Eta-Ricci solitons on para-Kenmotsu manifolds

A. M. Blaga

Abstract. In the context of paracontact geometry, η-Ricci solitons are considered on manifolds satisfying certain curvature conditions: $R(\xi, X) \cdot S = 0$, $S \cdot R(\xi, X) = 0$, $W_2(\xi, X) \cdot S = 0$ and $S \cdot W_2(\xi, X) = 0$. We prove that on a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$, the existence of an η-Ricci soliton implies that (M, g) is quasi-Einstein and if the Ricci curvature satisfies $R(\xi, X) \cdot S = 0$, then (M, g) is Einstein. Conversely, we give a sufficient condition for the existence of an η-Ricci soliton on a para-Kenmotsu manifold.

M.S.C. 2010: 53C21, 53C44, 53C25.
Key words: η-Ricci solitons; almost paracontact structure.

1 Introduction

Ricci solitons represent a natural generalization of Einstein metrics on a Riemannian manifold, being generalized fixed points of Hamilton's Ricci flow $\frac{\partial}{\partial t} g = -2S$ [17]. The evolution equation defining the Ricci flow is a kind of nonlinear diffusion equation, an analogue of the heat equation for metrics. Under the Ricci flow, a metric can be improved to evolve into a more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold: it will expand in the directions of negative Ricci curvature and shrink in the positive case. Ricci solitons have been studied in many contexts: on Kähler manifolds [10], on contact and Lorentzian manifolds [1], [7], [19], [26], [29], on Sasakian [15], [18], α-Sasakian [19] and K-contact manifolds [26], on Kenmotsu [2], [23] and f-Kenmotsu manifolds [7] etc. In paracontact geometry, Ricci solitons firstly appeared in the paper of G. Calvaruso and D. Perrone [5]. Recently, C. L. Bejan and M. Crasmareanu studied Ricci solitons on 3-dimensional normal paracontact manifolds [4].

A more general notion is that of *η-Ricci soliton* introduced by J. T. Cho and M. Kimura [9], which was treated by C. Călin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [6].

In the present paper we shall consider η-Ricci solitons in the context of paracontact geometry, precisely, on a para-Kenmotsu manifold which satisfies certain curvature

Balkan Journal of Geometry and Its Applications, Vol.20, No.1, 2015, pp. 1-13.

properties: $R(\xi, X) \cdot S = 0$, $S \cdot R(\xi, X) = 0$, $W_2(\xi, X) \cdot S = 0$ and $S \cdot W_2(\xi, X) = 0$, respectively. Remark that in [23] H. G. Nagaraja and C. R. Premalatha have obtained some results on Ricci solitons satisfying conditions of the following type: $R(\xi, X) \cdot \tilde{C} = 0$, $P(\xi, X) \cdot \tilde{C} = 0$, $H(\xi, X) \cdot S = 0$, $\tilde{C}(\xi, X) \cdot S = 0$ and in [2] C. S. Bagewadi, G. Ingalahalli and S. R. Ashoka treated the cases: $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \bar{P} = 0$ and $\bar{P}(\xi, X) \cdot S = 0$.

2 Para-Kenmotsu manifolds

Let M be a $(2n+1)$-dimensional smooth manifold, φ a tensor field of $(1,1)$-type, ξ a vector field, η a 1-form and g a pseudo-Riemannian metric on M of signature $(n+1, n)$. We say that (φ, ξ, η, g) is an *almost paracontact metric structure* on M if [31]:

1. $\varphi\xi = 0$, $\eta \circ \varphi = 0$,
2. $\eta(\xi) = 1$, $\varphi^2 = I_{\mathfrak{X}(M)} - \eta \otimes \xi$,
3. φ induces on the $2n$-dimensional distribution $\mathcal{D} := \ker \eta$ an almost paracomplex structure P i.e. $P^2 = I_{\mathfrak{X}(M)}$ and the eigensubbundles $\mathcal{D}^+$, $\mathcal{D}^-$, corresponding to the eigenvalues 1, -1 of P respectively, have equal dimension n; hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$,
4. $g(\varphi\cdot, \varphi\cdot) = -g + \eta \otimes \eta$.

We call $(M, \varphi, \xi, \eta, g)$ *almost paracontact metric manifold*, φ *the structural endomorphism*, ξ *the characteristic vector field* and η *the paracontact form*. Examples of almost paracontact metric structures are given in [20] and [13].

From the definition it follows that η is the g-dual of ξ:

$$\eta(X) = g(X, \xi), \tag{2.1}$$

ξ is a unitary vector field:

$$g(\xi, \xi) = 1 \tag{2.2}$$

and φ is a g-skew-symmetric operator:

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.3}$$

Remark that the canonical distribution $\mathcal{D}$ is φ-invariant since $\mathcal{D} = Im\varphi$. Moreover, ξ is orthogonal to $\mathcal{D}$ and therefore the tangent bundle splits orthogonally:

$$TM = \mathcal{D} \oplus \langle \xi \rangle. \tag{2.4}$$

An analogue of the Kenmotsu manifold [22] in paracontact geometry will be further considered.

Definition 2.1. [24] We say that the almost paracontact metric structure (φ, ξ, η, g) is para-Kenmotsu if the Levi-Civita connection ∇ of g satisfies $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$, for any X, $Y \in \mathfrak{X}(M)$.

Note that the para-Kenmotsu structure was introduced by J. Wełyczko in [30] for 3-dimensional normal almost paracontact metric structures. A similar notion called P-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [28].

We shall further give some immediate properties of this structure.

Proposition 2.1. *On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$, the following relations hold:*

$$\nabla\xi = I_{\mathfrak{X}(M)} - \eta \otimes \xi \tag{2.5}$$

$$\eta(\nabla_X \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$\eta(R(X,Y)Z) = -\eta(X)g(Y,Z) + \eta(Y)g(X,Z), \quad \eta(R(X,Y)\xi) = 0, \tag{2.8}$$

$$\nabla\eta = g - \eta \otimes \eta, \quad \nabla_\xi \eta = 0, \tag{2.9}$$

$$L_\xi \varphi = 0, \quad L_\xi \eta = 0, \quad L_\xi(\eta \otimes \eta) = 0, \quad L_\xi g = 2(g - \eta \otimes \eta), \tag{2.10}$$

where R is the Riemann curvature tensor field and ∇ is the Levi-Civita connection associated to g. Moreover, η is closed, the distribution $\mathcal{D}$ is involutive and the Nijenhuis tensor field of φ vanishes identically.

Proof. Taking $Y := \xi$ in $\nabla_X \varphi Y - \varphi(\nabla_X Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X$ follows $\varphi(\nabla_X \xi) = \varphi X$ and applying φ we obtain $\nabla_X \xi - \eta(\nabla_X \xi)\xi = X - \eta(X)\xi$. But $X(g(\xi,\xi)) = 2g(\nabla_X \xi, \xi)$ and so $\eta(\nabla_X \xi) = g(\nabla_X \xi, \xi) = 0$. Therefore, $\nabla_X \xi = X - \eta(X)\xi$. In particular, $\nabla_\xi \xi = 0$.

Replacing now the expression of $\nabla\xi$ in $R(X,Y)\xi := \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$, from a direct computation we get $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$. Also $\eta(R(X,Y)Z) = g(R(X,Y)Z, \xi) = -g(R(X,Y)\xi, Z) = -[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]$. In particular, $\eta(R(X,Y)\xi) = 0$.

Compute $(\nabla_X \eta)Y := X(\eta(Y)) - \eta(\nabla_X Y) = X(g(Y,\xi)) - g(\nabla_X Y, \xi) = g(Y, \nabla_X \xi) = g(X,Y) - \eta(X)\eta(Y)$. In particular, $(\nabla_\xi \eta)Y = 0$.

Express the Lie derivatives along ξ as follows:

$$(L_\xi \varphi)(X) := [\xi, \varphi X] - \varphi([\xi, X]) = \nabla_\xi \varphi X - \nabla_{\varphi X}\xi - \varphi(\nabla_\xi X) + \varphi(\nabla_X \xi) =$$
$$= \nabla_\xi \varphi X - \varphi(\nabla_\xi X) := (\nabla_\xi \varphi)X = 0,$$
$$(L_\xi \eta)(X) := \xi(\eta(X)) - \eta([\xi, X]) = \xi(g(X,\xi)) - g(\nabla_\xi X, \xi) + g(\nabla_X \xi, \xi) =$$
$$= g(X, \nabla_\xi \xi) + \eta(\nabla_X \xi) = 0,$$
$$(L_\xi(\eta \otimes \eta))(X,Y) := \xi(\eta(X)\eta(Y)) - \eta([\xi, X])\eta(Y) - \eta(X)\eta([\xi, Y]) =$$

$$= \eta(X)\xi(\eta(Y))+\eta(Y)\xi(\eta(X))-\eta(\nabla_\xi X)\eta(Y)+\eta(\nabla_X\xi)\eta(Y)-\eta(X)\eta(\nabla_\xi Y)+\eta(X)\eta(\nabla_Y\xi) =$$
$$= \eta(X)[\xi(g(Y,\xi)) - g(\nabla_\xi Y,\xi)] + \eta(Y)[\xi(g(X,\xi)) - g(\nabla_\xi X,\xi)] =$$
$$= \eta(X)g(Y,\nabla_\xi\xi) - \eta(Y)g(X,\nabla_\xi\xi) = 0$$

and

$$(L_\xi g)(X,Y) := \xi(g(X,Y)) - g([\xi,X],Y) - g(X,[\xi,Y]) =$$
$$= \xi(g(X,Y)) - g(\nabla_\xi X,Y) + g(\nabla_X\xi,Y) - g(X,\nabla_\xi Y) + g(X,\nabla_Y\xi) =$$
$$= g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi) = 2[g(X,Y) - \eta(X)\eta(Y)].$$

From $\nabla_X\xi = X - \eta(X)\xi$ and $\nabla_X\varphi Y - \varphi(\nabla_X Y) = g(\varphi X,Y)\xi - \eta(Y)\varphi X$ we consequently obtain:

$$(d\eta)(X,Y) := X(\eta(Y))-Y(\eta(X))-\eta([X,Y]) = X(g(Y,\xi))-Y(g(X,\xi))-g([X,Y],\xi) =$$
$$= X(g(Y,\xi)) - g(\nabla_X Y,\xi) - Y(g(X,\xi)) + g(\nabla_Y X,\xi) = g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi) = 0$$

and

$$N_\varphi(X,Y) := \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] =$$
$$= \varphi^2(\nabla_X Y)-\varphi(\nabla_X\varphi Y)-\varphi^2(\nabla_Y X)+\varphi(\nabla_Y\varphi X)+\nabla_{\varphi X}\varphi Y-\varphi(\nabla_{\varphi X}Y)-\nabla_{\varphi Y}\varphi X+\varphi(\nabla_{\varphi Y}X) =$$
$$= [g(\varphi^2 X,Y) - g(X,\varphi^2 Y)]\xi = 0.$$

□

Example 2.2. Let $M = \mathbb{R}^3$ and (x,y,z) be the standard coordinates in $\mathbb{R}^3$. Set

$$\varphi := \frac{\partial}{\partial y}\otimes dx + \frac{\partial}{\partial x}\otimes dy, \quad \xi := -\frac{\partial}{\partial z}, \quad \eta := -dz,$$
$$g := dx\otimes dx - dy\otimes dy + dz\otimes dz.$$

Then (φ,ξ,η,g) is a para-Kenmotsu structure on $\mathbb{R}^3$. Indeed, being sufficiently to verify the conditions in the definition on a linearly independent system of vector fields, consider it,

$$E_1 := \frac{\partial}{\partial x}, \quad E_2 := \frac{\partial}{\partial y}, \quad E_3 := -\frac{\partial}{\partial z}.$$

Follows

$$\varphi E_1 = E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0,$$
$$\eta(E_1) = 0, \quad \eta(E_2) = 0, \quad \eta(E_3) = 1,$$
$$[E_1,E_2] = 0, \quad [E_2,E_3] = 0, \quad [E_3,E_1] = 0$$

and the Levi-Civita connection ∇ is deduced from Koszul's formula

$$2g(\nabla_X Y,Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))-$$
$$-g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),$$

precisely,

$$\nabla_{E_1}E_1 = -E_3, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = E_1,$$
$$\nabla_{E_2}E_1 = 0, \quad \nabla_{E_2}E_2 = E_3, \quad \nabla_{E_2}E_3 = E_2,$$
$$\nabla_{E_3}E_1 = E_1, \quad \nabla_{E_3}E_2 = E_2, \quad \nabla_{E_3}E_3 = 0.$$

In this setting, we shall study η-Ricci solitons for the cases: $R(\xi,X)\cdot S = 0$, $S\cdot R(\xi,X) = 0$, $W_2(\xi,X)\cdot S = 0$ and $S\cdot W_2(\xi,X) = 0$.

3 η-Ricci solitons on $(M, \varphi, \xi, \eta, g)$

Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. Consider the equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y), \tag{3.2}$$

for any X, $Y \in \mathfrak{X}(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (3.1) is said to be an *η-Ricci soliton* on M [9]; in particular, if $\mu = 0$, (g, ξ, λ) is a *Ricci soliton* [17] and it is called *shrinking*, *steady* or *expanding* according as λ is negative, zero or positive, respectively [11].

An important geometrical object in studying Ricci solitons is well-known to be a symmetric $(0, 2)$-tensor field which is parallel with respect to the Levi-Civita connection, some of its geometrical properties being described in [3], [12] etc. In the same manner as in [6] we shall state the existence of η-Ricci solitons in our settings.

Consider now α such a symmetric $(0, 2)$-tensor field which is parallel with respect to the Levi-Civita connection ($\nabla\alpha = 0$). From the Ricci identity $\nabla^2\alpha(X, Y; Z, W) - \nabla^2\alpha(X, Y; W, Z) = 0$, one obtains for any X, Y, Z, $W \in \mathfrak{X}(M)$ [27]

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0. \tag{3.3}$$

In particular, for $Z = W := \xi$ from the symmetry of α follows $\alpha(R(X, Y)\xi, \xi) = 0$, for any X, $Y \in \mathfrak{X}(M)$.

If (φ, ξ, η, g) is a para-Kenmotsu structure on M, from Proposition 2.1 we have $R(X, Y)\xi = \eta(X)Y - \eta(Y)X$ and replacing this expression in α we get:

$$\alpha(Y, \xi) - \eta(Y)\alpha(\xi, \xi) = 0, \tag{3.4}$$

for any $Y \in \mathfrak{X}(M)$, equivalent to:

$$\alpha(Y, \xi) - \alpha(\xi, \xi)g(Y, \xi) = 0, \tag{3.5}$$

for any $Y \in \mathfrak{X}(M)$. Differentiating the equation (3.5) covariantly with respect to the vector field $X \in \mathfrak{X}(M)$ we obtain:

$$\alpha(\nabla_X Y, \xi) + \alpha(Y, \nabla_X \xi) = \alpha(\xi, \xi)[g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi)]$$

and substituting the expression of $\nabla_X \xi = X - \eta(X)\xi$ we get:

$$\alpha(Y, X) = \alpha(\xi, \xi)g(Y, X), \tag{3.6}$$

for any X, $Y \in \mathfrak{X}(M)$. As α is ∇-parallel, follows $\alpha(\xi, \xi)$ is constant and we conclude:

Proposition 3.1. *Under the hypotheses above, any parallel symmetric* $(0,2)$*-tensor field is a constant multiple of the metric.*

Because on a para-Kenmotsu manifold (M,φ,ξ,η,g), $\nabla_X\xi = X-\eta(X)\xi$ and $L_\xi g = 2(g-\eta\otimes\eta)$, the equation (3.2) becomes:

$$S(X,Y) = -(\lambda+1)g(X,Y) - (\mu-1)\eta(X)\eta(Y). \tag{3.7}$$

In particular, $S(X,\xi) = S(\xi,X) = -(\lambda+\mu)\eta(X)$. But it is known [31] that on a $(2n+1)$-dimensional paracontact manifold M, $S(X,\xi) = -(\dim(M)-1)\eta(X) = -2n\eta(X)$, so:

$$\lambda+\mu = 2n. \tag{3.8}$$

In this case, the Ricci operator Q defined by $g(QX,Y) := S(X,Y)$ has the expression:

$$QX = -(2n+1-\mu)X - (\mu-1)\eta(X)\xi. \tag{3.9}$$

Now we shall apply the previous results to η-Ricci solitons.

Theorem 3.2. *Let* (M,φ,ξ,η,g) *be a para-Kenmotsu manifold. Assume that the symmetric* $(0,2)$*-tensor field* $\alpha := L_\xi g + 2S + 2\mu\eta\otimes\eta$ *is parallel with respect to the Levi-Civita connection associated to* g*. Then* (g,ξ,μ) *yields an* η*-Ricci soliton.*

Proof. Compute

$$\alpha(\xi,\xi) = (L_\xi g)(\xi,\xi) + 2S(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi) = -2\lambda,$$

so $\lambda = -\frac{1}{2}\alpha(\xi,\xi)$. From (3.6) we get $\alpha(X,Y) = -2\lambda g(X,Y)$, for any X, $Y \in \mathfrak{X}(M)$. Therefore, $L_\xi g + 2S + 2\mu\eta\otimes\eta = -2\lambda g$. □

For $\mu = 0$ follows $L_\xi g + 2S + 4ng = 0$ and we conclude:

Corollary 3.3. *On a para-Kenmotsu manifold* (M,φ,ξ,η,g) *with the property that the symmetric* $(0,2)$*-tensor field* $\alpha := L_\xi g + 2S$ *is parallel with respect to the Levi-Civita connection associated to* g*, the relation (3.1), for* $\mu = 0$ *and* $\lambda = 2n$*, defines a Ricci soliton on* M*.*

Conversely, we shall study the consequences of the existence of η-Ricci solitons on a para-Kenmotsu manifold. From (3.7) we deduce:

Proposition 3.4. *If (3.1) defines an* η*-Ricci soliton on the para-Kenmotsu manifold* (M,φ,ξ,η,g)*, then* (M,g) *is quasi-Einstein.*

Recall that the manifold is called *quasi-Einstein* if the Ricci curvature tensor field S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero 1-form η satisfying $\eta(X) = g(X,\xi)$, for ξ a unit vector field [8] and respectively, *Einstein* if S is collinear with g.

Proposition 3.5. *If* (φ,ξ,η,g) *is a para-Kenmotsu structure on* M *and (3.1) defines an* η*-Ricci soliton on* M*, then:*

1. $Q \circ \varphi = \varphi \circ Q$*;*

2. Q *and* S *are parallel along* ξ*.*

Proof. The first statement follows from a direct computation and for the second one, note that $(\nabla_\xi Q)X := \nabla_\xi QX - Q(\nabla_\xi X)$ and $(\nabla_\xi S)(X,Y) := \xi(S(X,Y)) - S(\nabla_\xi X, Y) - S(X, \nabla_\xi Y)$ and replace Q and S from (3.9) and (3.7). □

A particular case arise when the manifold is *φ-Ricci symmetric*, which means that $\varphi^2 \circ \nabla Q = 0$, fact stated in the next proposition:

Proposition 3.6. *Let $(M, \varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If M is φ-Ricci symmetric and (3.1) defines an η-Ricci soliton on M, then $\mu = 1$, $\lambda = 2n - 1$ and (M, g) is Einstein manifold.*

Proof. Replacing Q from (3.9) in $(\nabla_X Q)Y := \nabla_X QY - Q(\nabla_X Y)$ and applying φ^2 we obtain:

$$(\mu - 1)\eta(Y)[X - \eta(X)\xi] = 0,$$

for any X, $Y \in \mathfrak{X}(M)$. Follows $\mu = 1$, $\lambda = 2n - 1$ and $S = -2ng$. □

In particular, the existence of an η-Ricci soliton on a para-Kenmotsu manifold which is *Ricci symmetric* (i.e. $\nabla S = 0$) implies that (M, g) is Einstein manifold. Remark that the class of Ricci symmetric manifolds represents an extension of the class of Einstein manifolds to which belong also the locally symmetric manifolds (i.e. those satisfying $\nabla R = 0$). The condition $\nabla S = 0$ implies $R \cdot S = 0$ and the manifolds satisfying this condition are called Ricci semisymmetric [21].

We end these considerations by giving an example of η-Ricci soliton on a para-Kenmotsu manifold.

Example 3.1. Let $M = \mathbb{R}^3$ and (x, y, z) be the standard coordinates in $\mathbb{R}^3$. Set

$$\varphi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -\frac{\partial}{\partial z}, \quad \eta := -dz,$$

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz$$

and consider the linearly independent system of vector fields

$$E_1 := \frac{\partial}{\partial x}, \quad E_2 := \frac{\partial}{\partial y}, \quad E_3 := -\frac{\partial}{\partial z}.$$

Follows

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1, \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = E_3,$$

$$\nabla_{E_2} E_3 = E_2, \quad \nabla_{E_3} E_1 = E_1, \quad \nabla_{E_3} E_2 = E_2, \quad \nabla_{E_3} E_3 = 0.$$

Then the Riemann and the Ricci curvature tensor fields are given by:

$$R(E_1, E_2)E_2 = E_1, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_1)E_1 = -E_2,$$

$$R(E_2,E_3)E_3=-E_2,\quad R(E_3,E_1)E_1=E_3,\quad R(E_3,E_2)E_2=-E_3,$$

$$S(E_1,E_1)=0,\quad S(E_2,E_2)=0,\quad S(E_3,E_3)=-2.$$

In this case, from (3.7), for $\lambda=-1$ and $\mu=3$, the data (g,ξ,λ,μ) is an η-Ricci soliton on $(\mathbb{R}^3,\varphi,\xi,\eta,g)$.

In what follows we shall consider η-Ricci solitons requiring for the curvature to satisfy $R(\xi,X)\cdot S=0$, $S\cdot R(\xi,X)=0$, $W_2(\xi,X)\cdot S=0$ and $S\cdot W_2(\xi,X)=0$, respectively, where the W_2-curvature tensor field is the curvature tensor introduced by G. P Pokhariyal and R. S. Mishra in [25]:

$$W_2(X,Y)Z:=R(X,Y)Z+\frac{1}{\dim(M)-1}[g(X,Z)QY-g(Y,Z)QX]=$$

$$=R(X,Y)Z+\frac{1}{2n}[g(X,Z)QY-g(Y,Z)QX]. \tag{3.10}$$

3.1 η-Ricci solitons on para-Kenmotsu manifolds satisfying $R(\xi,X)\cdot S=0$

The condition that must be satisfied by S is:

$$S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0, \tag{3.11}$$

for any X, Y, $Z\in\mathfrak{X}(M)$.

Replacing the expression of S from (3.7) and from the symmetries of R we get:

$$(\mu-1)[\eta(Y)g(X,Z)+\eta(Z)g(X,Y)-2\eta(X)\eta(Y)\eta(Z)]=0, \tag{3.12}$$

for any X, Y, $Z\in\mathfrak{X}(M)$.

For $Z:=\xi$ we have:

$$(\mu-1)g(\varphi X,\varphi Y)=0, \tag{3.13}$$

for any X, $Y\in\mathfrak{X}(M)$. But $\lambda+\mu=2n$ and we can state:

Theorem 3.7. *If (φ,ξ,η,g) is a para-Kenmotsu structure on the $(2n+1)$-dimensional manifold M, (g,ξ,λ,μ) is an η-Ricci soliton on M and $R(\xi,X)\cdot S=0$, then $\mu=1$, $\lambda=2n-1$ and (M,g) is Einstein manifold.*

Corollary 3.8. *On a para-Kenmotsu manifold (M,φ,ξ,η,g) satisfying $R(\xi,X)\cdot S=0$, there is no Ricci soliton with the potential vector field ξ.*

3.2 η-Ricci solitons on para-Kenmotsu manifolds satisfying $S(\xi, X) \cdot R = 0$

The condition that must be satisfied by S is:

$$S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W-$$

$$-S(\xi, Y)R(X, Z)W + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W+$$

$$+S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0, \tag{3.14}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

Taking the inner product with ξ, the relation (3.14) becomes:

$$S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\eta(X)+$$

$$+S(X, Y)\eta(R(\xi, Z)W) - S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W)-$$

$$-S(\xi, Z)\eta(R(Y, X)W) + S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0, \tag{3.15}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

Replacing the expression of S from (3.7), we get:

$$(\lambda + 1)[g(X, R(Y, Z)W) - 2\eta(X)\eta(Z)g(Y, W) + 2\eta(X)\eta(Y)g(Z, W)-$$

$$-g(X, Y)g(Z, W) + g(X, Z)g(Y, W)]+$$

$$+(\mu - 1)[\eta(Y)\eta(W)g(X, Z) - \eta(Z)\eta(W)g(X, Y)] = 0, \tag{3.16}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

For $W := \xi$ we have:

$$(2\lambda + \mu + 1)[\eta(Y)g(X, Z) - \eta(Z)g(X, Y)] = 0, \tag{3.17}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, which is equivalent to

$$(2\lambda + \mu + 1)g(X, R(Y, Z)\xi) = 0, \tag{3.18}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. But $\lambda + \mu = 2n$, so $4n + 1 - \mu = 0$ and we can state:

Theorem 3.9. *If (φ, ξ, η, g) is a para-Kenmotsu structure on the $(2n+1)$-dimensional manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $S(\xi, X) \cdot R = 0$, then $\mu = 4n+1$ and $\lambda = -2n - 1$.*

Corollary 3.10. *On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ satisfying $S(\xi, X) \cdot R = 0$, there is no Ricci soliton with the potential vector field ξ.*

3.3 η-Ricci solitons on para-Kenmotsu manifolds satisfying $W_2(\xi, X) \cdot S = 0$

The condition that must be satisfied by S is:

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0, \tag{3.19}$$

for any X, Y, $Z \in \mathfrak{X}(M)$.

Replacing the expression of S from (3.7) we get:

$$\frac{(\mu-1)(2\lambda+\mu+1-2n)}{2n}[\eta(Y)g(X,Z)+\eta(Z)g(X,Y)-2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{3.20}$$

for any X, Y, $Z \in \mathfrak{X}(M)$.

For $Z := \xi$ we have:

$$(\mu-1)(2\lambda+\mu+1-2n)g(\varphi X, \varphi Y) = 0, \tag{3.21}$$

for any X, $Y \in \mathfrak{X}(M)$. But $\lambda + \mu = 2n$, so $(\mu-1)(2n+1-\mu) = 0$ and we can state:

Theorem 3.11. *If (φ, ξ, η, g) is a para-Kenmotsu structure on the $(2n+1)$-dimensional manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $W_2(\xi, X) \cdot S = 0$, then $\mu = 1$ and $\lambda = 2n-1$ or $\mu = 2n+1$ and $\lambda = -1$.*

Corollary 3.12. *On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ satisfying $W_2(\xi, X) \cdot S = 0$, there is no Ricci soliton with the potential vector field ξ.*

3.4 η-Ricci solitons on para-Kenmotsu manifolds satisfying $S(\xi, X) \cdot W_2 = 0$

The condition that must be satisfied by S is:

$$\begin{aligned} &S(X, W_2(Y,Z)V)\xi - S(\xi, W_2(Y,Z)V)X + S(X,Y)W_2(\xi,Z)V - \\ &-S(\xi,Y)W_2(X,Z)V + S(X,Z)W_2(Y,\xi)V - S(\xi,Z)W_2(Y,X)V + \\ &+S(X,V)W_2(Y,Z)\xi - S(\xi,V)W_2(Y,Z)X = 0, \end{aligned} \tag{3.22}$$

for any X, Y, Z, $V \in \mathfrak{X}(M)$.

Taking the inner product with ξ, the relation (3.22) becomes:

$$\begin{aligned} &S(X, W_2(Y,Z)V) - S(\xi, W_2(Y,Z)V)\eta(X) + \\ &+S(X,Y)\eta(W_2(\xi,Z)V) - S(\xi,Y)\eta(W_2(X,Z)V) + S(X,Z)\eta(W_2(Y,\xi)V) - \\ &-S(\xi,Z)\eta(W_2(Y,X)V) + S(X,V)\eta(W_2(Y,Z)\xi) - S(\xi,V)\eta(W_2(Y,Z)X) = 0, \end{aligned} \tag{3.23}$$

for any X, Y, Z, $V \in \mathfrak{X}(M)$.

Replacing the expression of S from (3.7), we get:

$$(\lambda+1)[g(X,R(Y,Z)V)-\frac{2\lambda+\mu+1-2n}{2n}(g(X,Z)g(Y,V)-g(X,Y)g(Z,V))+$$

$$+\frac{2\lambda+\mu+1-4n}{2n}(\eta(X)\eta(Z)g(Y,V)-\eta(X)\eta(Y)g(Z,V))+$$

$$+\frac{(\mu-1)(\lambda+\mu-2n)}{2n}(\eta(Z)\eta(V)g(X,Y)-\eta(Y)\eta(V)g(X,Z))=0, \tag{3.24}$$

for any X, Y, Z, $V\in\mathfrak{X}(M)$.

For $V:=\xi$ we have:

$$[(\lambda+1)^2+(\lambda+\mu)^2-2n(2\lambda+\mu+1)][\eta(Y)g(X,Z)-\eta(Z)g(X,Y)]=0, \tag{3.25}$$

for any X, Y, $Z\in\mathfrak{X}(M)$, which is equivalent to

$$[(\lambda+1)^2+(\lambda+\mu)^2-2n(2\lambda+\mu+1)]g(X,R(Y,Z)\xi)=0, \tag{3.26}$$

for any X, Y, $Z\in\mathfrak{X}(M)$. But $\lambda+\mu=2n$, so $\mu^2-2(n+1)\mu+2n+1=0$ and we can state:

Theorem 3.13. *If (φ,ξ,η,g) is a para-Kenmotsu structure on the $(2n+1)$-dimensional manifold M, (g,ξ,λ,μ) is an η-Ricci soliton on M and $S(\xi,X)\cdot W_2=0$, then $\mu=1$ and $\lambda=2n-1$ or $\mu=2n+1$ and $\lambda=-1$.*

Corollary 3.14. *On a para-Kenmotsu manifold (M,φ,ξ,η,g) satisfying $S(\xi,X)\cdot W_2=0$, there is no Ricci soliton with the potential vector field ξ.*

Acknowledgement. The author acknowledges the support by the research grant PN-II-ID-PCE-2011-3-0921.

References

[1] C. S. Bagewadi, G. Ingalahalli, G., *Ricci solitons in Lorentzian α-Sasakian manifolds*, Acta Math. Academiae Paedagogicae Nyíregyháziensis 28, 1 (2012), 59-68.

[2] C. S. Bagewadi, G. Ingalahalli, S. R. Ashoka, *A Study on Ricci Solitons in Kenmotsu Manifolds*, ISRN Geometry, vol. 2013, Article ID 412593, 6 pages (2013).

[3] C. L. Bejan, M. Crasmareanu, *Ricci solitons in manifolds with quasi-constant curvature*, Publ. Math. Debrecen 78, 1 (2011), 235–243.

[4] C. L. Bejan, M. Crasmareanu, *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Anal. Global Anal. Geom., DOI: 10.1007/s10455-014-9414-4.

[5] G. Calvaruso, D. Perrone, *Geometry of H-paracontact metric manifolds*, arxiv:1307.7662v1.2013.

[6] C. Călin, M. Crasmareanu, *Eta-Ricci solitons on Hopf hypersurfaces in complex space forms*, Revue Roumaine de Mathematiques pures et appliques 57, 1 (2012), 55-63.

[7] C. Călin, M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malaysian Math. Sci. Soc. 33, 3 (2010), 361-368.

[8] M. C. Chaki, R. K. Maity, *On quasi Einstein manifolds*, Publ. Math. Debrecen 57 (2000), 297-306.

[9] J. T. Cho, M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. 61, 2 (2009), 205-212.

[10] O. Chodosh, F. T.-H. Fong, *Rotational symmetry of conical Kähler-Ricci solitons*, arxiv:1304.0277v2.2013.

[11] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci Flow, Graduate Studies in Mathematics*, 77, AMS, Providence, RI, USA (2006).

[12] M. Crasmareanu, *Parallel tensors and Ricci solitons in $N(k)$-quasi Einstein manifolds*, Indian J. Pure Appl. Math. 43, 4 (2012).

[13] P. Dacko, Z. Olszak, *On weakly para-cosymplectic manifolds of dimension* 3, J. Geom. Phys. 57 (2007), 561-570.

[14] S. Deshmukh, H. Al-Sodais, H. Alodan, *A note on Ricci solitons*, Balkan J. Geom. Appl. 16, 1 (2011), 48-55.

[15] A. Futaki, H. Ono, G. Wang, *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds*, J. Diff. Geom. 83, 3 (2009), 585-636.

[16] S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor. N. S. 29 (1975), 293-301.

[17] R. S. Hamilton, *The Ricci flow on surfaces*, Math. and general relativity (Santa Cruz, CA, 1986), 237-262, Contemp. Math. 71, AMS (1988).

[18] C. He, M. Zhu, *The Ricci solitons on Sasakian manifolds*, arxiv:1109.4407v2.2011.

[19] G. Ingalahalli, C. S. Bagewadi, *Ricci solitons in α-Sasakian manifolds*, ISRN Geometry, vol. 2012, Article ID 421384, 13 pages (2012).

[20] S. Ivanov, D. Vassilev, S. Zamkovoy, *Conformal paracontact curvature and the local flatness theorem*, Geom. Dedicata 144 (2010), 79-100.

[21] D. H. Jin, *The curvatures of lightlike hypersurfaces of an indefinite Kenmotsu manifold*, Balkan J. Geom. Appl. 17, 1 (2012), 49-57.

[22] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. 24 (1972), 93-103.

[23] H. G. Nagaraja, C. R. Premalatha, *Ricci solitons in Kenmotsu manifolds*, J. Math. Anal. 3, 2 (2012), 18-24.

[24] Z. Olszak, *The Schouten-van Kampen affine connection adapted to an almost (para) contact metric structure*, Publ. Inst. Math. nouv. sér. 94(108) (2013), 31-42.

[25] G. P. Pokhariyal, R. S. Mishra, *The curvature tensors and their relativistic significance*, Yokohama Math. J. 18 (1970), 105-108.

[26] R. Sharma, *Certain results on K-contact and (k,μ)-contact manifolds*, J. Geom. 89, 1-2 (2008), 138-147.

[27] R. Sharma, *Second order parallel tensor in real and complex space forms*, Internat. J. Math. Math. Sci. 12, 4 (1989), 787-790.

[28] B. B. Sinha, K. L. Sai Prasad, *A class of almost para contact metric manifolds*, Bull. Cal. Math. Soc. 87 (1995), 307-312.

[29] M. M. Tripathi, *Ricci solitons in contact metric manifolds*, http://arxiv.org/abs/0801.4222.

[30] J. Wełyczko, *Slant curves in 3-dimensional normal almost paracontact metric manifolds*, Mediterr. J. Math., DOI 10.1007/s00009-013-0361-2 (2013).

[31] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom. 36, 1 (2008), 37-60.

Author's address:

Adara-Monica Blaga
Department of Mathematics and Computer Science,
V. Parvan 4, Timisoara, 300223, Romania.
E-mail: adarablaga@yahoo.com