complete linear Weingarten hypersurfaces in locally symmetric manifolds and obtain some rigidity theorems. More precisely, we have

Theorem 1.1. *Let M^n be an n-dimensional complete orientable hypersurface immersed in the locally symmetric manifold N^{n+1} $(n \geq 3)$ satisfying $\frac{1}{2} < \delta \leq K_N \leq 1$ and $K_{n+1in+1i} = c_0$. Assume that M^n has bounded mean curvature and $r = aH + b, a, b \in \mathbb{R}$, $a \leq 0$, $b > 1$. If $S \leq 2\sqrt{n-1}(2\delta - c_0)$, then either M^n is a totally umbilical hypersurface or $\sup S = 2\sqrt{n-1}(2\delta - c_0)$. Moreover, if $\sup S = 2\sqrt{n-1}(2\delta - c_0)$ and this supremum is attained at some point of M^n, then M^n is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.*

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When $\delta = c_0 = 1$, N^{n+1} is the unit sphere $S^{n+1}(1)$, so we have the following corollary.

Corollary 1.2. *Let M^n be an n-dimensional complete orientable hypersurface immersed in $S^{n+1}(1)$. Assume that M^n has bounded mean curvature and $r = aH + b, a, b \in \mathbb{R}$, $a \leq 0$, $b > 1$. If $S \leq 2\sqrt{n-1}$, then either M^n is a totally umbilical hypersurface or $\sup S = 2\sqrt{n-1}$. Moreover, if $\sup S = 2\sqrt{n-1}$ and this supremum is attained at some point of M^n, then M^n is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.*

2 Preliminaries

Let N^{n+1} be a locally symmetric manifold and M^n be an n-dimensional complete orientable hpersurface in N^{n+1}. For any $p \in M$, we choose a local orthonormal frame $e_1, \cdots, e_{n+1}$ in N^{n+1} around p such that $e_1, \cdots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. Let $\omega_1, \cdots, \omega_{n+1}$ be the corresponding dual coframe. Then the Riemannian metric tensor h of N^{n+1} is given by $h = \sum\limits_A \omega_A \otimes \omega_A$. Here and in the sequel, we use the following standard convention for indices:

$$1 \leq A, B, C, \cdots \leq n+1, \qquad 1 \leq i, j, k, \cdots \leq n.$$

Associated with the frame field $\{e_A\}$, there exist 1-forms $\{\omega_{AB}\}$ which are usually called as connection forms on N^{n+1} so that they satisfy the structure equations of N^{n+1}:

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

where K_{ABCD} are the components of the curvature tensor of N^{n+1}.

Restricting these forms to M^n, we have $\omega_{n+1} = 0$ and the induced Riemannian metric tensor g of M^n is given by $g = \sum\limits_i \omega_i \otimes \omega_i$. Since $0 = d\omega_{n+1} = -\sum_i \omega_{n+1i} \wedge \omega_i$, from Cartan lemma, we have

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.3}$$

The quadratic form $B = \sum\limits_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$ with values in the normal bundle is called the second fundamental form of M^n. The mean curvature vector h is defined by

$$h = \frac{1}{n}\sum_i h_{ii}e_{n+1}.$$

The length of the mean curvature vector is called the mean curvature of M^n, denote by H. When $h \neq 0$, we choose e_{n+1} such that $H = |h| = \frac{1}{n}\sum\limits_i h_{ii}$.

It follows from the structure equations of N^{n+1} that the structure equations of M^n are

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.4}$$

$$d\omega_{ij} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l=1}^{n} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.5}$$

where R_{ijkl} are the components of the curvature tensor of M^n. Then the Gauss equations are

$$R_{ijkl} = K_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.6}$$

$$n(n-1)r = \sum_{i,j} K_{ijij} + n^2H^2 - S, \tag{2.7}$$

where r and $S = \sum_{i,j} h_{ij}^2$ are the normalized scalar curvature and the square norm of the second fundamental form of M^n, respectively.

The Codazzi and Ricci equations are

$$h_{ijk} - h_{ikj} = -K_{n+1ijk}, \tag{2.8}$$

$$K_{n+1ijkl} = K_{n+1in+1k}h_{jl} + K_{n+1ijn+1}h_{kl} - \sum_m K_{mijk}h_{ml}, \tag{2.9}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj}. \tag{2.10}$$

Similarly, the components h_{ijkl} of the second derivative $\nabla^2 h$ are given by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}. \tag{2.11}$$

The Laplacian $\triangle h_{ij}$ of h_{ij} is defined by

$$\triangle h_{ij} = \sum_k h_{ijkk}.$$

By a simple and direct calculation, we have

$$\begin{aligned} \triangle h_{ij} &= \sum_k \big[(h_{ijkk} - h_{ikjk}) + (h_{ikjk} - h_{ikkj}) + (h_{ikkj} - h_{kkij}) + h_{kkij}\big] \\ &= \sum_k K_{n+1ikjk} + \sum_{k,m}(h_{mi}R_{mkjk} + h_{mk}R_{mijk}) + \sum_k K_{n+1kkij} + \sum_k h_{kkij} \\ &= (nH)_{ij} + nHK_{n+1in+1j} - \sum_k h_{ij}K_{n+1kn+1k} + nH\sum_k h_{ik}h_{kj} \\ &\quad - Sh_{ij} + \sum_k \big[h_{mi}K_{mkjk} + h_{mj}K_{mkik} + 2h_{km}K_{mijk}\big]. \end{aligned} \tag{2.12}$$

Since (h_{ij}) is symmetric, we may choose a local orthonormal frame $\{e_i\}$ such that at arbitrary fixed point p on M^n

$$h_{ij} = \lambda_i \delta_{ij}, \tag{2.13}$$

where $\lambda_i's$ are the principal curvatures of M^n. Then it follows, at p, that

$$\begin{aligned}\frac{1}{2}\triangle S =&\frac{1}{2}\sum_{i,j}\triangle h_{ij}^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\triangle h_{ij}\\ =&\sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} - S^2 + nH\sum_i \lambda_i^3\\ &+ nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i K_{n+1in+1i}\\ &+\sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ijij}.\end{aligned} \tag{2.14}$$

Set $\phi_{ij} = h_{ij} - H\delta_{ij}$, it is easy to check that ϕ is traceless and

$$|\phi|^2 = \sum_{i,j}(\phi_{ij})^2 = S - nH^2, \tag{2.15}$$

where ϕ denotes the matrix (ϕ_{ij}). Moreover, $|\phi|^2 = S - nH^2 \geq 0$ with equality holds if and only if M^n is totally umbilical.

Lemma 2.1 ([6]). *Let $u_1, u_2, \cdots, u_n$ be real numbers such that $\sum_i u_i = 0$ and $\sum_i u_i^2 = \beta$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i u_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3,$$

and equality holds if and only if at least $n-1$ of $u_i's$ are equal.

Lemma 2.2. *Let N^{n+1} be a locally symmetric manifold satisfying $\frac{1}{2} < \delta \leq K_N \leq 1$ and M^n be an n-dimensional complete orientable hypersurface immersed in N^{n+1} with $r = aH + b, a, b \in \mathbb{R}$ and $(n-1)a^2 + 4n(b-1) \geq 0$. Then we have*

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2, \tag{2.16}$$

and equality holds if and only if $|\nabla H|^2 = 0$ or $4n^2S = (2n^2H - n(n-1)a)^2$.

Proof. From Gauss equation (2.7), we have

$$\begin{aligned}S =& \sum_{i,j} K_{ijij} + n^2H^2 - n(n-1)r\\ =& \sum_{i,j} K_{ijij} + n^2H^2 - n(n-1)(aH+b).\end{aligned} \tag{2.17}$$

Since N^{n+1} is locally symmetric, taking the covariant derivative on both sides of the above equation, we have

$$2\sum_{i,j} h_{ij}h_{ijk} = 2n^2HH_k - n(n-1)aH_k.$$

Therefore,

$$4S\sum_{i,j,k} h_{ijk}^2 \geq 4\sum_k \Big(\sum_{i,j} h_{ij}h_{ijk}\Big)^2 = (2n^2H - n(n-1)a)^2|\nabla H|^2. \tag{2.18}$$

We know from $0 < \delta \leq K_{ijij} \leq 1$ that

$$\begin{aligned}
&(2n^2H - n(n-1)a)^2 - 4n^2S\\
=&4n^4H^2 + n^2(n-1)^2a^2 - 4n^3(n-1)aH\\
&- 4n^2\Big(\sum_{i,j} K_{ijij} + n^2H^2 - n(n-1)(aH+b)\Big)\\
\geq&4n^4H^2 + n^2(n-1)^2a^2 - 4n^3(n-1)aH\\
&- 4n^2\Big(n(n-1) + n^2H^2 - n(n-1)(aH+b)\Big)\\
=&n^2(n-1)^2a^2 + 4n^3(n-1)(b-1)\\
=&n^2(n-1)\Big((n-1)a^2 + 4n(b-1)\Big) \geq 0. \qquad (2.19)
\end{aligned}$$

It follows (2.18) and (2.19) that

$$4S\sum_{i,j,k} h_{ijk}^2 \geq (2n^2H - n(n-1)a)^2|\nabla H|^2 \geq 4n^2S|\nabla H|^2.$$

Thus either $S = 0$ and $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$ or $\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2$.

If $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$, from (2.17) and (2.18), we have

$$\begin{aligned}
0 &\leq n^2(n-1)\Big((n-1)a^2 + 4n(b-1)\Big)|\nabla H|^2\\
&\leq (2n^2H - n(n-1)a)^2|\nabla H|^2 - 4n^2S|\nabla H|^2\\
&\leq 4S\sum_{i,j,k} h_{ijk}^2 - 4n^2S|\nabla H|^2 = 4S\Big(\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2\Big) = 0.
\end{aligned}$$

Then we conclude that $|\nabla H|^2 = 0$ or $4n^2S = (2n^2H - n(n-1)a)^2$. □

Following Cheng-Yau [3], as in [2], we introduce a modified operator □ acting on any C^2- function f by

$$\Box(f) = \sum_{i,j} \Big(\big(nH - \frac{n-1}{2}a\big)\delta_{ij} - h_{ij}\Big)f_{ij}, \tag{2.20}$$

where f_{ij} is given by the following

$$\sum_j f_{ij}\omega_j = df_i + f_j\omega_{ij}.$$

Lemma 2.3. *Let N^{n+1} be a locally symmetric manifold satisfying $\frac{1}{2} < \delta \leq K_N \leq 1$ and M be an n-dimensional orientable linear Weingarten hypersurface with $r = aH+b$ immersed in N^{n+1}. If $a \leq 0$ and $b > 1$, then $\Box$ is elliptic.*

Proof. Since $r = aH + b$ and $K_N \leq 1$, from Gauss equation (2.7), we have

$$n(n-1)(aH+b) \leq n(n-1) + n^2H^2 - S,$$

i.e.

$$S \leq n^2H^2 - n(n-1)(b-1) - n(n-1)aH. \tag{2.21}$$

Then it follows from $b > 1$ that

$$n^2H^2 - n(n-1)aH - S \geq n(n-1)(b-1) > 0. \tag{2.22}$$

Therefore $H \neq 0$. Thus we can assume $H > 0$ on M. So $\Box$ is elliptic if and only if $nH - \frac{n-1}{2}a - \lambda_i > 0$ for $i = 1, 2, \cdots, n$, where $\lambda_i' s$ are the principal curvatures of M. If, for some i, $nH - \frac{n-1}{2}a - \lambda_i \leq 0$ holds, then $0 < nH - \frac{n-1}{2}a \leq \lambda_i$ and

$$(nH - \frac{n-1}{2}a)^2 \leq \lambda_i^2 \leq S,$$

$$n^2H^2 - n(n-1)aH + \frac{1}{4}(n-1)^2a^2 \leq S.$$

This together with (2.22) gives

$$S < n^2H^2 - n(n-1)aH \leq S,$$

which is a contradiction. So $\Box$ is an elliptic operator. □

Proposition 2.4. *Let N^{n+1} $(n \geq 3)$ be a locally symmetric manifold satisfying $\frac{1}{2} < \delta \leq K_N \leq 1$, $K_{n+1in+1i} = c_0$ and M^n be an n-dimensional complete orientable hypersurface immersed in N^{n+1} with $r = aH+b$, $a, b \in \mathbb{R}$ and $(n-1)a^2+4n(b-1) \geq 0$. Then*

$$\Box(nH) \geq -\frac{n}{2\sqrt{n-1}}[S - 2\sqrt{n-1}(2\delta - c_0)]|\phi|^2. \tag{2.23}$$

Proof. First, (2.20) gives

$$\begin{aligned}
\Box(nH) &= \sum_{i,j}((nH - \frac{1}{2}(n-1)a)\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} \\
&= (nH - \frac{1}{2}(n-1)a)\triangle(nH) - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\
&= (nH - \frac{1}{2}(n-1)a)\triangle(nH - \frac{1}{2}(n-1)a) - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\
&= \frac{1}{2}\triangle(nH - \frac{1}{2}(n-1)a)^2 - |\nabla(nH + \frac{1}{2}(n-1)a)|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\
&= \frac{1}{2}\triangle(nH - \frac{1}{2}(n-1)a)^2 - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}. \qquad (2.24)
\end{aligned}$$

Since the scalar curvature $\bar{R}$ of a locally symmetric manifold is constant. Then it follows from

$$\bar{R} = 2\sum_i K_{n+1in+1i} + \sum_{i,j} K_{ijij} = 2nc_0 + \sum_{i,j} K_{ijij},$$

that $\sum_{i,j} K_{ijij}$ is constant. Therefore, from Gauss equation (2.7) and $r = aH + b$, we have

$$\begin{aligned}\triangle S &= \triangle\Big(\sum_{i,j} K_{ijij} + n^2H^2 - n(n-1)r\Big)\\ &= \triangle(n^2H^2 - n(n-1)(aH+b))\\ &= \triangle(n^2H^2 - n(n-1)aH)\\ &= \triangle(nH - \frac{1}{2}(n-1)a)^2. \end{aligned}\tag{2.25}$$

Combining (2.14) (2.24) and (2.25), we get

$$\begin{aligned}\Box(nH) =& \frac{1}{2}\triangle S - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}\\ =& \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 - S^2 + nH\sum_i \lambda_i^3 + \sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ijij}\\ &+ nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i K_{n+1in+1i}. \end{aligned}\tag{2.26}$$

Set $\mu_i = \lambda_i - H$, it is easy to check that

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = |\phi|^2 = S - nH^2, \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3HS + 2nH^3.$$

Then, for any $\varepsilon > 0$, we have

$$\begin{aligned}-S^2 + nH\sum_i \lambda_i^3 &= -S^2 + nH\sum_i \mu_i^3 + 3nH^2S - 2n^2H^4\\ &\geq -\frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|^3 + nH^2|\phi|^2 - |\phi|^4\\ &\geq -\frac{n-2}{2\sqrt{n-1}}\Big(n\varepsilon H^2 + \frac{1}{\varepsilon}|\phi|^2\Big)|\phi|^2 + nH^2|\phi|^2 - |\phi|^4, \end{aligned}\tag{2.27}$$

where the second inequality uses the absorbing inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2$. When $n \geq 3$, taking $\varepsilon = \frac{n+2\sqrt{n-1}}{n-2}$ in (2.27), we get

$$-S^2 + nH\sum_i \lambda_i^3 \geq -\frac{n}{2\sqrt{n-1}}(nH^2|\phi|^2 + |\phi|^4) = -\frac{n}{2\sqrt{n-1}}S|\phi|^2. \tag{2.28}$$

Since N is a δ-pinched manifold, we have

$$\sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ijij} \geq \delta\sum_{i,j}(\lambda_i - \lambda_j)^2 = 2n\delta|\phi|^2, \tag{2.29}$$

At the same time, using the curvature condition, we have

$$nH\sum_i \lambda_i K_{n+1in+1i} - S\sum_i K_{n+1in+1i} = nc_0(n^2H^2 - S) = -nc_0|\phi|^2. \tag{2.30}$$

From (2.26) (2.28) (2.29) (2.30) and Lemma 2.2, we see that

$$\begin{aligned} \square(nH) &\geq -nc_0|\phi|^2 + 2n\delta|\phi|^2 - \frac{n}{2\sqrt{n-1}}S|\phi|^2 \\ &= -\frac{n}{2\sqrt{n-1}}[S - 2\sqrt{n-1}(2\delta - c_0)]|\phi|^2. \end{aligned} \tag{2.31}$$

□

We also need the well known generalized Maximum Principle due to H. Omori.

Lemma 2.5 ([7]). *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $f : M^n \to \mathbb{R}$ be a smooth function which is bounded from above on M^n. Then there is a sequence of points $\{p_k\}$ in M^n such that*

$$\lim_{k\to\infty} f(p_k) = \sup f; \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0; \quad \limsup_{k\to\infty}(\triangle f(p_k)) \leq 0.$$

Proposition 2.6. *Let M^n be a n-dimensional complete orientable hypersurface of locally symmetric manifold N^{n+1} $(n \geq 3)$ satisfying $\frac{1}{2} < \delta \leq K_N \leq 1$ and $K_{n+1in+1i} = c_0$. If M has bounded mean curvature and $r = aH + b, a, b \in \mathbb{R}$, $a \leq 0$, $(n-1)a^2 + 4n(b-1) \geq 0$. Then there is sequence of points $\{p_k\} \in M^n$ such that*

$$\lim_{k\to\infty} nH(p_k) = n\sup H; \quad \lim_{k\to\infty} |\nabla nH(p_k)| = 0; \quad \limsup_{k\to\infty}(\square(nH)(p_k)) \leq 0.$$

Proof. Choose a local orthonormal frame field $e_1, \ldots, e_n$ at $p \in M^n$ such that $h_{ij} = \lambda_i\delta_{ij}$. Thus

$$\square(nH) = \sum_i \Big[(nH - \frac{1}{2}(n-1)a) - \lambda_i\Big](nH)_{ii}.$$

If $H \equiv 0$ the proposition holds trivially. Now we may assume $\sup H > 0$ if H is not identically zero by choosing the appropriate orientation of M^n. From

$$\begin{aligned} \lambda_i^2 \leq S &= n^2H^2 + \sum_{i,j} K_{ijij} - n(n-1)(aH + b) \\ &= (nH)^2 - (n-1)a(nH) - n(n-1)b + \sum_{i,j} K_{ijij} \\ &\leq \big(nH - \frac{1}{2}(n-1)a\big)^2 - \frac{1}{4}(n-1)((n-1)a^2 + 4nb - 4n) \\ &\leq \big(nH - \frac{1}{2}(n-1)a\big)^2, \end{aligned}$$

we have

$$|\lambda_i| \leq \big|nH - \frac{1}{2}(n-1)a\big|. \tag{2.32}$$

Then

$$R_{ijij} = K_{ijij} + \lambda_i\lambda_j \geq c - \big(nH - \frac{1}{2}(n-1)a\big)^2. \tag{2.33}$$

Since H is bounded, it follows from (2.33) that the sectional curvatures are bounded from below. Then we may obtain a sequence of points $\{p_k\} \in M^n$, by applying Lemma 2.5 to nH, such that

$$\lim_{k\to\infty} nH(p_k) = n\sup H;\ \ \lim_{k\to\infty} |\nabla nH(p_k)| = 0;\ \ \limsup_{k\to\infty}((nH)_{ii}(p_k)) \leq 0. \tag{2.34}$$

Since H is bounded, taking subsequences if necessary, we can arrive to a sequence $\{p_k\} \in M^n$ which satisfies (2.34) and such that $H(p_k) \geq 0$. This together with (2.32) gives

$$\begin{aligned} 0 \leq nH(p_k) - \frac{1}{2}(n-1)a - |\lambda_i(p_k)| &\leq nH(p_k) - \frac{1}{2}(n-1)a + |\lambda_i(p_k)| \\ &\leq 2nH(p_k) - (n-1)a. \end{aligned} \tag{2.35}$$

Using once more the fact that H is bounded, from (2.35) we infer that $nH(p_k) - \frac{1}{2}(n-1)a - \lambda_i(p_k)$ is non-negative and bounded. By applying $\Box(nH)$ at p_k, taking the limit and using (2.34) and (2.35), we have

$$\limsup_{k\to\infty}(\Box(nH)(p_k)) \leq \sum_i \limsup_{k\to\infty} \Big[(nH - \frac{1}{2}(n-1)a) - \lambda_i\Big](p_k)(nH)_{ii}(p_k) \leq 0.$$

□

3 Proof of Theorem 1.1

From the assumption of theorem 1.1, we may assume that $H > 0$ on M^n. Then Proposition 2.6 gives that there exist a sequence of points $\{p_k\} \in M^n$ such that

$$\limsup_{k\to\infty}(\Box(nH)(p_k)) \leq 0, \quad \lim_{k\to\infty} H(p_k) = \sup H > 0. \tag{3.1}$$

On the other hand, from Gauss equation (2.7), we have

$$|\phi|^2 = S - nH^2 = n(n-1)(H^2 - aH - b) + \sum_{i,j} K_{ijij}. \tag{3.2}$$

In view of $\lim_{k\to\infty} H(p_k) = \sup H$ and $a \leq 0$, (3.2) implies that $\lim_{k\to\infty} |\phi|^2(p_k) = \sup|\phi|^2$ and $\lim_{k\to\infty} S(p_k) = \sup S$. Evaluating (2.23) at the points p_k of the sequence, taking the limit and using (3.1), we obtain that

$$0 \geq \limsup_{k\to\infty}(\Box(nH)(p_k)) \geq -\frac{n}{2\sqrt{n-1}}[\sup S - 2\sqrt{n-1}(2\delta - c_0)]\sup|\phi|^2 \geq 0.$$

Then it follows that either $\sup|\phi|^2 = 0$ and M^n is totally umbilical or $\sup S = 2\sqrt{n-1}(2\delta - c_0)$.

From Gauss equation (2.7), (2.23) and $\sup S \leq 2\sqrt{n-1}(2\delta - c_0)$, we have

$$\begin{aligned}\Box(S) &= \Box(n^2H^2) - n(n-1)\Box(aH+b)\\ &= [2nH-(n-1)a]\Box(nH) + 2(nH - \frac{1}{2}(n-1)a - \lambda_i)(nH_i)^2\\ &\geq -[2nH-(n-1)a]\frac{n}{2\sqrt{n-1}}[S - 2\sqrt{n-1}(2\delta - c_0)]|\phi|^2 \geq 0.\end{aligned}$$

On the other hand, from lemma 2.3, we know that $\Box$ is an elliptic operator. If $\sup S = 2\sqrt{n-1}(2\delta - c_0)$ and this supremum is attained at some point of M^n, then, by maximum principle, S mus be constant and $S = 2\sqrt{n-1}(2\delta - c_0)$. Then H is also constant by using Gauss equation. Thus (2.23) become an equality and all inequalities in the proof of Proposition 2.6 must be equalities. By lemma 2.1 and (2.27), we obtain that M^n is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.

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