Five–Dimensional φ–Symmetric Spaces

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Abstract

It is still an open problem whether Riemannian manifolds all of whose local geodesic symmetries are volume–preserving (i.e., D'Atri spaces) or more generally, ball–homogeneous spaces, and C-spaces are locally homogeneous or not. We provide some partial positive answers by proving that five–dimensional locally φ–symmetric spaces can be characterized as Sasakian spaces which are ball–homogeneous with η-parallel Ricci tensor or D'Atri spaces or C–spaces. We also prove that all K–contact metric manifolds, and hence all Sasakian manifolds, which are harmonic have constant curvature one.

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1 Introduction

Locally homogeneous Riemannian manifolds have the property that the volumes of small geodesic spheres or balls only depend on the radius. Riemannian manifolds having this property are called *ball-homogeneous spaces* [16]. It is still an intringuing open problem whether the converse holds or not for manifolds of dimension greater than two. This converse problem has been treated for several special classes of Riemannian manifolds. First we note that harmonic spaces, or more generally, Riemannian manifolds all of whose local geodesic symmetries are volume–preserving (that is, *D'Atri spaces*) (see [15] for a survey and further references) are ball-homogeneous [12]. Even for this subclass, and also for that of the harmonic spaces, the mentioned converse is still open in general, although only locally homogeneous examples are known. For two- and three-dimensional D'Atri spaces the converse holds but for four-dimensional D'Atri spaces only partial results are given in [7], [9], [18], [20]. In [18] it is proved that four-dimensional Kählerian D'Atri manifolds are locally symmetric and a similar result holds for four-dimensional 2-stein [20] and Hermitian Einstein spaces [9]. The last results have been extended to the broader class of four-dimensional ball-homogeneous Einstein spaces [7].

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The *C-spaces* are introduced in [1] (see [2]) as Riemannian manifolds such that the eigenvalues of the Jacobi operator are constant along each geodesic. Their geometry shares some properties with that of the D'Atri spaces. First, also here, only locally homogeneous examples are known. Further, it has been proved in [1] that two- and three-dimensional C-spaces are D'Atri spaces, and conversely. In dimension four, a similar result is derived in [7] for Kähler manifolds, for 2-stein spaces and for compact Hermitian-Einstein spaces by showing that these spaces are again locally symmetric.

The main purpose of this paper is to study ball-homogeneous and C-spaces in the framework of Sasakian geometry. There, locally symmetric spaces are replaced by φ-symmetric spaces [22]. They are endowed with a naturally reductive structure [5] and this yields that they are D'Atri spaces [15] and also C-spaces [1]. The converse holds in dimension three [5]. In Section 4 and Section 5 we shall prove that the converse also holds for five-dimensional Sasakian manifolds.

To prove our result we first collect some basic material and formulas in Section 2. In Section 3 we derive the useful result stating that a five-dimensional Sasakian manifold is locally φ-symmetric if and only if it is ball-homogeneous and has η-parallel Ricci tensor. Finally, in Section 6 we prove that any Sasakian 2-stein space (or more generally, any K-contact 2-stein space) and hence, any Sasakian harmonic space, is locally symmetric and so, has constant curvature one. This contrasts with Kähler geometry where it is still an open problem whether Kählerian harmonic spaces are locally symmetric or not.

2 Preliminaries

In this section we collect some basic material about the geometry of Sasakian manifolds and locally φ-symmetric spaces which we shall need to prove our results. We refer to [4], [22], [26] for more details.

2.1 Sasakian manifolds

A smooth n-dimensional manifold M^n is said to be an *almost contact manifold* if the structural group of its tangent bundle is reducible to $U(k) \times 1$ where $n = 2k + 1$. Such a manifold admits a tensor field φ of type $(1,1)$, a vector field ξ and a one-form η satisfying

$$\eta(\xi) = 1, \qquad \varphi^2 = -I + \eta \otimes \xi.$$

These conditions imply that $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Moreover, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any tangent vector fields X, Y. Note that this implies $\eta(X) = g(X, \xi)$. M together with these structure tensors (φ, ξ, η, g) is said to be an *almost contact metric manifold.*

Next, let ∇ denote the Levi Civita connection on (M, g). Then $(M, g, \varphi, \xi, \eta)$ is said to be a *Sasakian manifold* if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$$

for X, $Y \in \mathcal{X}(M)$, the Lie algebra of smooth vector fields on M. This condition implies

$$\nabla_X \xi = -\varphi X$$

from which it follows that ξ is a Killing vector field. Further, the Riemannian curvature tensor

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

of a Sasakian manifold satisfies

$$\begin{aligned} R_{X\xi}Y &= g(X,Y)\xi - \eta(Y)X, \\ R_{XY}\xi &= \eta(X)Y - \eta(Y)X \end{aligned}$$

for all X, Y, $Z \in \mathcal{X}(M)$. These curvature properties imply that the sectional curvature $K(X,\xi)$ of the two-plane spanned by X, ξ equals $+1$ and hence (M,g) has constant sectional curvature c if and only if $c = 1$. A plane section of T_mM, $m \in M$, is called a *φ-section* if it possesses an orthonormal basis of the form $\{X, \varphi X\}$ where $X \in T_mM$ is orthogonal to ξ (that is, X is *horizontal*). The corresponding sectional curvature is called the *φ–sectional curvature*. If this is a global constant, then the Sasakian manifold is called a *Sasakian space form*. The Ricci tensor of such a space form satisfies

$$\rho(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y).$$

Sasakian manifolds satisfying this condition are said to be *η-Einstein* manifolds. Moreover, ρ is said to be *η- parallel* if it satisfies the weaker condition

$$(\nabla_X \rho)(Y,Z) = 0$$

for all horizontal X, Y, Z.

2.2 Local submersions and locally φ–symmetric spaces

As already noted, on any Sasakian manifold $(M, g, \varphi, \xi, \eta)$, ξ is a unit Killing vector field and hence, M is equipped with a Riemannian foliation with one-dimensional geodesic leaves. So, these manifolds are locally Riemannian submersions and this feature may be used to study the Sasakian geometry. Indeed, at each $m \in M$, there exists a cubic neighborhood U such that ξ is regular on U. Then $\pi : U \to \tilde{U} = U/\xi$ is a submersion. Let $\tilde{g}$ be the induced metric on the base space $\tilde{U}$ such that π becomes a Riemannian submersion. Further, let X^*, Y^*, ... denote the horizontal lifts of X, Y, ... $\in \mathcal{X}(\tilde{U})$ with respect to the connection form η. On $(\tilde{U}, \tilde{g})$ there is an induced Kähler structure $\tilde{J}$ defined by $(\tilde{J}X)^* = \varphi X^*$. Further, the Levi Civita connections and the Riemannian curvature tensors on $(\tilde{U}, \tilde{g})$ and (U, g) are related by

$$\begin{aligned} (\tilde{\nabla}_X Y)^* &= \nabla_{X^*}Y^* - \eta(\nabla_{X^*}Y^*)\xi, \\ (\tilde{R}_{XY}Z)^* &= R_{X^*Y^*}Z^* + g(\varphi X^*, Z^*)\varphi Y^* - g(\varphi Y^*, Z^*)\varphi X^* \\ &\quad +2g(\varphi X^*, Y^*)\varphi Z^*, \end{aligned} \tag{1}$$

$X, Y, Z \in \mathcal{X}(\tilde{U})$. Much information about the Sasakian geometry may be derived from the corresponding one about the Kähler geometry on the base space of each local submersion π by making a systematic use of the previous relations. For example, it follows at once that a Sasakian manifold $(M, g, \varphi, \xi, \eta)$ has constant φ-sectional curvature or is η–Einstein, respectively η-parallel, if and only if the base space $(\tilde{U}, \tilde{g}, \tilde{J})$ of each local fibration π has constant holomorphic sectional curvature or is Einstein, respectively Ricci-parallel.

In what follows we shall also need some relations between some scalar curvature invariants or order one and two on (M, g) and $(\tilde{U}, \tilde{g})$. Let τ, $\tilde{\tau}$ denote the scalar curvatures on these manifolds. Then we obtain from (1):

$$\begin{aligned} \tau &= \{\tilde{\tau} - (n-1)\} \circ \pi, \\ \|\rho\|^2 &= \{\|\tilde{\rho}\|^2 - 4\tilde{\tau} + 4(n-1) + (n-1)^2\} \circ \pi, \\ \|R\|^2 &= \{\|\tilde{R}\|^2 - 12\tilde{\tau} + 10(n-1) + 6(n-1)^2\} \circ \pi. \end{aligned} \tag{2}$$

Remark 1. Since a k-dimensional Riemannian manifold is an Einstein space if and only if $k\|\rho\|^2 = \|\tau\|^2$, it follows at once from (2) and the previous considerations that the n-dimensional Sasakian manifold is η-Einstein if and only if

$$\|\rho\|^2 = \frac{1}{n-1}\{\tau - (n-1)\}^2 + (n-1)^2.$$

Similarly, since a Kähler manifold of complex dimension $k > 1$ has constant holomorphic sectional curvature if and only if $(k+1)\|R\|^2 = 4\|\rho\|^2$, it follows that $(M^n, g, \varphi, \xi, \eta)$, $n \geq 5$, has constant φ-sectional curvature if and only if

$$\|R\|^2 = \frac{8}{n+1}\|\rho\|^2 + \left(\frac{32}{n+1} - 12\right)\tau - 2(n-1) + \left(6 - \frac{8}{n+1}\right)(n-1)^2.$$

(See also [21] for another proof.)

As is well–known, locally symmetric Sasakian manifolds have constant sectional curvature +1. In [22], T. Takahashi weakened this condition and introduced *locally φ-symmetric spaces* by the condition

$$\varphi^2(\nabla_V R)_{XY} Z = 0$$

for all horizontal V, X, Y, Z. These manifolds play a similar role as the Kähler manifolds which are locally isometric to Hermitian symmetric spaces.
The relation is given by
Theorem 2 .[22] *Let $(M, g, \varphi, \xi, \eta)$ be a Sasakian manifold. Then it is a locally φ–symmetric space if and only if each base space $\tilde{U}$ of the local fibration $\pi : U \to \tilde{U}$ is a locally symmetric Hermitian space.*

The local φ-symmetry is also equivalent to the fact that each local reflection with respect to the flow lines of ξ is an isometry. When ξ is complete and each local reflection is extendable to a global one, then the Sasakian manifold is said to be a *globally φ-symmetric space*. Any complete and simply connected locally φ–symmetric space is globally φ-symmetric.

There is no scarcity of examples. We refer to [6] for further details and references. In particular, we refer to [14], [17] for complete classifications of simply connected globally φ-symmetric spaces.

Finally, we note that the *Okumura connection* D on a Sasakian manifold $(M, g, \varphi, \xi, \eta)$ is defined by

$$D_X Y = \nabla_X Y + d\eta(X, Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X.$$

Then it follows that the Sasakian manifold is locally φ-symmetric if and only if the torsion of this connection or equivalently, if the $(1, 2)$-tensor field T given by

$$T_X Y = d\eta(X, Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X$$

defines a naturally reductive homogeneous structure [23] on it. As already noted in the introduction, this implies that locally φ-symmetric spaces are D'Atri spaces and also C-spaces. Since they are locally homogeneous, they are trivially ball-homogeneous.

3 Ball-homogeneneous Sasakian manifolds

We recall that a Riemannian manifold (M, g) is said to be ball-homogeneous if the volumes of small geodesic spheres or balls are independent of the center, that is, only depend on the radius. Clearly, any locally homogeneous manifold has this property. We will now concentrate on the converse for five-dimensional Sasakian manifolds and start by mentioning a result for four-dimensional Kähler manifolds. We note that all Riemannian manifolds are supposed to be connected.

Theorem 3. [7] *Let (M, g) be a four-dimensional Kähler manifold. Then (M, g) is locally symmetric if and only if it is ball-homogeneous and Ricci-parallel.*

The proof of it uses the following useful unpublished result of A.Derdziński (see [18]):

Theorem 4. *Let (M, g) be a four-dimensional Einstein space such that its Weyl tensor $W \in C^\infty(End \bigwedge^2 M)$ has constant eigenvalues (that is, which is curvature homogeneous). Then (M, g) is locally symmetric.*

Moreover, one makes extensive use of the following power series expansion for the volume $V_m(r)$ of a small geodesic ball $B_m(r)$ of radius r and center m.

Theorem 5. [11] *For any point $m \in M$ and any small radius $r > 0$ we have*

$$V_m(r) = V_0^n(r)\left\{1 + A(m)r^2 + B(m)r^4 + C(m)r^6 + O(r^8)\right\}, \tag{3}$$

where $V_0^n(r)$ denotes the volume of a Euclidean ball of dimension $n = dimM$ and radius r. The coefficients A, B and C are given by

$$A(m) = -\frac{\tau(m)}{6(n+2)}, \tag{4}$$

$$B(m) = \frac{(-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 - 18\tau\Delta\tau)(m)}{360(n+2)(n+4)}, \tag{5}$$

$$C(m) = \frac{1}{720(n+2)(n+4)(n+6)} \Big\{ -\frac{5}{9}\tau^3 - \frac{8}{3}\tau\|\rho\|^2 + \tau\|R\|^2 + \frac{64}{63}\check{\rho}$$

$$-\frac{64}{21}g(\rho\otimes\rho,\bar{R}) + \frac{32}{7}g(\rho,\dot{R}) - \frac{110}{63}\check{R} - \frac{200}{63}\check{\check{R}} + \frac{45}{7}\|\nabla\tau\|^2$$

$$+\frac{45}{14}\|\nabla\rho\|^2 + \frac{45}{7}\alpha(\rho) - \frac{45}{14}\|\nabla R\|^2 + 6\tau\Delta\tau + \frac{48}{7}g(\Delta\rho,\rho)$$

$$+\frac{54}{7}g(\nabla^2\tau,\rho) - \frac{30}{7}g(\Delta R,R) - \frac{45}{7}\Delta^2\tau\Big\}(m). \tag{6}$$

We refer to [11] for the explicit expressions of the cubic scalar curvature invariants used in (6). We only mention

$$\check{R} = \sum_{i,j,k,l,p,q} R_{ijkl}R_{klpq}R_{pqij},$$
$$\check{\check{R}} = \sum_{i,j,k,l,p,q} R_{ikjl}R_{kplq}R_{piqj}$$

where $\{e_i, i = 1,\ldots,n\}$ is an arbitrary orthonormal basis of each tangent space, and note the relation

$$\frac{1}{2}\Delta\|R\|^2 = g(\Delta R,R) + \|\nabla R\|^2.$$

Now, we prove the main result of this section.

Theorem 6. *Let (M,g,φ,ξ,η) be a five-dimensional Sasakian manifold. Then it is locally φ-symmetric if and only if it is ball-homogeneous and η-parallel.*

Proof. Since a locally φ-symmetric space is clearly ball-homogeneous and η-parallel, we concentrate on the "if" part in the statement.

So, let (M,g) be ball-homogeneous and η-parallel. Then each $(\tilde{U},\tilde{g})$ is Ricci-parallel. If it is locally reducible, then $(\tilde{U},\tilde{g})$ is trivially locally symmetric and then the result follows from Theorem 2. So, we suppose that $(\tilde{U},\tilde{g})$ is locally irreducible. Then it is an Einstein space of dimension 4 and hence (U,g) is η-Einsteinian, that is,

$$\rho(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y).$$

It then follows that

$$\tau = n\lambda + \mu, \quad \|\rho\|^2 = n\lambda^2 + \mu^2 + 2\lambda\mu$$

where λ, μ are constant since $\dim M \geq 5$. Moreover, a detailed computation, which we omit, yields

$$
\begin{aligned}
\|\nabla\rho\|^2 &= 2(n-1)\mu^2, \\
\alpha(\rho) &= -(n-1)\mu^2, \\
\check{\rho} &= n\lambda^3 + \mu^3 + 3\lambda\mu(\lambda+\mu), \\
g(\rho\otimes\rho, \overline{R}) &= \lambda\{n\lambda^2 + \mu\lambda + 2(n-1)\mu\}, \\
g(\rho, \dot{R}) &= \lambda\|R\|^2 + 2(n-1)\mu, \\
g(\nabla^2\rho, \overline{R}) &= \mu\{3\tau - (n-1)(3n-1)\}, \\
g(\Delta\rho, \rho) &= \frac{1}{2}\Delta\|\rho\|^2 - \|\nabla\rho\|^2 = -2(n-1)\mu^2.
\end{aligned}
$$

Since M is ball–homogeneous, it follows from Theorem 5 that B is constant on M and hence $\|R\|^2$ is constant. This implies that all the above mentioned cubic invariants are constant on M. Since also C is constant, we have

$$22\check{R} + 40\check{\overline{R}} - \frac{27}{2}\|\nabla R\|^2 = \text{const.} \tag{7}$$

Moreover, from [11, (2.18)] we get

$$g(\nabla^2\rho, \overline{R}) = \frac{1}{4}g(\Delta R, R) - \frac{1}{2}g(\rho, \dot{R}) + \check{\overline{R}} + \frac{1}{4}\check{R}$$

and hence,

$$\check{R} + 4\check{\overline{R}} - \|\nabla R\|^2 = \text{const.} \tag{8}$$

Finally, since $\dim M = 5$, the six-dimensional Gauss-Bonnet integrand vanishes, that is [11, (2.24)],

$$\tau^3 + 3\tau\|R\|^2 - 12\tau\|\rho\|^2 + 16\check{\rho} + 4\check{R} - 8\check{\overline{R}} + 24g(\rho\otimes\rho, \overline{R}) - 24g(\rho, \dot{R}) = 0,$$

from which we get

$$\check{R} - 2\check{\overline{R}} = \text{const.} \tag{9}$$

Hence, from (7), (8), (9) we may conclude that $\check{R}$, $\check{\overline{R}}$ and $\|\nabla R\|^2$ are constant.

Next, we consider $(\tilde{U}, \tilde{g}, \tilde{J})$. Since this is Einstein, $\tilde{\tau}$ and $\|\tilde{\rho}\|^2$ are constant and from (2) it follows that $\|\tilde{R}\|^2$ is constant. Moreover, proceeding as for (2), we get

$$\check{R} = \{\check{\tilde{R}} - 6\|\tilde{R}\|^2 - 24\|\tilde{\rho}\|^2 + 12(2n+3)\tilde{\tau} - 8n^3 + 4n + 4\} \circ \pi \tag{10}$$

from which it follows that $\check{\tilde{R}}$ is constant on $\tilde{U}$. Further, we have [11, (11.4)]

$$\frac{\tilde{\tau}^3}{2} - \tilde{\tau}\|\tilde{R}\|^2 + 8\check{\tilde{R}} - 8\check{\overline{\tilde{R}}} = 0$$

and [11, (11.3)]

$$\|\tilde{\nabla}\tilde{R}\|^2 = \check{\tilde{R}} - 4\check{\overline{\tilde{R}}} - \frac{\tilde{\tau}}{2}\|\tilde{R}\|^2.$$

Hence $\tilde{\check{\tilde{R}}}$ and $\|\tilde{\nabla}\tilde{R}\|^2$ are also constant on $\tilde{U}$.

Finally, since $(\tilde{U}, \tilde{g}, \tilde{J})$ is a four-dimensional Kähler-Einstein manifold, we may choose at each $m \in \tilde{U}$ an adapted Singer–Thorpe basis $\{e_1, e_2 = \tilde{J}e_1, e_3, e_4 = \tilde{J}e_3\}$ [18], [3]. Then the components of $\tilde{R}$ with respect to this basis are given by

$$\left.\begin{array}{rcl} \tilde{R}_{1212} &=& \tilde{R}_{3434} = a,\ \tilde{R}_{1313} = \tilde{R}_{2424} = b,\ \tilde{R}_{1414} = \tilde{R}_{2323} = c, \\ \tilde{R}_{1234} &=& \alpha,\ \tilde{R}_{1342} = \beta,\ \tilde{R}_{1423} = \gamma, \\ \tilde{R}_{ijkl} &=& 0 \text{ whenever three of the indices } i,j,k,l \text{ are distinct} \end{array}\right\} \tag{11}$$

and where

$$\alpha = b + c = \frac{\tilde{\tau}}{4} - a, \quad \beta = -b, \quad \gamma = -c.$$

Then we get

$$\begin{array}{rcl} a + b + c &=& \frac{1}{4}\tilde{\tau} \\ 8(a^2 + 3b^2 + 3c^2 + 2bc) &=& \|\tilde{R}\|^2 \\ 48bc(a - b - c) &=& \frac{1}{48}\tilde{\tau}^3 - \frac{1}{24}\tilde{\tau}\|\tilde{R}\|^2 + \frac{1}{6}\|\tilde{\nabla}\tilde{R}\|^2, \end{array}$$

from which it follows that a, b, c are constant on $\tilde{U}$ [7]. This shows that $(\tilde{U}, \tilde{g}, \tilde{J})$ is curvature homogeneous and hence, as a consequence of Theorem 4, locally symmetric. This completes the proof by using Theorem 2. □

4 Five-dimensional Sasakian D'Atri spaces

A Riemannian manifold is said to be a *D'Atri space* if the local geodesic symmetries are volume-preserving up to sign, or equivalently, divergence-preserving. If $\theta_m = \sqrt{\det g_{ij}}$ denotes the volume-density function with respect to normal coordinates, then the D'Atri condition is equivalent to $\theta_m(\exp_m(rx)) = \theta_m(\exp_m(-rx))$ for any unit vector $x \in T_mM$ and any sufficiently small $r > 0$. Since

$$\theta_m(\exp_m(rx)) = 1 - \frac{r^2}{6}\rho_{xx}(m) - \frac{r^3}{12}(\nabla_x\rho_{xx})(m) + O(r^4), \tag{12}$$

this condition implies $(\nabla_x\rho)(x, x) = 0$ for any vector $x \in T_mM$ and hence, a D'Atri space has constant scalar curvature and is analytic in normal coordinates (see for example [15]).

Watanabe [25] observed that any three-dimensional Sasakian D'Atri space is locally φ-symmetric. Moreover, using the conditions for the curvature tensor following from the condition on θ_m, he showed that any five-dimensional Sasaki-Einstein D'Atri space of non-negative sectional curvature is locally φ-symmetric. Our main result in this section will yield an extension of this result by using

Theorem 7 [12]. *Any connected D'Atri space is ball-homogeneous.*

Now we prove

Theorem 8. *Any five-dimensional Sasakian manifold is locally φ-symmetric if and only if it is a D'Atri space.*
Proof. In Section 2 we have already mentioned that any locally φ-symmetric space is a D'Atri space.

Conversely, let $(M, g, \varphi, \xi, \eta)$ be a D'Atri space. Then, as mentioned above, $(\nabla_x \rho)(x, x) = 0$ and hence, using(1), we get that each base space $(\tilde{U}, \tilde{g}, \tilde{J})$ is a Kähler manifold with cyclic-parallel Ricci tensor. It then follows that $\tilde{\rho}$ is parallel [19] and so, $(M, g, \varphi, \xi, \eta)$ has η-parallel Ricci tensor. The result follows now from Theorem [6] and Theorem [7]. □

5 Sasakian C-spaces

The main purpose of this section is to extend Theorem [8] to the class of C-spaces. A Riemannian manifold (M, g) is said to be a C-*space* if for any geodesic γ the eigenvalues of the Jacobi operator $R_\gamma := R_{\gamma'} . \gamma'$ are constant along γ. This is equivalent to the condition that traceR_γ^k is constant along γ for all $k \in N$. For $k = 1$ this condition means that ρ is cyclic-parallel. This implies that the scalar curvature is constant and hence any three-dimensional Sasakian C-space is localy φ–symmetric. Note that we already mentioned in Section 2 that any locally φ-symmetric space is a C-space.

Now we consider the case of five-dimensional Sasakian C-spaces and prove
Theorem 9. *Any five-dimensional Sasaki manifold is locally φ-symmetric if and only if it is a C-space.*
Proof. It suffices to prove the "if" part. So, let $(M, g, \varphi, \xi, \eta)$ be a Sasakian C-space of dimension 5. Since ρ is cyclic-parallel, $\tilde{\rho}$ is cyclic parallel and hence parallel on each base space $(\tilde{U}, \tilde{g}, \tilde{J})$. As in the proof of Theorem [6] we may restrict to the case where $(\tilde{U}, \tilde{g})$ is an Einsein space, that is, (U, g) is η–Einsteinian.

Now, since traceR_γ^2 is constant along γ, we have

$$G(x) = \sum_{a,b} R_{xaxb} \nabla_x R_{xaxb} = 0.$$

Further, let x be a horizontal vector and choose an orthonormal basis $\{e_1, \ldots, e_4, \xi\}$ of $T_m M$. Then we get by using the formulas in Section 2:

$$0 = G(x) = \sum_{a,b=1}^{5} R_{xaxb} \nabla_x R_{xaxb} = \tilde{G}(\tilde{x}) - 3 \tilde{\nabla}_{\tilde{x}} \tilde{R}_{\tilde{x}\tilde{J}\tilde{x}\tilde{x}\tilde{J}\tilde{x}} \tag{13}$$

where $\tilde{x} = \pi_* x$ and

$$\tilde{G}(\tilde{x}) = \sum_{i,j=1}^{4} \tilde{R}_{\tilde{x}i\tilde{x}j} \tilde{\nabla}_{\tilde{x}} \tilde{R}_{\tilde{x}i\tilde{x}j}.$$

Now, denote by D the Laplacian on R^n. Then it follows at once from (13) that $D\tilde{G}(\tilde{x}) = 0$. Since for any Riemannian manifold with parallel Ricci tensor we have [20] $D^2 G(x) = 12x\|R\|^2$, it follows that $\|\tilde{R}\|^2$ is constant on $\tilde{U}$. Further, since $(\tilde{U}, \tilde{g}, \tilde{J})$ is a four-dimensional Kähler-Einstein space, we have [9]

$$\|\tilde{R}\|^2 = 16\tilde{F}(\tilde{x}) + \frac{1}{2}\tilde{\tau}^2 - 4\tilde{\tau}\tilde{H}(\tilde{x}) \tag{14}$$

where $\tilde{H}(\tilde{x})$ denotes the holomorphic sectional curvature of the two-plane determined by the unit vector $\tilde{x}$ and

$$\tilde{F}(\tilde{x}) = \sum_{i,j} \tilde{R}^2_{\tilde{x}i\tilde{x}j}.$$

Hence, by differentiating (14), we get

$$(24 - \tilde{\tau})\tilde{\nabla}_{\tilde{x}}\tilde{R}_{\tilde{x}\tilde{J}\tilde{x}\tilde{x}\tilde{J}\tilde{x}} = 0.$$

If $\tilde{\tau} \neq 24$, then $(\tilde{U}, \tilde{g}, \tilde{J})$ is locally symmetric [19] and (U, g) locally φ-symmetric.

So, we are left with the case $\tilde{\tau} = 24$, or equivalently, $\tau = 20$. This means that (U, g) is Einsteinian. In this case, put

$$A_{xyzw} = A(x, y, z, w) = R(x, y, z, w)$$

for all horizontal x, y, z, w. Then A is a curvature-like tensor and its Ricci tensor $\rho(A)$ is proportional to g since $\rho(A)(x, x) = \rho(x, x) - g(x, x)$. So A is Einsteinian. For the corresponding scalar curvature we have $\tau(A) = 12$. Further,

$$F(A)(x) = \sum_{a,b} A^2_{xaxb} = F(x) - 1$$

and since

$$F(x) = \tilde{F}(\tilde{x}) - 6\tilde{R}_{\tilde{x}\tilde{J}\tilde{x}\tilde{x}\tilde{J}\tilde{x}} + 10,$$

we have

$$F(A)(x) = \tilde{F}(\tilde{x}) - 6\tilde{R}_{\tilde{x}\tilde{J}\tilde{x}\tilde{x}\tilde{J}\tilde{x}} + 9.$$

Since $\tilde{\tau} = 24$, (14) yields

$$\|\tilde{R}\|^2 = 16\tilde{F}(\tilde{x}) + 288 - 96\tilde{R}_{\tilde{x}\tilde{J}\tilde{x}\tilde{x}\tilde{J}\tilde{x}}$$

and so, $F(A)(x)$ is independent of the horizontal unit vector x. Since A is already Einsteinian, this means that A is a 2-stein curvature-like tensor (see for example [10], [20] and Section 6). Moreover, since the horizontal subspace is four-dimensional, A is pointwise Osserman and hence 3-stein [3], [10], that is,

$$\sum_{a,b,c} A_{xaxb}A_{xbxc}A_{xcxa}$$

is independent of the horizontal unit vector x. Now, with the method used in [11], integration over the unit sphere of $\{\xi\}^\perp = R^4$ yields

$$\sum_{a,b,c} A_{xaxb}A_{xbxc}A_{xcxa} = \frac{1}{48}\left\{\check{\rho}(A) + \frac{9}{2}g(\rho(A), \dot{A}) - \check{\bar{A}} + \frac{7}{2}\check{A}\right\}. \tag{15}$$

Moreover, since A is Einsteinian, we have

$$\check{\rho}(A) = \frac{\tau(A)^3}{16}, \qquad g(\rho(A), \dot{A}) = \frac{\tau(A)}{4}\|A\|^2.$$

Since A is 2-stein and $G(x) = 0$, we get that $\|A\|^2 = \|R\|^2 - 32$ is constant along horizontal curves, and hence also $\|R\|^2$. Using this and since traceR_γ^3 is constant along γ, we get by differentiating (15) that $\check{\bar{A}} - \frac{7}{2}\check{A}$ is constant along horizontal curves. A straightforward computation yields

$$\check{A} = \check{R} - 32 \qquad \check{\bar{A}} = \check{\bar{R}} - 40$$

and hence, $\check{\bar{R}} - \frac{7}{2}\check{R}$ is also constant along horizontal curves.

Finally, on a five-dimensional Einstein manifold we have [11, (11.6)]

$$\frac{\tau^3}{5} - \frac{9}{5}\tau\|R\|^2 + 4\check{R} - 8\check{\bar{R}} = 0$$

and so, in our case, $\check{R} - 2\check{\bar{R}}$ is constant along horizontal curves. Since [11, (11.6)]

$$\|\nabla R\|^2 = 3\check{R} + \frac{1}{10}\tau^3 - \frac{13}{10}\tau\|R\|^2,$$

we obtain that $\|R\|^2$, $\check{R}$, $\check{\bar{R}}$ and $\|\nabla R\|^2$ are constant along horizontal curves. From this one then easily derives that $\|\tilde{R}\|^2$, $\check{\tilde{R}}$, $\check{\bar{\tilde{R}}}$ and $\|\tilde{\nabla}\tilde{R}\|^2$ are constant on $(\tilde{U}, \tilde{g}, \tilde{J})$. This again shows that $(\tilde{U}, \tilde{g})$ is curvature homogeneous and hence locally symmetric. This completes the proof of the theorem. $\square$

6 Sasakian 2-stein and harmonic spaces

We mentioned already in the introduction that all known example of *harmonic manifolds*, that is, Riemannian manifolds such that the volume-density function θ_m is a radial function (see [2], [3], [8] for more details), are locally homogeneous. It is still unknown whether this holds in general or not. Surprisingly, this is even the case for harmonic Kähler manifolds. In this section, we shall give a positive answer for general Sasakian manifolds. To do this, we first prove a result about 2-*stein manifolds*, that is, Riemannian manifolds which are Einsteinian and satisfy

$$\sum_{a,b} R_{xaxb}^2 = \lambda g(x,x)g(x,x).$$

In this case we have [8]

$$\sum_{a,b} R_{xaxb}^2 = \frac{1}{n(n+2)}\left\{\frac{3}{2}\|R\|^2 + \|\rho\|^2\right\} g(x,x)g(x,x). \tag{16}$$

Theorem 10. *A Sasakian manifold (M, g) is a 2-stein space if and only if it has constant sectional curvature $+1$.*

Proof. First, let (M, g) be a 2-stein space. Since $K(x, \xi) = 1$, we get from (16), by putting $x = \xi$,

$$\frac{1}{n(n+2)}\left\{\frac{3}{2}\|R\|^2+\|\rho\|^2\right\}=n-1. \tag{17}$$

Since (M,g) is Einsteinian, we have $\|\rho\|^2=\dfrac{\tau^2}{n}=n(n-1)^2$ and so, (17) yields

$$\|R\|^2=\frac{2}{n-}\|\rho\|^2.$$

This implies that (M,g) has constant sectional curvature c (see for example [8]) and hence $c=1$.

The converse is trivial. □

Since any harmonic space is a 2-stein space [12] and any real space form is harmonic, we get

Corollary 11. *A Sasakian manifold is harmonic if and only if it is a space of constant curvature* $+1$.

We finish this paper with some remarks.

Remark. A Riemannian manifold is said to be *disk-homogeneous* if the volumes of all small geodesic disks $D_m^x(r)$ do not depend on the unit vector $x \in T_mM$ for all $m \in M$. If, in addition, the volumes do not depend on m, then (M,g) is said to be *strongly disk-homogeneous*. See [16].

Moreover, (M,g) is called a *tube-homogeneous manifold* if the volumes of all small circumscribing tubes $U_\sigma(r)$ are independent of the axial geodesic σ [24].

It is still an open problem whether disk- and tube-homogeneous spaces are locally homogeneous. Since it is proved in [16], [24] that in both cases (M,g) must be a 2-stein space, Theorem 10 yields that a *Sasakian disk-homogeneous or tube-homogeneous space has constant sectional curvature* $+1$. The converse holds for any real space form [16], [24].

Remark. An almost contact metric manifold (M,g,φ,ξ,η) is said to be a K-*contact manifold* if it is a contact metric manifold and ξ is a Killing vector field. For these manifolds we also have $K(x,\xi)=1$ and so the proof of Theorem 10 applies. So, we get: *A* K-*contact manifold is a* 2-*stein space if and only if it is a space of constant sectional curvature* $+1$. Also the results about harmonicity, disk- and tube-homogeneity still hold.

Finally, we note that these properties are still valid for the class of $C(\alpha)$-manifolds, introduced in [13]. This broader class includes the co-Kähler, Kenmotsu, Sasaki and K-contact spaces.

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