Characterization of warped product submanifolds in Kenmotsu manifolds

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Abstract. In the present paper, we show the existence of warped product semi-slant submanifolds in a Kenmotsu manifold by an example. We locally characterize the warped product semi-slant submanfiolds in a Kenmotsu manifold. Such submanifold does not exist in Kähler, Sasakian and cosymplectic manifolds. Further, we search some geometric properties to construct an inequality for second fundamental form of the immersion of warped product submanifolds in Kenmotsu space forms. The equality case is also discussed.

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1 Introduction

The idea of slant submanifolds of an almost Hermitian manifold was given by Chen [9] as a generalization of holomorphic and totally real submanifolds. Later on, N. Papaghiuc [21] introduced another class of submanifolds, called semi-slant submanifolds which generalize CR as well as slant submanifolds. On the other hand, warped products appeared in differential geometry, generalizing the class of Riemannian product manifolds [4]. The study of warped products are applied in general relativity to model the standard space time, specially in the neighborhood of massive stars and black holes. Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and $f > 0$ be a differential function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. Then their warped product $N_1 \times_f N_2$ is the Riemannian manifold $(N_1 \times N_2, g)$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_{1\star}(X)\|^2 + (f \circ \pi_1)^2 \|\pi_{2\star}(X)\|^2,$$

for any tangent vector X on M, where $\star$ is the symbol for the tangent maps. It was proved in [4] that for a warped product manifold $M = N_1 \times_f N_2$, we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \tag{1.1}$$

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for any $X \in TN_1$ and $Z \in TN_2$, where ∇ denotes the Levi-Civita connection on M. A warped product manifold $N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant. For a survey on warped products as Riemannian submanifolds we refer to ([2], [10], [12], [17]). Recently, Sahin [22] introduced the notion of semi-slant warped product in complex geometry.

On the other hand, Kenmotsu [15] studied a class of almost contact metric manifolds, so called Kenmotsu manifolds. He showed that Kenmotsu manifold is locally a warped product $I \times_f N$ of an interval I and a Kähler manifold N with warping function $f(t) = se^t$; where s is a nonzero constant. Kenmotsu manifolds were studied by many authors such as De [13], Binh, Tamassy, De and Tarafdar [3], Ozgur ([19], [20]). Since Kenmotsu manifolds are themselves locally warped product spaces, it is interesting to study geometry of warped product submanifolds in the context of Kenmotsu manifolds. Several authors has studied warped product submanifolds of Kenmotsu manifolds (see for example, [1], [2], [18], [24] and the references therein). Non-existence of warped product semi-slant submanifolds in Kähler, cosymplectic and Sasakian manifolds was shown in [22], [16] and [23], respectively. On the contrary, there do exist warped product semi-slant submanifolds in Kenmotsu manifolds as given in Example 3.1.

Chen used Codazzi equation to construct a relation between the second fundamental form and the warping function for a CR-warped product in complex space forms [10]. Later on, it was extended for Sasakian and Kenmotsu space forms in [17] and [1], respectively. We use Gauss equation to establish an inequality in terms of the second fundamental form and the scalar curvatures.

This paper is organized as: In section 2, we enlist the basic definitions and equations which we need for the next sections. In Section 3, warped product semi-slant submanifolds in Kenmotsu manifolds are characterized. We also prove the existence of warped product semi-slant subamnifolds in Kenmotsu manifolds by an example. In the last section, we discuss some geometric properties, specially N_T-minimality and using this result, we derive a general inequality. Finally, we establish an inequality for a more general type of warped product submanifolds $N_T \times_f N$ in a Kenmotsu space form.

2 Preliminaries

A $(2m+1)-$dimensional C^∞ manifold $\bar{M}$ is said to have an *almost contact metric structure* if there exist on $\bar{M}$ a tensor field ϕ of type $(1,1)$, a vector field ξ, a $1-$form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1,$$
$$\eta(X) = g(X, \xi), \qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

where X and Y are vector fields on $\bar{M}$ [5].

A Riemannian manifold $\bar{M}$ with an almost contact metric structure (ϕ, ξ, η, g) is called a *Kenmotsu manifold* if [15]

$$(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.1}$$

for all $X, Y \in T\bar{M}$. From (2.1) we also have $\bar{\nabla}_X \xi = X - \eta(X)\xi$, for all $X \in T\bar{M}$.

Let M be a Riemannian manifold isometrically immersed in a Kenmotsu manifold $\bar{M}$. Then, Gauss and Weingarten formulae are respectively given by [8]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for all vector fields X, Y tangent to M, where ∇ is the induced Riemannian connection on M, N is a vector field normal to M, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator corresponding to N. Clearly, $g(A_N X, Y) = g(h(X, Y), N)$, where g denotes the Riemannian metric on $\bar{M}$ as well as the metric induced on M.

The *equation of Gauss* for the submanifold M is given by [8]

$$\begin{aligned} R(X, Y, Z, W) &= \bar{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) \\ &\quad - g(h(X, Z), h(Y, W)), \end{aligned} \tag{2.2}$$

for all $X, Y, Z, W \in TM$, where $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M respectively.

For any $X \in TM$ and $N \in T^{\perp}M$, we write $\phi X = PX + FX$, and $\phi N = tN + fN$, where PX, tN are the tangential components and FX, fN are the normal components of ϕX and ϕN, respectively. If $PX = 0$ (resp. $FX = 0$), then M is called an invariant (resp. anti-invariant) submanifold.

The covariant derivatives of the tensor fields P and F are defined as

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.3}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y. \tag{2.4}$$

Also, we have

$$(\nabla_X P)Y = A_{FY} X + th(X, Y) - g(X, PY)\xi - \eta(Y)PX, \tag{2.5}$$

$$(\nabla_X F)Y = fh(X, Y) - h(X, PY) - \eta(Y)FX. \tag{2.6}$$

We recall that the Riemannian curvature tensor of a Kenmotsu space form $\bar{M}(c)$ of constant ϕ-sectional curvature c is given by [15]

$$\begin{aligned} \bar{R}(X, Y, Z, W) &= \frac{c-3}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ &- \frac{c+1}{4}\{\eta(Z)[\eta(Y)g(X, W) - \eta(X)g(Y, W)] \\ &+ \eta(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - g(\phi X, W)g(\phi Y, Z) \\ &+ g(\phi X, Z)g(\phi Y, W) + 2g(\phi X, Y)g(\phi Z, W)\}, \end{aligned} \tag{2.7}$$

for any vector fields X, Y, Z, W tangent to $\bar{M}$.

For an orthonormal frame $\{e_1, \cdots, e_n\}$ of TM, the *mean curvature vector* H is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $n = dimM$. The submanifold M is said to be *totally geodesic* in $\bar{M}$ if $h = 0$, and *minimal* if $H = 0$. If $h(X, Y) = g(X, Y)H$ for all $X, Y \in TM$, then M is called *totally umbilical.* It is easy to check that a totally umbilical submanifold in Kenmotsu manifold is always totally geodesic. For differentiable function ψ on M, the *gradient* $\vec{\nabla}\psi$ and the *Laplacian* $\Delta\psi$ of ψ are defined respectively by

$$g(\vec{\nabla}\psi, X) = X\psi,$$

and

$$\Delta\psi = \sum_{i=1}^{n}((\nabla_{e_i} e_i)\psi - e_i e_i \psi),$$

for any vector field X tangent to M. The *scalar curvature* τ of M is defined by

$$\tau(TM) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the *sectional curvature* of the plane section spanned by e_i and e_j. Let Π_k be a k-plane section of TM and $\{e_1, \cdots, e_k\}$ any orthonormal frame of Π_k. The scalar curvature $\tau(\Pi_k)$ of Π_k is given by

$$\tau(\Pi_k) = \sum_{1 \le i < j \le k} K(e_i \wedge e_j).$$

Let M be a submanifold of an almost contact metric manifold $\bar{M}$. For each non zero vector X tangent to M at x, such that X is not proportional to ξ, if the angle $\theta(X)$ $(0 \le \theta(X) \le \pi/2)$ between ϕX and $T_x M$ is constant for all $X \in T_x M - \langle\xi\rangle$ and $x \in M$, then M is said to be a slant submanifold [6]. Obviously, if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant. For $\xi \in T^{\perp}M$, following the same procedure as in [6] we can easily verify the following:

Theorem 2.1. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in T^{\perp}M$. Then M is slant if and only if there exists a constant $\delta \in [0, 1]$ such that $P^2 X = -\delta X$. Furthermore, if θ is slant angle, then $\delta = \cos^2\theta$. Also, for all $X, Y \in TM$,*

$$g(PX, PY) = \cos^2\theta g(X, Y), \tag{2.8}$$

$$g(FX, FY) = \sin^2\theta g(X, Y), \tag{2.9}$$

Definition 2.1. [7] A submanifold M of an almost contact manifold $\bar{M}$ is said to be a *proper semi-slant submanifold* if there exist two distributions $\mathcal{D}$ and $\mathcal{D}_\theta$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \langle\xi\rangle$
(ii) $\mathcal{D}$ is invariant i.e., $\phi\mathcal{D} \subseteq TM$.
(iii) $\mathcal{D}_\theta$ is slant with slant angle $0 \ne \theta \ne \frac{\pi}{2}$.

A semi-slant submanifold M is said to be *mixed geodesic* if $h(X, Z) = 0$, for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}_\theta$.

If ν_x is the maximal invariant subspace of the normal space $T_x^{\perp}M$, $x \in M$, then in the case of semi-slant submanifold, $\nu : x \to \nu_x$, $x \in M$, forms a subbundle of the normal bundle $T^{\perp}M$. Then, $T^{\perp}M$ can be decomposed as $T^{\perp}M = F\mathcal{D}_\theta \oplus \nu$.

3 Characterization for warped products

In this section, we prove a local characterization for warped product semi-slant submanifolds $M = N_T \times_f N_\theta$, with ξ tangent to N_T, in a Kenmotsu manifold. We first prove the following:

Lemma 3.1. *Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold in a Kenmotsu manifold $\bar{M}$ such that ξ is tangent to N_T, where N_T and N_θ are invariant and slant submanifolds of $\bar{M}$, respectively. Then*

(i) $X ln f - \eta(X) = 0$,

(ii) $A_{FZ}X = th(X, Z) = 0$,

for all $X \in TN_T$ and $Z \in TN_\theta$.

Proof. From (2.3) and (2.5) we have

$$A_{FZ}X + th(X, Z) = (\nabla_X P)Z = (X ln f)PZ - (X ln f)PZ = 0,$$

which implies

$$A_{FZ}X + th(X, Z) = 0, \tag{3.1}$$

and thus for all $Z, W \in TN_\theta$,

$$g(h(X, W), FZ) = g(h(X, Z), FW). \tag{3.2}$$

Again, from (2.3) and (2.5) we get

$$th(X, Z) - \eta(X)PZ = (\nabla_Z P)X = (PX ln f)Z - (X ln f)PZ,$$

which implies

$$\{X ln f - \eta(X)\}g(PZ, W) = (PX ln f)g(Z, W) + g(h(X, Z), FW). \tag{3.3}$$

Interchanging Z and W in the above equation we obtain

$$\{X ln f - \eta(X)\}g(PW, Z) = (PX ln f)g(Z, W) + g(h(X, W), FZ). \tag{3.4}$$

From (3.3) and (3.4) we conclude that $X ln f - \eta(X) = 0$, and it directly implies $PX ln f = 0$, for all $X \in TN_T$. Therefore, from (3.1) and (3.3) we obtain the lemma. □

Theorem 3.2. *Let M be a proper semi-slant submanifold of Kenmotsu manifold $\bar{M}$. Then M is locally a warped product of invariant and slant submanifolds if and only if*

$$A_{FZ}X = 0, \tag{3.5}$$

for any $X \in \mathcal{D} \oplus \langle\xi\rangle$ and any $Z \in \mathcal{D}_\theta$, where $\mathcal{D} \oplus \langle\xi\rangle$ and $\mathcal{D}_\theta$ are invariant and slant distributions of M, respectively.

Proof. Let M be a semi-slant submanifold of a Kenmotsu manifold $\bar{M}$ such that (3.5) holds. Let Y be a vector field in $D \oplus \langle\xi\rangle$, $Z \in D_\theta$ and $V \in TM$. Then, from (2.9) we have

$$\begin{aligned} \sin^2\theta g(\nabla_V Y, Z) &= g(F\nabla_V Y, FZ) \\ &= -g((\nabla_V F)Y, FZ) \\ &= g(fh(V,Y) - h(V,PY) - \eta(Y)FV, FZ) \\ &= -g(A_{fFZ}Y + A_{FZ}PY, V) + \sin^2\theta\eta(Y)g(V,Z). \end{aligned} \tag{3.6}$$

Now, $-Z = \phi^2 Z = P^2 Z + tFZ + fFZ + FPZ$, implies that $fFZ = -FPZ$. Hence, from (3.5) and (3.6) we conclude $g(\nabla_V Y, Z) = \eta(Y)g(V,Z)$, since $\sin^2\theta \neq 0$.

So, if $V \in D \oplus \langle\xi\rangle$, we obtain $\nabla_V Y \perp D_\theta$, which implies $D \oplus \langle\xi\rangle$ is integrable and each of its leaves N_T is totally geodesic in M.

Next, if we consider $V \in D_\theta$, then $g(\nabla_V Z, Y) = -g(V,Z)g(\xi,Y)$, for all $Y \in D \oplus \langle\xi\rangle$. Hence, D_θ is integrable. Let us consider N_θ to be a leaf of D_θ and h^θ be the second fundamental form of the immersion of N_θ in M. Then we have $g(h^\theta(V,Z),Y) = g(\nabla_V Z, Y) = -g(V,Z)g(\xi,Y)$, for all $Y \in D \oplus \langle\xi\rangle$. Hence N_θ is totally umbilical in M with mean curvature vector ξ. Moreover, if $\overset{\perp}{\nabla}$ is the normal connection of the immersion N_θ in M, then $g(\overset{\perp}{\nabla}_Z\xi, Y) = g(\nabla_Z\xi, Y) = g(Z - \eta(Z)\xi, Y) = 0$, implying the mean curvature vector of N_θ is parallel. Thus the leaves of $D \oplus \langle\xi\rangle$ are totally geodesic and the leaves of D_θ are totally umbilical with parallel mean curvature vector. Hence by a result of Hiepko [14], M is a warped product of the type $M = N_T \times_f N_\theta$, ξ tangent to N_T, for some function f defined on N_T.

The converse part is obvious from Lemma 3.1. □

Now we provide an example of warped product semi-slant submanifold of a Kenmotsu manifold.

Example 3.1. Let $\bar{M}_1$, $\bar{M}_2$ be two Kähler manifolds. Then $\bar{M}_1 \times \bar{M}_2$ is a Kähler manifold. Let $N_\theta \subset \bar{M}_2$ be a slant submanifold. Note that, $\bar{M} = \mathbb{R} \times_f (\bar{M}_1 \times \bar{M}_2)$, $f = e^t$ is a Kenmotsu manifold.

Then $N_T = \mathbb{R} \times_f \bar{M}_1 \subset \bar{M}$ is an invariant submanifold of $\bar{M}$ and $N_\theta \subset \bar{M}$ is a slant submanifold of $\bar{M}$.

Therefore, $N_T \times_f N_\theta = \mathbb{R} \times_f (\bar{M}_1 \times N_\theta) \subset \bar{M}$ is a warped product semi-slant submanifold of the Kenmotsu manifold $\bar{M}$.

4 Inequality for warped products $N_T \times_f N$

In this section, we consider an n-dimensional warped product submanifold $M = N_T \times_f N$ of a $(2m+1)$-dimensional Kenmotsu manifold $\bar{M}$ such that ξ is tangent to N_T, where N_T is an invariant submanifold of $\bar{M}$ and N is a Riemannian submanifold of $\bar{M}$. Let the dimension of N_T be n_1 and the dimension of N be n_2, then $n = n_1 + n_2$. We consider the orthonormal frame $\{e_1, \cdots, e_{2m+1}\}$ of $T\bar{M}$ where $e_1, \cdots, e_s, e_{s+1} = \phi e_1, \cdots, e_{n_1-1} = \phi e_s, e_{n_1} = \xi$ are tangent to N_T, $e_{n_1+1}, \cdots, e_n = e_{n_1+n_2}$ are tangent to N, $e_{n+1} = Fe_{n_1+1}, \cdots, e_{n+n_2} = Fe_n$.

In the beginning of this section, we prove the following lemma for later use.

Lemma 4.1. *Let $M = N_T \times_f N$, with ξ tangent to N_T be a warped product submanifold in a Kenmotsu manifold $\bar{M}$, such that N_T is invariant in $\bar{M}$, and N is a Riemannian submanifold of $\bar{M}$. Then, for any $X, Y \in TN_T$, $Z \in TN$ and the normal vector $\zeta \in \nu$, the following holds:*
(i) $g(h(X,Y), FZ) = 0$,
(ii) $g(h(X,X),\zeta) = -g(h(\phi X,\phi X),\zeta)$.

Proof. We have,

$$\begin{aligned} g(h(X,Y),FZ) &= g(\bar{\nabla}_X Y, FZ) \\ &= g(\bar{\nabla}_X Y, \phi Z) - g(\bar{\nabla}_X Y, PZ). \end{aligned} \tag{4.1}$$

Since, $g(Y,PZ) = g(Y,\phi Z) = -g(\phi Y, Z) = 0$, for $Y \in TN_T$, $Z \in TN$, from the above equation (4.1) and the Kenmotsu structure equation (2.1) we obtain

$$\begin{aligned} g(h(X,Y),FZ) &= g(\phi Y, \bar{\nabla}_X Z) + g(Y, \bar{\nabla}_X PZ) \\ &= g(\phi Y, \nabla_X Z) + g(Y, \nabla_X PZ). \end{aligned} \tag{4.2}$$

Hence, using (1.1) in the above equation, we get the required result *(i)*. To prove the second part, we make use of (1.1) and (2) to obtain

$$\nabla_X \phi X + h(\phi X, X) - \phi \nabla_X X - \phi h(X,X) = -g(X,X)\xi + \eta(X)X.$$

Taking the inner product with $\phi\zeta$, we deduce

$$g(h(\phi X, X), \phi\zeta) = g(h(X,X),\zeta). \tag{4.3}$$

Interchanging X with ϕX in (4.3) and using (1.1), we obtain

$$g(h(\phi X,\phi X),\zeta) = -g(h(X,\phi X),\phi\zeta) + \eta(X) g(h(\xi,\phi X),\phi\zeta).$$

Since for a Kenmotsu manifold $h(\xi, \phi X) = 0$, hence we get

$$(h(\phi X,\phi X),\zeta) = -g(h(X,\phi X),\phi\zeta). \tag{4.4}$$

Thus the result follows from (4.3) and (4.4). □

Definition 4.1. [11]*An immersion $\varphi : N_1 \times_f N_2 \to \bar{M}$ is called N_i-totally geodesic if the partial second fundamental form h_i vanishes identically. It is called N_i-minimal if the partial mean curvature vector H_i vanishes, for $i = 1$ or 2.*

Theorem 4.2. *In a Kenmotsu manifold $\bar{M}$, every isometric immersion $\varphi : M = N_T \times_f N \longrightarrow \bar{M}$, with ξ tangent to N_T, is N_T-minimal, where N_T is an invariant submanifold of $\bar{M}$, and N is a Riemannian submanifold of $\bar{M}$.*

Proof. By definition, the squared norm of the mean curvature vector H restricted to N_T is given by

$$||H_1||^2 = \frac{1}{n_1^2} \sum_{r=n+1}^{2m+1} (h_{11}^r + \cdots + h_{n_1 n_1}^r)^2.$$

Using the frames of TN_T and TN and the fact that for a Kenmotsu manifold $h(\xi,\xi) = 0$, the above definition can be expanded as

$$\|H_1\|^2 = \frac{1}{n_1^2}\sum_{r=n+1}^{2m+1}(h^r_{11} + \cdots + h^r_{ss} + h^r_{s+1s+1} + \cdots + h^r_{2s2s})^2, \tag{4.5}$$

where

$$h^r_{ij} = g(h(e_i,e_j),e_r), 1 \leq i,j \leq n, n+1 \leq r \leq 2m+1.$$

Using Lemma 4.1, we obtain $\|H_1\|^2 = 0$. □

From the above proof, we obtain the following result.

Corollary 4.3. *Let φ be an isometric immersion $\varphi : M = N_T \times_f N \longrightarrow \bar{M}$, with ξ tangent to N_T, such that N_T is an invariant submanifold of $\bar{M}$, and N is a Riemannian submanifold of $\bar{M}$. Then, the squared mean curvature of M is*

$$\|H\|^2 = \frac{1}{n^2}\sum_{r=n+1}^{2m+1}(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2.$$

Now, we construct a general inequality for the warped product submanifold $M = N_T \times_f N$ in a Kenmotsu manifold $\bar{M}$ by applying Gauss equation and the preceding theory.

Theorem 4.4. *Let $\varphi : M = N_T \times_f N \longrightarrow \bar{M}$, be an isometric immersion of an n-dimensional warped product submanifold M into a $(2m+1)$-dimensional Kenmotsu manifold $\bar{M}$ such that N_T is an n_1-dimensional invariant submanifold tangent to ξ and N is an n_2-dimensional Riemannian submanifold of $\bar{M}$. Then, we have*

(i) $\frac{1}{2}\|h\|^2 \geq \bar{\tau}(TM) - \bar{\tau}(TN_T) - \bar{\tau}(TN) - \frac{n_2\Delta f}{f}$.

(ii) If the equality in (i) holds, then N_T and N are totally geodesic and totally umbilical submanifolds in $\bar{M}$, respectively.

Proof. Putting $X = W = e_i, Y = Z = e_j$ in the Gauss equation (2.2) and taking summation over $1 \leq i,j \leq n (i \neq j)$, we obtain

$$2\tau(TM) = 2\bar{\tau}(TM) + n^2\|H\|^2 - \|h\|^2. \tag{4.6}$$

In view of (2.8), we get

$$\|h\|^2 = -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i \wedge e_j) - 2\tau(TN_T) - 2\tau(TN) + 2\bar{\tau}(TM) + n^2\|H\|^2. \tag{4.7}$$

Again, using Gauss equation (2.2), we calculate

$$\tau(TN_T) = \bar{\tau}(TN_T) + \sum_{r=n+1}^{2m+1}\sum_{1\leq i<k\leq n_1}(h^r_{ii}h^r_{kk} - (h^r_{ik})^2), \tag{4.8}$$

and

$$\tau(TN) = \bar{\tau}(TN) + \sum_{r=n+1}^{2m+1}\sum_{n_1+1\leq j<t\leq n}(h^r_{jj}h^r_{tt} - (h^r_{jt})^2). \tag{4.9}$$

In view of (4.8) and (4.9), the equation (4.7) transforms into

$$
\begin{aligned}
||h||^2 = -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i\wedge e_j) - 2\bar{\tau}(TN_T) - 2\bar{\tau}(TN) + 2\bar{\tau}(TM)\\
-2\sum_{r=n+1}^{2m+1}\sum_{1\le i<k\le n_1}(h^r_{ii}h^r_{kk} - (h^r_{ik})^2) + n^2||H||^2\\
-2\sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j<t\le n}(h^r_{jj}h^r_{tt} - (h^r_{jt})^2),
\end{aligned}
$$

which is equivalent to the following form

$$
\begin{aligned}
||h||^2 \;=\; & -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i\wedge e_j) - 2\bar{\tau}(TN_T) - 2\bar{\tau}(TN) + 2\bar{\tau}(TM)\\
& -\sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}(h^r_{ii}h^r_{kk} - (h^r_{ik})^2) + n^2||H||^2\\
& -\sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j\neq t\le n}(h^r_{jj}h^r_{tt} - (h^r_{jt})^2).
\end{aligned}
\tag{4.10}
$$

Since φ is an N_T-minimal immersion, we have

$$
\sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}(h^r_{ii}h^r_{kk} - (h^r_{ik})^2) = -\sum_{r=n+1}^{2m+1}\sum_{i,k=1}^{n_1}(h^r_{ik})^2.
$$

Hence, (4.10) takes the following form:

$$
\begin{aligned}
||h||^2 = & -2\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i\wedge e_j) - 2\bar{\tau}(TN_T) - 2\bar{\tau}(TN) + 2\bar{\tau}(TM)\\
& +\sum_{r=n+1}^{2m+1}\sum_{1\le i\neq k\le n_1}(h^r_{ik})^2 + n^2||H||^2\\
& - n^2\|H\|^2 + \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j\neq t\le n}(h^r_{jt})^2.
\end{aligned}
\tag{4.11}
$$

Next, we use the following formula for general warped products [4]

$$
\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} K(e_i\wedge e_j) = \frac{n_2\Delta f}{f}.
\tag{4.12}
$$

Then from (4.11) and (4.12), the inequality (i) follows immediately.

Now, if the equality holds in (i), then we must have $h(X,Y)=0$, for both $X,Y\in TN_T$, and for both $X,Y\in TN$. Hence, the immersion $N_T\to M$ is totally geodesic and the immersion $N\to M$ is totally umbilical. $\square$

For a warped product semi-slant submanifold $M = N_T \times_f N_\theta \longrightarrow \bar{M}$, we have $\vec{\nabla} lnf = \xi$ by Lemma 3.1. Further, since $\frac{\Delta f}{f} = \Delta lnf - g(\vec{\nabla} lnf, \vec{\nabla} lnf)$, we calculate

$$\begin{aligned} \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} K(e_i \wedge e_j) &= n_2 \Delta lnf - n_2\, g(\vec{\nabla} lnf, \vec{\nabla} lnf) \\ &= -n_2 \sum_{i=1}^{n_1} g(\nabla_{e_i}\xi, e_i) - n_2 g(\xi, \xi) \\ &= -n_1 n_2. \end{aligned} \tag{4.13}$$

Hence we obtain the following:

Corollary 4.5. *Let $\varphi : M = N_T \times_f N_\theta \longrightarrow \bar{M}$, be an isometric immersion of an n-dimensional warped product semi-slant submanifold M into a $(2m+1)$-dimensional Kenmotsu manifold $\bar{M}$ such that N_T is an n_1-dimensional invariant submanifold tangent to ξ and N_θ is an n_2-dimensional proper slant submanifold of $\bar{M}$. Then, we have*

(i) $\frac{1}{2}||h||^2 \geq \bar{\tau}(TM) - \bar{\tau}(TN_T) - \bar{\tau}(TN) + n_1 n_2$.

(ii) If the equality in (i) holds, then N_T and N are totally geodesic and totally umbilical submanifolds in $\bar{M}$, respectively.

Now, we can derive the following relation for a Kenmotsu space form:

Theorem 4.6. *Let $\varphi : M = N_T \times_f N \longrightarrow \bar{M}(c)$ be an isometric immersion from a warped product submanifold M into a Kenmotsu space form $\bar{M}(c)$ with constant ϕ-sectional curvature c where N_T is an n_1-dimensional invariant submanifold tangent to ξ and N is an n_2-dimensional Riemannian submanifold of $\bar{M}(c)$. Then, we have*

(i) $||h||^2 \geq \frac{(c-3)}{2} n_1 n_2 + \frac{(c+1)}{2} n_2 - \frac{2n_2 \Delta f}{f}$,

(ii) If the equality in (i) holds, then N_T and N are totally geodesic and totally umbilical submanifolds in $\bar{M}$, respectively.

Proof. Putting $X = W = e_i, Y = Z = e_j$ in the curvature equation (2.7) for Kenmotsu space form, we obtain

$$\begin{aligned} 2\bar{\tau}(TN_T) &= \frac{(c-3)}{4} \sum_{1\leq i\neq j\leq n_1} \{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_j, e_i)\} \\ &- \frac{(c+1)}{4}\{\eta(e_{n_1}) \sum_{1\leq i\leq n_1} [\eta(e_{n_1})g(e_i, e_i) - \eta(e_i)g(e_{n_1}, e_i)] \\ &+ \eta(e_{n_1}) \sum_{1\leq j\leq n_1} [g(e_j, e_j)\eta(e_{n_1}) - g(e_{n_1}, e_j)\eta(e_j)] \\ &- \sum_{1\leq i\neq j\leq n_1} g(\phi e_i, e_i)g(\phi e_j, e_j) + 3 \sum_{1\leq i\neq j\leq n_1} g(\phi e_i, e_j)g(\phi e_j, e_i)\} \\ &= \frac{(c-3)}{4} n_1(n_1 - 1) + \frac{(c+1)}{4}(n_1 - 1). \end{aligned} \tag{4.14}$$

Similarly, we obtain

$$2\bar{\tau}(TN) = \frac{(c-3)}{4} n_2(n_2 - 1) + 3\frac{(c+1)}{4} \sum_{n_1+1\leq \alpha\neq\beta\leq n} g(\phi e_\alpha, e_\beta)^2. \tag{4.15}$$

and

$$2\bar{\tau}(TM) = \frac{(c-3)}{4}n(n-1) + \frac{(c+1)}{4}\left[(n-1) + 3\sum_{n_1+1\leq\alpha\neq\beta\leq n} g(\phi e_\alpha, e_\beta)^2\right]. \tag{4.16}$$

Therefore, using Theorem 4.4, we obtain the required results. □

We have the following consequence of the above theorem.

Corollary 4.7. *Let $\varphi : M = N_T \times_f N_\theta \longrightarrow \bar{M}(c)$ be an isometric immersion from a warped product submanifold M into a Kenmotsu space form $\bar{M}(c)$ with constant ϕ-sectional curvature c where N_T is an n_1-dimensional invariant submanifold tangent to ξ and N_θ is an n_2-dimensional proper slant submanifold of $\bar{M}(c)$. Then, we have*

(i) $||h||^2 \geq \frac{(c+1)}{2}n_2(n_1+1)$,

(ii) If the equality in (i) holds, then N_T and N are totally geodesic and totally umbilical submanifolds in $\bar{M}$, respectively.

References

[1] K. Arsalan, E. Ridvan I. Mihai and C. Murathan, *Contact CR-warped product submanifolds in Kenmotsu space forms,* J. Korean Math. Soc. 42 (2005), 1101-1110.

[2] M. Atceken, *Warped product semi-slant submanifolds in Kenmotsu manifolds*, Turk. J. Math., 34 (2010), 425-433.

[3] T.Q. Binh, L. Tamassy, U. C. De and M. Tarafder, Some Remarks on almost Kenmotsu manifolds, Math Pannon. 13(2002), 31-39.

[4] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature,* Trans. Amer. Math. Soc. 145 (1969), 1-49.

[5] D.E. Blair, *Contact manifolds in Riemannian geometry,* Lecture Notes in Math. 509, Springer Verlag, Berlin, 1976.

[6] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds,* Glasgow Math. J., 42 (2000), 125-138.

[7] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata 78 (1999), 183-199.

[8] B.Y. Chen, *Geometry of submanifolds*, Pure and Applied Mathematics, No. 22. Marcel Dekker, Inc. New York, 1973.

[9] B.Y. Chen, *Slant immersions*, Bull. Austral. Math. Soc., 41 (1990), 135-147.

[10] B.Y. Chen, *Pseudo-Riemannian geometry, delta-Invariants and applications*, World Scientific press, Singapore (2011).

[11] B.Y. Chen, *On isometric minimal immersions from warped products into real space forms,* Proc. Edinb. Math. Soc., 45 (2002), 579-587.

[12] B.Y. Chen, *Geometry of warped products as Riemannian submanifolds and related problems,* Soochow J. Math., 28 (2002), 125-156.

[13] A. De, *On Kenmotsu manifold*, Bulletin of Mathematical Analysis and Applications, 3 (2010), 1-6.

[14] S. Hiepko, *Eine inner kennzeichungder verzerrten produkte*, Math. Ann. 241 (1979), 209-215.

[15] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24 (1972), 93-103.

[16] K.A. Khan, V.A. Khan and S. Uddin, *Warped product submanifolds of cosymplectic manifolds*, Balkan J. Geom. Appl., 13 (2008), 55-65.

[17] M.I. Munteanu, *Warped product contact CR-submanifolds of Sasakian space forms,* Publ. Math. Debrecen, 66 (2005), 75-120.

[18] C. Murathan, K. Arslan, R. Ezentas and I. Mihai, *Warped product submanifolds in Kenmotsu spaceforms*, Taiwanese J. of Mathematics, 10 (2006), 1431-1441.

[19] C. Ozgur, *On weakly symmetric Kenmotsu manifolds*, Differ. Geom. Dyn. Syst., 8 (2006), 204-209.

[20] C. Ozgur, *On generalized recurrent Kenmotsu manifolds*, World Applied Sciences Journal, (2007), 29-33.

[21] N. Papaghuic, *Semi-slant submanifolds of a Kählerian manifold*, An. Stiint. Al. I. Cuza. Univ. Iasi, 40 (1994), 55-61.

[22] B. Sahin, *Non-existence of warped product semi-slant submanifolds of Kähler manifold*, Geom. Dedicata, 117 (2006), 195-202.

[23] S. Uddin, V.A. Khan and H.H. Khan, *Some results on warped product submanifolds of a Sasakian manifold*, Int. J. Math. Math. Sci., (2010), Article ID 743074, 9 pages.

[24] S. Uddin V.A. Khan and K.A. Khan, *Warped product submanifolds of a Kenmotsu manifold,* Turk. J. Math., 36 (2012) 319-330.

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