A Study of the Rimmer Bifurcation of Symmetric Fixed Points of Reversible Diffeomorphisms

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Abstract

We prove that the study of Rimmer bifurcation of symmetric fixed points in two–dimensional discrete reversible dynamical systems can be achieved analysing either bifurcation of critical points of a symmetric Hamiltonian function or the bifurcation of symmetric equilibrium points for a nonconservative reversible vector field. We give the normal forms for generating functions of area preserving reversible diffeomorphisms and the normal forms for nonconservative reversible vector fields associated to Rimmer bifurcation.

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1 Introduction

In classical mechanics a lot of dynamical systems possess time–reversal symmetry, i.e. the equations of evolution are invariant under the transformation $t \to -t$. In connection to the study of the 3–body problem, Moser [5] generalized the reversibility of a system on $\mathbf{R}^{2n}$, defining time–reversal symmetry with respect to a linear reflection (involution) R $(R \circ R = id)$. Namely, given a dynamical system defined by the complete vector field X on $\mathbf{R}^{2n}$, the system is called R–reversible if $RX = -XR$. Devaney [1] studied reversible dynamical systems on even dimensional compact manifolds M, reversibility being introduced by a nonlinear smooth involution R having the fixed point set, $Fix(R)$, of dimension $\dim(M)/2$. In the nonlinear context R–reversibility means

1.1 $$TR \circ X = -X \circ R,$$

where T is the tangent functor on the category of smooth manifolds. Denoting by Φ_t the flow of the vector field X defined on M, condition (1.1) implies:

1.2 $$R\Phi_t = \Phi_{-t}R, \ \ \forall t \in \mathbf{R}$$

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Let X be a smooth complete R–reversible vector field on $\mathbf{R}^n$, having the flow Φ_t. The orbit $\Phi_t x_0$ of the point $x_0 \in \mathbf{R}^n$ is called symmetric with respect to R if $\Phi_t x_0 = R\,\Phi_{-t} x_0$. Taking $t = 0$ we get $R\,x_0 = x_0$, i.e. a symmetric orbit is the orbit of a point in the set $Fix(R)$. An equilibrium point of X is called symmetric if it lies on $Fix(R)$. If x_0 is a nonsymmetric equilibrium point then $R(x_0)$ is also an equilibrium point.

In the discrete context, if R is a smooth involution of the smooth manifold M_{2n}, with $\dim(Fix(R)) = n$, a diffeomorphism f of M is called R–reversible diffeomorphism if $I = f \circ R$ is also an involution. So $f = I \circ R$, and $f^{-1} = R \circ f \circ R$.

Definition 1.1 Let R be a linear involution of $\mathbf{R}^{2n}$. A linear map L of $\mathbf{R}^{2n}$ is called R–reversible if $I = L \circ R$ is also an involution of $\mathbf{R}^{2n}$. A linear map A is called infinitesimally R–reversible if $A\,R = -R\,A$.

If $L : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is a R–reversible linear map then λ is an eigenvalue for L iff λ^{-1} is also an eigenvalue for L.

For an infinitesimally R–reversible map $A : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ if λ is an eigenvalue of A of multiplicity k then $-\lambda$ is also an eigenvalue of multiplicity k. Moreover, the eigenvalue 0 has even multiplicity if it occurs.

Now we are able to characterize the eigenvalues of a linear R–reversible map $L : \mathbf{R}^2 \to \mathbf{R}^2$: The only nontrivial linear involutions of $\mathbf{R}^2$ are $S(x, y) = (-x, y)$, $S(x, y) = (x, -y)$ and $S(x, y) = (-x, -y)$. So their determinant is ± 1. Since $L = I \circ R$, it follows that $\det(L) = \pm 1$. If L is orientation preserving then $\det(L) = 1$, i.e. the product of their eigenvalues is 1: $\lambda\lambda^{-1} = 1$. Therefore the normal forms for orientation preserving linear R–reversibile systems on $\mathbf{R}^2$ are:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

If x_0 is a fixed point of the R–reversible diffeomorphism $f : M \to M$, where $M = \mathbf{R}^2$ or a two dimensional smooth manifold, then the linear map $d_{x_0} f$ is also R–reversible. Therefore we can clasify fixed points of a two–dimensional R–reversible smooth mapping according to the properties of the eigenvalues λ, λ^{-1} of its linear part $d_{x_0} f$:

1. x_0 is a hyperbolic fixed point if $\lambda \in \mathbf{R}$, and $\lambda \neq 1$
2. elliptic if $\lambda, \lambda^{-1} \in C$, and $|\lambda| = 1$;
3. parabolic if $\lambda = \lambda^{-1} = \pm 1$. x_0 is called $1:1$ resonant fixed point if the both eigenvalues are 1, and $1:2$ resonant if the both eigenvalues are -1.

2 Bifurcation of symmetric $1:1$ resonant fixed points in two–dimensional reversible dynamical systems

During the last decade much attention was paid to the study of the dynamics of a reversible system (see [9] and references therein) and particularly to the bifurcation of points of equilibrium of revesible vector fields [3] or bifurcation of fixed points of reversible diffeomorphisms [7].

Next we study the so called Rimmer bifurcation of a 1 : 1 resonant equilibrium/fixed point of a planar R–reversible system. Rimmer [8] studied bifurcation of a 1 : 1 fixed point in area preserving R–reversible maps using a Poincaré generating function

The Rimmer's result states:

If $(x, \mu) \to f(x, \mu), \mu \in I$ is a smooth family of area preserving R–reversible maps of an open set of the plane $(R(x, y) = \pm(-x, y))$ and (x_0, μ_0) is a 1 : 1 resonant symmetric fixed point of f, and the Poincaré generating function satisfies some conditions, then at $\mu = 0$ f undergoes a symmetry breaking bifurcation, that is (x_0, μ_0) is embedded in a family of symmetric fixed points that change the type from hyperbolic to elliptic or conversely, when traverse the bifurcation point, and two further families of asymmetric fixed points bifurcate from (x_0, μ_0).

The generating function involved in the treatment of Rimmer bifurcation is somewhat artificial and the proofs are very laborious.

In the following, we shall use a different generating function suggested by Meyer [4] in the study of bifurcation of fixed points in area preserving (non-reversible) maps. In fact we establish a local correspondence between R–reversible flows and R–reversible diffeomorphism.

Let Ψ be the fractional linear transformation of $\mathbf{C}$, defined by: $\Psi(z) = (1 + z)(1 - z)^{-1}$. Its inverse is defined as $\Psi^{-1}(z) = (z - 1)(z + 1)^{-1}$. It is straightforward that $\Psi(\{z|Re(z) = 0\}) = S^1$ (S^1 is the unit circle), and the left half complex plane is mapped to the interior of the unit disc. Denote by $\mathcal{L}_1(\mathbf{R}^n)$, respectively $\mathcal{L}_{-1}(\mathbf{R}^n)$ the subset of linear transformations of $\mathbf{R}^n$ with no eigenvalue 1, respectively with no eigenvalue -1. Meyer [4] proved:

1. Ψ maps $\mathcal{L}_1(\mathbf{R}^n)$ onto $\mathcal{L}_{-1}(\mathbf{R}^n)$, and if $\lambda_i, i = \overline{1, n}$ are eigenvalues of $A \in \mathcal{L}_1(\mathbf{R}^n)$, then $\Psi(\lambda_i)$ are the eigenvalues of $\Psi(A)$.

2. If $A \in \mathcal{L}_1(\mathbf{R}^{2n})$ is a Hamiltonian matrix then $\Psi(A)$ is symplectic, and conversely, if $B \in \mathcal{L}_{-1}(\mathbf{R}^{2n})$ is a symplectic transformation then $\Psi^{-1}(B)$ is a Hamiltonian linear transformation.

Next we study the action of the map Ψ on the subspace of infinitesimally R–reversible linear maps.

Proposition 2.1 *If A is an infinitesimally R–reversible linear map of $\boldsymbol{R}^n$ having no eigenvalue* 1*, then $\Psi(A)$ is a linear R–reversible map, and conversely, if B is a linear R–reversible map having no eigenvalue -1, then $\Psi^{-1}(B)$ is infinitesimally R–reversible.*

Proof. Let $B = \Psi(A)$. Then $R\,B\,R = R\,(I + A)\,(I - A)^{-1}\,R = R\,(R^2 - R\,A\,R)(R^2 + R\,A\,R)^{-1} = (I - A)(I + A)^{-1} = B^{-1}$, that is B is R–reversible.

Conversely, let $A = \Psi^{-1}(B) = (B - I)(B + I)^{-1}$. B being R–reversible, it is conjugated to its inverse: $B = R\,B^{-1}\,R$. Hence $R\,A = R\,(R\,B^{-1}\,R - R^2)(R\,B^{-1}\,R + R^2) = R^2\,(B^{-1} - I)\,(B^{-1} + I)\,R = \Psi^{-1}(B^{-1})\,R = -\Psi(B)\,R = -A\,R$. q.e.d.

In order to discuss the bifurcation of a $1:1$ fixed point of a R–reversible diffeomorphism we extend the action of the application Ψ to the nonlinear maps.

Let V be an open neighbourhood of 0 in $\mathbf{R}^2$ and X a R–reversible vector field defined on V having 0 as equilibrium point such that the linear vector field d_0X has no eigenvalue 1. Hence $id - X$ is locally invertible in a neibourhood of 0, and we can associate to X the map f defined in that neighbourhood by $f = \Psi(X) = (id + X)(id - X)^{-1}$. So f is a R–reversible local diffeomorphism, having 0 as fixed point. If X depends smoothly on a parameter then so does f, and thus bifurcation of the R–reversible map f reduces to the bifurcation of the equilibrium points of X.

Exploiting this correspondence we give a simpler proof for the Rimmer bifurcations of a $1:1$ resonant fixed point of an area preserving R–reversible diffeomorphism of the plane.

It is well known in the theory of area preserving diffeomorphisms that fixed points are critical points for a generating function. We associate a generating function in the following way: Consider a smooth area preserving R–reversible map f ($R(x,y) = (x,-y)$) defined on a simply connected neighbourhood of the origin of $\mathbf{R}^2$. Suppose that $(0,0)$ is $1:1$ resonant fixed point for f. Then the vector field $X = (f-id)\circ(f+id)^{-1}$ is well defined on the same neighbourhood of the origin, it is R–reversible, and has $(0,0)$ as double zero equilibrium point. One verifies that the vector field $Y = J^{-1}X$, ($J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$) has the property that $\frac{\partial Y_1}{\partial y} = \frac{\partial Y_2}{\partial x}$. Therefore there exists a smooth real function H, such that $Y = \text{grad}H$. Hence the local vector field $X = JY$ is a hamiltonian vector field, and $(0,0)$ is a critical point for the Hamiltonian function H. Moreover, $H \circ R = H$. Conversely, to a vector field $X = J\text{grad}H$, where $H \circ R = H$, $\text{grad}H(0) = 0$, and $J\,d^2H(0)$ has no eigenvalue 1, one associates through Ψ an area preserving R–reversible diffeomorphism f, having 0 as fixed point.

Thus the bifurcation of symmetric fixed points of f is reduced to the bifurcation of critical points of the Hamiltonian function H.

Proposition 2.2 *Let f_μ, $\mu \in (-\epsilon, \epsilon)$ be a family of area preserving R–reversible smooth local diffeomorphisms , defined on a simply connected neighbouhood of the origin, having for $\mu = 0$ the $1:1$ resonant fixed point $(0,0)$. If the normal form of the Hamiltonian generating function of f_μ is $H(x,y) = x^2 - \mu y^2 \pm y^4$, then the family f_μ undergoes a Rimmer bifurcation at $\mu = 0$*

Proof

$$\frac{\partial H}{\partial x} = 2x \quad \frac{\partial H}{\partial y} = \pm 4y^3 - 2\mu y$$

and

$$d^2H = \begin{bmatrix} 2 & 0 \\ 0 & \pm 12y^2 - 2\mu \end{bmatrix}$$

For any $\mu \in (-\epsilon, \epsilon)$, $x = 0, y = 0$ is a critical point of the function H, hence an equilibrium point for the associated vector field X, and a symmetric fixed point for the local symplectomorphism f. The eigenvalues of the linear vector field $A = J\,d_0^2H$ are the solution of equation:

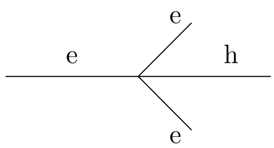
$$\lambda^2 + \det d_0^2 H = 0$$

For $\mu = 0$, $\det(d_0^2 H) = 0$, hence $\Psi(0) = 1$ and $(0,0)$ is a parabolic fixed point for f.

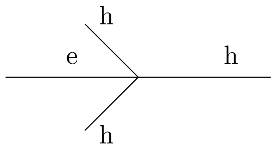
If $\mu < 0$, $\det(d_0^2 H) > 0$, and $(0,0)$ is an elliptic fixed point for f, while for $\mu > 0$ $\det(d_0^2 H) < 0$, i.e. $(0,0)$ is a hyperbolic fixed point for f. Observe that when μ traverses the value 0 two assymetric fixed points (critical points for H) arise:

- in the case " + " and $\mu > 0$ $(0, \sqrt{\mu/2})$ and $(0, -\sqrt{\mu/2}$.

  The determinant of the hessian d^2H evaluated at these points is $8\mu > 0$, hence the fixed points are of elliptic type. Therefore a curve of symmetric fixed points passes through $(0,0)$ and further two curves of elliptic asymmetric fixed points bifurcates from $(0,0)$.



- in the case " − " and $\mu < 0$ the two assymetric fixed points are $(0, \sqrt{-\mu/2})$ and $(0, -\sqrt{-\mu/2}$, and they are of hyperbolic type.



q.e.d.

An example of a family of area preserving reversible diffeomorphisms exhibiting this type of bifurcation of symmetric periodic points is analysed in [6] in connection to disappearence/reappearence of some KAM invariant curves.

Next we show that a Rimmer type bifurcation holds for a nonconservative R–reversible diffeomorphism of the plane. In order to do that we consider a R–reversible vector field X defined on the neighbourhood of $(0,0) \in \mathbf{R}^2$, $R(x,y) = (-x, y)$ (This choice is not a restriction because by a change of coordinates we can change the reversor).

Proposition 2.3. *Let R be the reversor defined by $R(x_1, x_2) = (-x_1, x_2)$. Then the normal forms of a R–reversible vector field $X = (X_1, X_2)$ defined on an open set $U \subset \boldsymbol{R}^2$, $(0,0) \in U$, having origin as a double-zero symmetric equilibrium point are:*

$$X_1(x_1, x_2) = x_2 + \mathcal{O}(\mid x \mid^4)$$
$$X_2(x_1, x_2) = x_1 x_2 \pm x_1^3 + \mathcal{O}(\mid x \mid^4)$$

Proof

R–reversibility of the vector field ensures that X_1 is even in x_1, while X_2 is odd in the same argument. Moreover the the origin being a double zero equilibrium point the jacobian matrix is of the form $[d_0X] = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$, $\alpha \neq 0$. That is, a double zero equilibrium point of a planar reversible vector field is a codimension one equilibrium point.

The system of differential equations associated to the vector field X is:

$$\dot{x} = d_0X(x) + V(x),$$

where $V = X - d_0X$. By the change of coordinates $y_1 = x_1/\alpha$, $y_2 = x_2$ the system becomes: $\dot{y} = Jy + F(y)$, where $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is the Jordan canonical form of the matrix $[d_0X]$. The R–reversibility was preserved by the chosen change of coordinates. Expanding F as a Taylor series one obtains: $\dot{y} = Jy + F_2(y) + F_3(y) + \mathcal{O}(|y|^4)$, where the terms $F_k(y)$ are terms of degree k in the Taylor expansion.

Consider the real vector space P_k generated by 2–vector valued monomials of degree k. Namely, if (e_1, e_2) is the standard basis in $\mathbf{R}^2$ then the corresponding basis in P_k is $x_{k_1}y_{k_2}e_i, \sum_{j=1}^{2} k_j = k$.

Our goal is to seek succesive changes of coordinates preserving R–reversibility of the system, and such that to reduce as much as possible from the k-order terms of the system. We choose the changes of the form $y = z + h_2(z)$, and then $z = u + h_3(u)$, where $h_k \in P_k, k = 2, 3$. In order to preserve R–reversibility, the double valued polynomials h_k must have the first/second component odd/even in the first argument. From the normal form theory [] it is known that can be eliminated by such changes of coordinates the terms of F_k that are in the image of the linear operator $L_J : P_k \to P_k$ defined by $L_J(Q) = JQ - dQ\,JQ$, where J is the above Jordan matrix. In our special case P_k splits as $H_e^o \oplus H_o^e$ (the super/subscripts e and o stand for even, respectively odd). For example the subspace $H_e^o \subset P_2$ is generated by the two vector valued monomials:

$$\left\{ E_1 = \begin{pmatrix} z_1z_2 \\ 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 \\ z_1^2 \end{pmatrix}, E_3 = \begin{pmatrix} 0 \\ z_2^2 \end{pmatrix} \right\}$$

and the subspace H_o^e by:

$$\left\{ E_4 = \begin{pmatrix} z_1^2 \\ 0 \end{pmatrix}, E_5 = \begin{pmatrix} z_2^2 \\ 0 \end{pmatrix}, E_6 = \begin{pmatrix} 0 \\ z_1z_2 \end{pmatrix} \right\}$$

Because we have to choose particular changes of coordinates that preserve the R–reversibility, we study only the efect of the operator L_J on the subspace H_e^o. It is easy to see that the image $\mathrm{Im}(L_{J|H_e^o}) \subset H_o^e$. Therefore the normal form of the considered vector field will contain only the terms of order $k, k = 1, 2$ that belong to the complement D_k of the subspace $L_J(H_e^o)$ in H_o^e. By straightforward computation we get that the terms $\begin{pmatrix} 0 \\ z_1z_2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ u_1^3 \end{pmatrix}$ can not be reduced by the chosen changes of coordinates.

After renaming the variables we have the normal form of the system of differential equations associated the vector field under the consideration:

$$\dot{x} = y + \mathcal{O}(|\ (x, y)\ |^4)$$
$$\dot{y} = a\,xy + b\,x^3 + \mathcal{O}(|\ (x, y)\ |^4)$$

We take the truncated normal form :

$$\dot{x} = y$$
$$\dot{y} = a\,xy + b\,x^3$$

and make the rescaling :

$$x \to \alpha\, x$$
$$y \to \beta\, y$$
$$t \to \gamma\, t$$

in order to get the simplest coefficients. Our system then becomes:

$$\dot{x} = \frac{\gamma\,\beta}{\alpha} y$$
$$\dot{y} = \gamma\,\alpha\,a\,xy + b\,\frac{\gamma\,\alpha^2}{\beta} x^3$$

Next we require

2.1 $$\frac{\gamma\,\beta}{\alpha} = 1, \text{i.e. } \gamma = \frac{\alpha}{\beta}$$

For the stability not be affected under the rescaling, α and β must have the same signs. We also require

2.2 $$\frac{\gamma\alpha^2\,b}{\beta} = 1,$$

and

2.3 $$\gamma\alpha\,a = 1$$

By (2.1), (2.3) becomes

$$a\beta\frac{\alpha^2}{\beta^2} = 1$$

while (2.2):

2.4 $$b\alpha\frac{\alpha^2}{\beta^2} = 1$$

If (2.4) is to hold, then a and b must have the same sign. But this is a too restrictive condition. In order to have a full generality we require $b\alpha\frac{\alpha^2}{\beta^2} = \pm 1$. So the normal form is:

$$\dot{x} = y$$
$$\dot{y} = xy \pm x^3$$

q.e.d.

Next we consider a candidate for a versal deformation:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\mu\, x + xy + s\, x^3, s = \pm 1 \end{aligned}$$

and study the local dynamics.

For $s = 1$, the origin is a symmetric equilibrium point for every μ. For $\mu < 0$ it is of hyperbolic type, while for $\mu > 0$ it is elliptic. When μ passes through zero (from negative to positive values) two hyperbolic asymmetric equilibrium points are born $(\pm\sqrt{\mu}, 0)$. In the case $s = -1$ when μ decreases from positive to negative values again two asymmetric equilibrium points are born: $(\sqrt{-\mu}, 0)$ – a repulsor, and $(-\sqrt{-\mu}, 0)$ an attractor. Hence a Rimmer type bifurcation occurs.

Remark. One verifies that the system

$$\begin{aligned} \dot{x} &= y + \mathcal{O}(\mid (x, y) \mid^4) \\ \dot{y} &= -\mu\, x + xy + s\, x^3 + \mathcal{O}(\mid (x, y) \mid^4) \end{aligned}$$

is locally topologically equivalent near the origin to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\mu\, x + xy + s\, x^3 \end{aligned}$$

Therefore () is indeed a versal deformation.

Remark. Given a family f_μ of non-area preserving R–reversible diffeomorfisms, such that f_0 has $(0, 0)$ as a $1:1$ resonant fixed point, then the associated family of R– reversible vectors fields $X_\mu = \Psi^{-1}(f_\mu)$ undergoes a Rimmer type bifurcation as we have seen above. By Prop. 2.1 the family f_μ also undergoes the same type of bifurcation. Moreover the type of created assymetric fixed points is preserved through Ψ.

Examples of families of nonconservative diffeomorphisms exhibiting this type of bifurcation are given in [7]. Post [7] proved that Rimmer type bifurcation also occurs in non-reversible systems if only a certain order of local reversibility is satisfied.

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