Propagation of inhomogeneous waves in monoclinic crystals subject to initial fields

Olivian Simionescu-Panait, Livia Harabagiu

Abstract. In this paper we investigate the conditions of inhomogeneous plane wave propagation in monoclinic crystals subject to initial electromechanical fields. We obtain here the components of the electroacoustic tensor for the class 2, resp. m, of the monoclinic system. For particular isotropic directional bivectors we derive the decomposition of the propagation condition, and we show that the specific coefficients are similar to the case of guided waves propagation in monoclinic crystals subject to a bias. We analyze the important particular case of polar anisotropic directional bivectors and we obtain a similar decomposition of the propagation condition, with specific coefficients.

M.S.C. 2010: 74J05, 74B15, 74E10, 74F15.
Key words: Inhomogeneous plane waves; monoclinic crystals; initial electromechanical fields; isotropic/anisotropic directional bivectors.

1 Introduction

Inhomogeneous plane waves arise in many areas of mechanics of continua, including Rayleigh, Love and Stoneley waves from the linear elasticity theory, TE and TM waves from electromagnetism, or viscoelastic waves. These are waves for which an attenuation of the amplitude occurs in a direction distinct from the direction of propagation. They are elliptically polarized plane waves, which generalize the classical homogeneous plane waves. Important examples using this concept may be found in papers [1, 2] for anisotropic elasticity, in [3] for electromagnetism, in [7] for viscoelasticity, or in [5, 6] for wave propagation in porous materials. The use of complex vectors (called *bivectors*) leads to a direct formulation of the condition of propagation for this kind of waves. The concept of bivector is due to W.R. Hamilton (see work [9]). The algebra of bivectors is well established (see, for example, the works [4, 8, 10, 20]).

In the present paper we derive the conditions of propagation for inhomogeneous plane waves in monoclinic crystals subject to initial electromechanical fields. We obtain the components of the electroacoustic tensor for the class 2, resp. m, of the

Balkan Journal of Geometry and Its Applications, Vol.20, No.1, 2015, pp. 109-119.

monoclinic system. In the particular case of isotropic directional bivectors we derive the decomposition of the propagation condition, and we show that the specific coefficients are similar to the case of guided waves propagation in monoclinic crystals subject to a bias (see papers [18, 19], to compare). It is obvious that the present particular inhomogenous plane wave generalizes the homogenous plane wave. Moreover, we study here the "free" wave propagation, while in [18, 19] we analyzed the guided waves case.

Furthermore, we analyze the important particular case of polar anisotropic directional bivectors and we find a similar decomposition of the propagation condition, with specific coefficients. These results generalize previous solutions concerning attenuated waves propagation in crystals subject to a electromechanical bias (see papers [12]-[14]). Some of the present results were conjectured in the abstract paper [17].

2 Basic equations. Condition of propagation

We review here the basic equations for the problems concerning the waves propagation in anisotropic crystals subject to initial electromechanical fields (see chapter [15], or paper [16], for details). For this problem we suppose to be in the case of incremental dynamic electromechanical fields superposed on large initial static electromechanical fields.

To describe this situation, we use here three different configurations : the *reference configuration* B_R in which, at time $t = 0$, the body is undeformed and free of all fields; the *initial configuration* $\overset{\circ}{B}$ in which the body is statically deformed and carries the initial fields; the *present (current) configuration* B_t obtained from $\overset{\circ}{B}$ by applying time dependent incremental deformations and fields. In what follows, all the fields related to the initial configuration $\overset{\circ}{B}$ will be denoted by a superposed "$\circ$".

We assume the material to be an elastic dielectric, which is non-magnetizable and conducts neither heat, nor electricity. Consequently, we shall use the quasi-electrostatic approximation of the equations of balance. Furthermore, we assume that the elastic dielectric is linear and homogeneous, and that the initial homogeneous deformations and electric fields are static and large.

Under these hypotheses *the homogeneous field equations* take the following form:

$$
\begin{aligned}
&\overset{\circ}{\rho}\, \ddot{\boldsymbol{u}} = \operatorname{div} \boldsymbol{\Sigma}, \ \operatorname{div} \boldsymbol{\Delta} = 0 \\
&\operatorname{rot} \boldsymbol{e} = 0 \ \Leftrightarrow \ \boldsymbol{e} = -\operatorname{grad} \varphi,
\end{aligned} \tag{2.1}
$$

where $\overset{\circ}{\rho}$ is the mass density, $\boldsymbol{u}$ is the incremental displacement, $\boldsymbol{\Sigma}$ is the incremental electromechanical nominal stress tensor, $\boldsymbol{\Delta}$ is the incremental electric displacement vector, $\boldsymbol{e}$ is the incremental electric field and φ is the incremental electric potential. All incremental fields involved into the above equations depend on the spatial variable $\boldsymbol{x}$ and on time t.

We suppose the following *incremental constitutive equations*:

$$\Sigma_{kl} = \overset{\circ}{\Omega}_{klmn}\, u_{m,n} + \overset{\circ}{\Lambda}_{mkl}\, \varphi_{,\,m}$$
(2.2)
$$\Delta_k = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} + \overset{\circ}{\epsilon}_{kl}\, e_l = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} - \overset{\circ}{\epsilon}_{kl}\, \varphi_{,\,l}.$$

In these equations $\overset{\circ}{\Omega}_{klmn}$ are the components of the instantaneous elasticity tensor, $\overset{\circ}{\Lambda}_{kmn}$ are the components of the instantaneous coupling tensor and $\overset{\circ}{\epsilon}_{kl}$ are the components of the instantaneous dielectric tensor. The instantaneous coefficients can be expressed in terms of the classical moduli of the material and on the initial applied fields as follows

$$\overset{\circ}{\Omega}_{klmn} = c_{klmn} + \overset{\circ}{S}_{kn}\, \delta_{lm} - e_{kmn}\, \overset{\circ}{E}_l - e_{nkl}\, \overset{\circ}{E}_m - \eta_{kn}\, \overset{\circ}{E}_l \overset{\circ}{E}_m,$$
(2.3)
$$\overset{\circ}{\Lambda}_{mkl} = e_{mkl} + \eta_{mk}\, \overset{\circ}{E}_l, \quad \overset{\circ}{\epsilon}_{kl} = \delta_{kl} + \eta_{kl},$$

where c_{klmn} are the components of the constant elasticity tensor, e_{kmn} are the components of the constant piezoelectric tensor, $\overset{\circ}{E}_i$ are the components of the initial applied electric field and $\overset{\circ}{S}_{kn}$ are the components of the initial applied symmetric (Cauchy) stress tensor.

From the relations (2.3) we find the symmetry relations

$$\overset{\circ}{\Omega}_{klmn} = \overset{\circ}{\Omega}_{nmlk}, \quad \overset{\circ}{\epsilon}_{kl} = \overset{\circ}{\epsilon}_{lk}\ . \tag{2.4}$$

Moreover, we see that $\overset{\circ}{\Omega}_{klmn}$ is not symmetric according to indices (k,l) and (m,n) and $\overset{\circ}{\Lambda}_{mkl}$ is not symmetric relative to indices (k,l).

Theorem 2.1. *Summing up, from the previous field and constitutive equations we obtain the following* fundamental system of equations*:*

$$\overset{\circ}{\rho}\, \ddot{u}_l = \overset{\circ}{\Omega}_{klmn}\, u_{m,nk} + \overset{\circ}{\Lambda}_{mkl}\, \varphi_{,mk}, \qquad \overset{\circ}{\Lambda}_{kmn}\, u_{n,mk} - \overset{\circ}{\epsilon}_{kn}\, \varphi_{,nk} = 0, \quad l = \overline{1,3}. \tag{2.5}$$

In this context, for our electromechanical problem, we define the *inhomogeneous plane wave* by:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a} \exp[\mathrm{i}\omega(\mathbf{s}\cdot\mathbf{x} - t)], \quad \varphi(\mathbf{x}, t) = a_4 \exp[\mathrm{i}\omega(\mathbf{s}\cdot\mathbf{x} - t)]. \tag{2.6}$$

Here $\mathbf{a} = \mathbf{a}^+ + \mathrm{i}\mathbf{a}^-$ is a complex vector defining the mechanical amplitude bivector, a_4 is the electric amplitude of the wave, and $\mathbf{s} = \mathbf{s}^+ + \mathrm{i}\mathbf{s}^-$ is a complex vector denoting the slowness bivector. In relations (2.6) ω defines the frequency of the wave, which is a real parameter. We suppose that this kind of wave propagates in an unbounded domain.

The real part of $\mathbf{u}$ is

$$\mathbf{u}^+ = [\mathbf{a}^+ \cos\omega(\mathbf{s}^+\cdot\mathbf{x} - t) - \mathbf{a}^- \sin\omega(\mathbf{s}^+\cdot\mathbf{x} - t)] \exp(-\omega\, \mathbf{s}^-\cdot\mathbf{x}).$$

The planes $\mathbf{s}^+ \cdot \mathbf{x} =$ constant are planes of constant phase, while $\mathbf{s}^- \cdot \mathbf{x} =$ constant are planes of constant amplitude. The previous relations represent a train of elliptically polarized plane waves. The waves travel in the direction of the vector $\mathbf{s}^+$, with the slowness $|\mathbf{s}^+|$, and are attenuated in the direction of the vector $\mathbf{s}^-$. The period is $2\pi/\omega$. For any fixed position vector $\mathbf{x}$, the displacement vector $\mathbf{u}^+$ describes an ellipse similar to the ellipse defined by the bivector $\mathbf{a}$ (see works [4, 10], for details).

A solution in the form (2.6) defines an *inhomogeneous plane wave* if the vector $\mathbf{s}^-$ is not parallel to the vector $\mathbf{s}^+$. We see that in the case of inhomogeneous plane waves the planes of constant phase are different from the planes of constant amplitude. In the particular case, when $\mathbf{s}^-$ is parallel to $\mathbf{s}^+$, we have an *attenuated homogeneous plane wave*. The phase speed is given by $V = |\mathbf{s}^+|^{-1}$, while $|\mathbf{s}^-|$ defines the attenuation coefficient.

In order to solve the problem of inhomogeneous plane wave propagation in the described material, we use the *directional ellipse method*, due to M. Hayes (see paper [10]). The slowness bivector is written in the form $\mathbf{s} = N\mathbf{C}$, where the directional bivector $\mathbf{C}$ has the form $\mathbf{C} = q\mathbf{m} + i\mathbf{n}$, with $\mathbf{m} \cdot \mathbf{n} = 0$, $|\mathbf{m}| = |\mathbf{n}| = 1$, and $q \geq 1$. N is called complex scalar slowness. Once the directional bivector $\mathbf{C}$ is prescribed, then the slowness $\mathbf{s}$, as well as the amplitudes $\mathbf{a}$ and a_4, are determined from the equations of motion. Thus, the main unknown of the inhomogeneous plane wave propagation problem is the complex scalar slowness N. A bivector $\mathbf{C}$ is said to be *isotropic* if $\mathbf{C} \cdot \mathbf{C} = 0$.

Inserting the relations (2.6) into the fundamental equations (2.5), we derive the *condition of propagation of inhomogeneous plane waves* in previously defined materials:

$$\begin{pmatrix} \overset{\circ}{Q}_{lm} & \overset{\circ}{Q}_{l4} \\ \overset{\circ}{Q}_{4m} & \overset{\circ}{Q}_{44} \end{pmatrix} \begin{pmatrix} a_m \\ a_4 \end{pmatrix} = 0, \quad l, m = \overline{1,3}. \tag{2.7}$$

Here the components of the electroacoustic tensor $\mathbf{Q}$ have the form:

$$\begin{gathered} \overset{\circ}{Q}_{lm} = N^2 \overset{\circ}{\Omega}_{klmn} C_k C_n - \overset{\circ}{\rho} \delta_{lm}, \quad \overset{\circ}{Q}_{l4} = N^2 \overset{\circ}{\Lambda}_{mkl} C_m C_k, \\ \overset{\circ}{Q}_{4m} = N^2 \overset{\circ}{\Lambda}_{klm} C_k C_l, \quad \overset{\circ}{Q}_{44} = -N^2 \overset{\circ}{\epsilon}_{kn} C_k C_n. \end{gathered} \tag{2.8}$$

Due to symmetry relations $\overset{\circ}{\Omega}_{klmn} = \overset{\circ}{\Omega}_{nmlk}$ and to the definition (2.8) of the electroacoustic tensor components it yields that the tensor $\mathbf{Q}$ is symmetric for the general anisotropy. In particular, it is symmetric in the monoclinic system.

3 Inhomogeneous plane waves propagation in monoclinic crystals

3.1 Direct dyad axis

In this case we suppose that x_3 is a direct dyad axis. Now, we are in the class 2 of the monoclinic system ($A_2 || x_3$). Then, following [11], the elastic constants with one index equal to 3 are zero and the piezoelectric constants with no index equal to 3 are zero.

Under this hypothesis, after a long, but elementary calculus, we find that the electroacoustic tensor **Q** is a symmetric tensor with complex components (see Appendix 1). Consequently, the inhomogeneous plane waves (2.6) may propagate in any direction, in a prestressed and prepolarized crystal from the class 2 of the monoclinic system.

On the other hand, after a short inspection the the components of the electroacoustic tensor, one can easily observe that, even if the initial fields are absent, the corresponding tensor has no zero components, for a general directional bivector **C**. Thus, we have no decomposition of the condition of propagation (2.7) in the general case (see Appendix 1 for the analysis of the components of the electroacoustic tensor).

3.1.1 Particular case: isotropic directional bivector

If we consider the particular case of *isotropic directional bivectors*, we may choose $\mathbf{C} = \mathbf{i} + \mathrm{i}\mathbf{j}$, where $\{\mathbf{i},\ \mathbf{j},\ \mathbf{k}\}$ represents an orthonormal basis of the three dimensional Euclidian space and i is the complex unit. Here the inhomogeneous wave is circularly polarized in a plane normal to the dyad axis x_3. Moreover, if $\overset{\circ}{E}_1=\overset{\circ}{E}_2= 0$, we obtain $\overset{\circ}{Q}_{14}=\overset{\circ}{Q}_{24}= 0$, respectively $\overset{\circ}{Q}_{13}=\overset{\circ}{Q}_{23}= 0$.

Then, the system (2.7) reduces to two independent subsystems, with the coefficients similar to those from the problem of guided wave propagation in monoclinic crystals (see papers [18, 19]). It is obvious that the present particular inhomogenous plane wave generalizes the homogenous guided wave.

- The first subsystem

$$\begin{pmatrix} \overset{\circ}{Q}_{11} & \overset{\circ}{Q}_{12} \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0. \tag{3.1}$$

defines a non-piezoelectric wave, polarized in the plane x_1x_2, which depends on the initial stress field, only. It corresponds to $\overset{\circ}{P}_2$ guided wave. These characteristics are due to the form of the involved coefficients:

$$\begin{gathered} \overset{\circ}{Q}_{11}= N^2[c_{11}+ \overset{\circ}{S}_{11} +2\mathrm{i}(c_{16}+ \overset{\circ}{S}_{12}) - (c_{66}+ \overset{\circ}{S}_{22})]- \overset{\circ}{\rho}, \\ \overset{\circ}{Q}_{12}=\overset{\circ}{Q}_{21}= N^2[c_{16} + \mathrm{i}(c_{12} + c_{66}) - c_{26}], \\ \overset{\circ}{Q}_{22}= N^2[c_{66}+ \overset{\circ}{S}_{11} +2\mathrm{i}(c_{26}+ \overset{\circ}{S}_{12}) - (c_{22}+ \overset{\circ}{S}_{22})]- \overset{\circ}{\rho}\ . \end{gathered} \tag{3.2}$$

- The second subsystem is

$$\begin{pmatrix} \overset{\circ}{Q}_{33} & \overset{\circ}{Q}_{34} \\ \overset{\circ}{Q}_{43} & \overset{\circ}{Q}_{44} \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = 0. \tag{3.3}$$

has as solution a transverse-horizontal wave, with polarization after the axis x_3, which is piezoelectric and electrostrictive active, and depends on the initial mechanical and electrical fields. This wave is linked with $\overline{\overset{\circ}{TH}}$ guided wave. The components involved into this equation have the form

$$
\begin{aligned}
&\overset{\circ}{Q}_{33}= N^2\{[c_{55}+ \overset{\circ}{S}_{11} +2\mathrm{i}(c_{45}+ \overset{\circ}{S}_{12}) - (c_{44}+ \overset{\circ}{S}_{22}) \\
&-2[e_{15} + \mathrm{i}(e_{14} + e_{25}) - e_{24}]\, \overset{\circ}{E}_3 -(\eta_{11} + 2\mathrm{i}\eta_{12} - \eta_{22})\, \overset{\circ}{E}{}_3^2\}- \overset{\circ}{\rho}, \\
&\overset{\circ}{Q}_{34}=\overset{\circ}{Q}_{43}= N^2[e_{15} + \mathrm{i}(e_{14} + e_{25}) - e_{24} + (\eta_{11} + 2\mathrm{i}\eta_{12} - \eta_{22})\, \overset{\circ}{E}_3], \\
&\overset{\circ}{Q}_{44}= -N^2(\eta_{11} + 2\mathrm{i}\eta_{12} - \eta_{22}).
\end{aligned}
\tag{3.4}
$$

3.1.2 Particular case: anisotropic directional bivector

The second particular case deals with the *anisotropic directional bivector* $\mathbf{C} = (C_1, C_2, 0)$, with

$$C_1 = \cos\alpha + \mathrm{i}\sin\alpha, \quad C_2 = \cos\alpha - \mathrm{i}\sin\alpha,\ \alpha \in [0, 2\pi).$$

This inhomogeneous wave is elliptically polarized in the plane normal to the dyad axis x_3, except the particular directions $\alpha \in \{\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4\}$, where it is circularly polarized.

If $\overset{\circ}{E}_1=\overset{\circ}{E}_2= 0$, we obtain $\overset{\circ}{Q}_{14}=\overset{\circ}{Q}_{24}= 0$ and $\overset{\circ}{Q}_{13}=\overset{\circ}{Q}_{23}= 0$. Then, the system (2.7) reduces to two independent subsystems, as follows:

- The first subsystem

$$
\begin{pmatrix} \overset{\circ}{Q}_{11} & \overset{\circ}{Q}_{12} \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.
\tag{3.5}
$$

defines a non-piezoelectric wave, polarized in the plane x_1x_2, which depends on the initial stress field, only. These characteristics are due to the form of the coefficients:

$$
\begin{aligned}
&\overset{\circ}{Q}_{11}= N^2[(c_{11}+ \overset{\circ}{S}_{11})(\cos 2\alpha + \mathrm{i}\sin 2\alpha) + 2(c_{16}+ \overset{\circ}{S}_{12}) + (c_{66}+ \overset{\circ}{S}_{22})(\cos 2\alpha - \mathrm{i}\sin 2\alpha)]- \overset{\circ}{\rho}, \\
&\overset{\circ}{Q}_{12}=\overset{\circ}{Q}_{21}= N^2[c_{16}(\cos 2\alpha + \mathrm{i}\sin 2\alpha) + (c_{12} + c_{66}) + c_{26}(\cos 2\alpha - \mathrm{i}\sin 2\alpha)], \\
&\overset{\circ}{Q}_{22}= N^2[(c_{66}+ \overset{\circ}{S}_{11})(\cos 2\alpha + \mathrm{i}\sin 2\alpha) + 2(c_{26}+ \overset{\circ}{S}_{12}) + (c_{22}+ \overset{\circ}{S}_{22})(\cos 2\alpha - \mathrm{i}\sin 2\alpha)]- \overset{\circ}{\rho}\,.
\end{aligned}
\tag{3.6}
$$

- The second subsystem

$$
\begin{pmatrix} \overset{\circ}{Q}_{33} & \overset{\circ}{Q}_{34} \\ \overset{\circ}{Q}_{43} & \overset{\circ}{Q}_{44} \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = 0.
\tag{3.7}
$$

has as solution a transverse-horizontal wave, with polarization after the axis x_3, which is piezoelectric and electrostrictive active, and depends on the initial mechanical and electrical fields. The components involved into this equation have the form

$$
\begin{aligned}
\overset{\circ}{Q}_{33} = {} & N^2\{(c_{55}+ \overset{\circ}{S}_{11} -2e_{15} \overset{\circ}{E}_3 -\eta_{11} \overset{\circ}{E}{}_3^2)(\cos 2\alpha + \mathrm{i}\sin 2\alpha) \\
& +2[c_{45}+ \overset{\circ}{S}_{12} -(e_{14}+e_{25}) \overset{\circ}{E}_3 -\eta_{12} \overset{\circ}{E}{}_3^2] \\
& +(c_{44}+ \overset{\circ}{S}_{22} -2e_{24} \overset{\circ}{E}_3 -\eta_{22} \overset{\circ}{E}{}_3^2)(\cos 2\alpha - \mathrm{i}\sin 2\alpha)\}- \overset{\circ}{\rho}, \\
\overset{\circ}{Q}_{34} = \overset{\circ}{Q}_{43} = {} & N^2[(e_{15}+\eta_{11} \overset{\circ}{E}_3)(\cos 2\alpha + \mathrm{i}\sin 2\alpha) \\
& +(e_{14}+e_{25}+2\eta_{12} \overset{\circ}{E}_3) + (e_{24}+\eta_{22} \overset{\circ}{E}_3)(\cos 2\alpha - \mathrm{i}\sin 2\alpha)], \\
\overset{\circ}{Q}_{44} = {} & -N^2[(1+\eta_{11})(\cos 2\alpha + \mathrm{i}\sin 2\alpha) + (1+\eta_{22})(\cos 2\alpha - \mathrm{i}\sin 2\alpha) + 2\eta_{12}].
\end{aligned}
\tag{3.8}
$$

3.2 Inverse dyad axis (mirror plane)

We suppose now that the plane x_1x_2 is normal to an inverse dyad axis (x_3 in our case) or, equivalently, that the plane x_1x_2 is parallel to a mirror plane M. It follows that the crystal belongs to the class m of the monoclinic system ($M \perp x_3$). In this particular case the elastic constants with one index equal to 3 are zero, as well as the piezoelectric constants with one index equal to 3, which vanish (see [11]).

As in the previous case, an elementary calculus shows us that the electroacoustic tensor $\mathbf{Q}$ is a symmetric tensor with complex components. Consequently, the inhomogeneous plane waves (2.6) may propagate in any direction in a prestressed and prepolarized crystal from the class m of the monoclinic system.

On the other hand, from a short inspection the the components of the electroacoustic tensor, one can easily observe that, even if the initial fields are absent, the corresponding tensor has no zero components, for a general directional bivector $\mathbf{C}$. Thus, we have no decomposition of the condition of propagation (2.7) (see Appendix 2 for the analysis of the components of the electroacoustic tensor).

3.2.1 Particular case: isotropic directional bivector

If we consider the particular case of *isotropic directional bivectors*, we can choose $\mathbf{C} = \mathbf{i} + \mathrm{i}\mathbf{j}$, where $\{\mathbf{i},\ \mathbf{j},\ \mathbf{k}\}$ represents an orthonormal basis of the three dimensional Euclidian space and i is the complex unit. Now, the inhomogeneous wave is circularly polarized in a plane normal to the inverse dyad axis x_3. Moreover, if $\overset{\circ}{E}_3 = 0$, we obtain $\overset{\circ}{Q}_{34} = 0$, respectively $\overset{\circ}{Q}_{13} = \overset{\circ}{Q}_{23} = 0$.

Then, the system (2.7) reduces to two independent subsystems, with the coefficients similar to the case of guided wave propagation in monoclinic crystals (see papers [18, 19]). Obviously, this particular inhomogenous plane wave generalizes the homogenous guided wave.

• The first subsystem has the form

$$\begin{pmatrix} \overset{\circ}{Q}_{11} & \overset{\circ}{Q}_{12} & \overset{\circ}{Q}_{14} \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} & \overset{\circ}{Q}_{24} \\ \overset{\circ}{Q}_{41} & \overset{\circ}{Q}_{42} & \overset{\circ}{Q}_{44} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_4 \end{pmatrix} = 0. \tag{3.9}$$

It has as solution a inhomogeneous plane wave, polarized into the plane x_1x_2, associated with the electric field (*via* the amplitude a_4 of the electric potential φ), providing piezoelectric and electrostrictive effects, and depending on the initial stress and electric fields. It corresponds to $\overset{\circ}{\overline{P}}_2$ wave form the problem of guided wave propagation. These features of the wave were obtained from the analysis of the corresponding coefficients

(3.10)

$$\begin{gathered} \overset{\circ}{Q}_{11} = N^2[(c_{11} + \overset{\circ}{S}_{11} - 2e_{11}\overset{\circ}{E}_1 - \eta_{11}\overset{\circ}{E}{}_1^2) + 2\mathrm{i}(c_{16} + \overset{\circ}{S}_{12} - e_{16}\overset{\circ}{E}_1 - e_{21}\overset{\circ}{E}_1 \\ - \eta_{12}\overset{\circ}{E}{}_1^2) - (c_{66} + \overset{\circ}{S}_{22} - 2e_{26}\overset{\circ}{E}_1 - \eta_{22}\overset{\circ}{E}{}_1^2)] - \overset{\circ}{\rho}, \end{gathered}$$

$$\begin{gathered} \overset{\circ}{Q}_{12} = \overset{\circ}{Q}_{21} = N^2\{(c_{16} - e_{16}\overset{\circ}{E}_1 - e_{11}\overset{\circ}{E}_2 - \eta_{11}\overset{\circ}{E}_1\overset{\circ}{E}_2) + \mathrm{i}[c_{12} + c_{66} - (e_{12} + e_{26})\overset{\circ}{E}_1 \\ - (e_{21} + e_{16})\overset{\circ}{E}_2 - 2\eta_{12}\overset{\circ}{E}_1\overset{\circ}{E}_2] - (c_{26} - e_{22}\overset{\circ}{E}_1 - e_{26}\overset{\circ}{E}_2 - \eta_{22}\overset{\circ}{E}_1\overset{\circ}{E}_2)\}, \end{gathered}$$

$$\begin{gathered} \overset{\circ}{Q}_{22} = N^2\{(c_{66} + \overset{\circ}{S}_{11} - 2e_{16}\overset{\circ}{E}_2 - \eta_{11}\overset{\circ}{E}{}_2^2) + 2\mathrm{i}[c_{26} + \overset{\circ}{S}_{12} - (e_{26} + e_{12})\overset{\circ}{E}_2 \\ - \eta_{12}\overset{\circ}{E}{}_2^2] - (c_{22} + \overset{\circ}{S}_{22} - 2e_{22}\overset{\circ}{E}_2 - \eta_{22}\overset{\circ}{E}{}_2^2)\} - \overset{\circ}{\rho}, \end{gathered}$$

respectively:

$$\begin{gathered} \overset{\circ}{Q}_{14} = \overset{\circ}{Q}_{41} = N^2[(e_{11} + \eta_{11}\overset{\circ}{E}_1) + \mathrm{i}(e_{16} + e_{21} + 2\eta_{12}\overset{\circ}{E}_1) - (e_{26} + \eta_{22}\overset{\circ}{E}_1)], \\ \overset{\circ}{Q}_{24} = \overset{\circ}{Q}_{42} = N^2[(e_{16} + \eta_{11}\overset{\circ}{E}_2) + \mathrm{i}(e_{12} + e_{26} + 2\eta_{12}\overset{\circ}{E}_2) - (e_{22} + \eta_{22}\overset{\circ}{E}_2)], \\ \overset{\circ}{Q}_{44} = -N^2(\eta_{11} + 2\mathrm{i}\eta_{12} - \eta_{22}). \end{gathered} \tag{3.11}$$

• The second subsystem reduces to a single equation

$$\overset{\circ}{Q}_{33}\, a_3 = 0. \tag{3.12}$$

Its root is linked to a transverse-horizontal wave, with polarization after the axis x_3, non-piezoelectric, and influenced by the initial stress field, only. It corresponds to $\overset{\circ}{TH}$ wave form the problem of guided wave propagation. Here:

$$\overset{\circ}{Q}_{33} = N^2[(c_{55} + \overset{\circ}{S}_{11}) + 2\mathrm{i}(c_{45} + \overset{\circ}{S}_{12}) - (c_{44} + \overset{\circ}{S}_{22})] - \overset{\circ}{\rho}\,. \tag{3.13}$$

3.2.2 Particular case: anisotropic directional bivector

In the second particular case we have the *anisotropic directional bivector* $\mathbf{C} = (C_1, C_2, 0)$, with

$$C_1 = \cos\alpha + \mathrm{i}\sin\alpha, \quad C_2 = \cos\alpha - \mathrm{i}\sin\alpha, \ \alpha \in [0, 2\pi).$$

This inhomogeneous wave is elliptically polarized in the plane normal to the dyad axis x_3, except the particular directions $\alpha \in \{\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4\}$, where it is circularly polarized.

If $\overset{\circ}{E}_3 = 0$, we obtain that $\overset{\circ}{Q}_{34} = 0$, and $\overset{\circ}{Q}_{13} = \overset{\circ}{Q}_{23} = 0$. Then, the system (2.7) reduces to two independent subsystems.

- The first subsystem has the form:

$$\begin{pmatrix} \overset{\circ}{Q}_{11} & \overset{\circ}{Q}_{12} & \overset{\circ}{Q}_{14} \\ \overset{\circ}{Q}_{21} & \overset{\circ}{Q}_{22} & \overset{\circ}{Q}_{24} \\ \overset{\circ}{Q}_{41} & \overset{\circ}{Q}_{42} & \overset{\circ}{Q}_{44} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_4 \end{pmatrix} = 0. \tag{3.14}$$

It has as solution an inhomogeneous plane wave, polarized into the plane x_1x_2, associated with the electric field (*via* the amplitude a_4 of the electric potential φ), providing piezoelectric and electrostrictive effects, and depending on the initial stress and electric fields. These properties of the wave were obtained from the analysis of the corresponding coefficients,

$$\begin{aligned}
\overset{\circ}{Q}_{11} = N^2\{&(c_{11} + \overset{\circ}{S}_{11} - 2e_{11}\overset{\circ}{E}_1 - \eta_{11}\overset{\circ}{E}{}_1^2)(\cos 2\alpha + \mathrm{i}\sin 2\alpha) \\
&+ 2[c_{16} + \overset{\circ}{S}_{12} - (e_{16} + e_{21})\overset{\circ}{E}_1 - 2\eta_{12}\overset{\circ}{E}{}_1^2] \\
&+ (c_{66} + \overset{\circ}{S}_{22} - 2e_{26}\overset{\circ}{E}_1 - \eta_{22}\overset{\circ}{E}{}_1^2)(\cos 2\alpha - \mathrm{i}\sin 2\alpha)\} - \overset{\circ}{\rho}, \\
\overset{\circ}{Q}_{12} = \overset{\circ}{Q}_{21} = N^2\{&(c_{16} - e_{16}\overset{\circ}{E}_1 - e_{11}\overset{\circ}{E}_2 - \eta_{11}\overset{\circ}{E}_1\overset{\circ}{E}_2)(\cos 2\alpha + \mathrm{i}\sin 2\alpha) \\
&+ [c_{12} + c_{66} - (e_{12} + e_{26})\overset{\circ}{E}_1 - (e_{21} + e_{16})\overset{\circ}{E}_2 - 2\eta_{12}\overset{\circ}{E}_1\overset{\circ}{E}_2] \\
&+ (c_{26} - e_{22}\overset{\circ}{E}_1 - e_{26}\overset{\circ}{E}_2 - \eta_{22}\overset{\circ}{E}_1\overset{\circ}{E}_2)(\cos 2\alpha - \mathrm{i}\sin 2\alpha)\}, \\
\overset{\circ}{Q}_{22} = N^2\{&(c_{66} + \overset{\circ}{S}_{11} - 2e_{16}\overset{\circ}{E}_2 - \eta_{11}\overset{\circ}{E}{}_2^2)(\cos 2\alpha + \mathrm{i}\sin 2\alpha) \\
&+ 2[c_{26} + \overset{\circ}{S}_{12} - (e_{26} + e_{12})\overset{\circ}{E}_2 - \eta_{12}\overset{\circ}{E}{}_2^2] \\
&+ (c_{22} + \overset{\circ}{S}_{22} - 2e_{22}\overset{\circ}{E}_2 - \eta_{22}\overset{\circ}{E}{}_2^2)(\cos 2\alpha - \mathrm{i}\sin 2\alpha)\} - \overset{\circ}{\rho},
\end{aligned} \tag{3.15}$$

respectively

(3.16)

$$\overset{\circ}{Q}_{14}=\overset{\circ}{Q}_{41}= N^2[(e_{11}+\eta_{11}\overset{\circ}{E}_1)(\cos 2\alpha+\mathrm{i}\sin 2\alpha)+(e_{16}+e_{21}+2\eta_{12}\overset{\circ}{E}_1)$$

$$+(e_{26}+\eta_{22}\overset{\circ}{E}_1)(\cos 2\alpha-\mathrm{i}\sin 2\alpha)],$$

$$\overset{\circ}{Q}_{24}=\overset{\circ}{Q}_{42}= N^2[(e_{16}+\eta_{11}\overset{\circ}{E}_2)(\cos 2\alpha+\mathrm{i}\sin 2\alpha)+(e_{12}+e_{26}+2\eta_{12}\overset{\circ}{E}_2)$$

$$+(e_{22}+\eta_{22}\overset{\circ}{E}_2)(\cos 2\alpha-\mathrm{i}\sin 2\alpha)],$$

$$\overset{\circ}{Q}_{44}= -N^2[(1+\eta_{11})(\cos 2\alpha+\mathrm{i}\sin 2\alpha)+(1+\eta_{22})(\cos 2\alpha-\mathrm{i}\sin 2\alpha)+2\eta_{12}].$$

- The second subsystem reduces to a single equation

$$\overset{\circ}{Q}_{33}\, a_3 = 0. \tag{3.17}$$

Its root is linked to a transverse-horizontal wave, with polarization after the axis x_3, non-piezoelectric, and influenced by the initial stress field, only. Here:

(3.18)

$$\overset{\circ}{Q}_{33}= N^2[(c_{55}+\overset{\circ}{S}_{11})(\cos 2\alpha+\mathrm{i}\sin 2\alpha)+2(c_{45}+\overset{\circ}{S}_{12})+(c_{44}+\overset{\circ}{S}_{22})(\cos 2\alpha-\mathrm{i}\sin 2\alpha)]-\overset{\circ}{\rho}\,.$$

4 Conclusions

In our work we obtained the conditions of inhomogeneous plane wave propagation in monoclinic crystals subject to initial electromechanical fields. For particular isotropic directional bivectors we derive the decomposition of the propagation condition, and we show that the specific coefficients are similar to the case of guided waves propagation in monoclinic crystals subject to a bias. Moreover, we analyze the important particular case of polar anisotropic directional bivectors and we obtain a similar decomposition of the propagation condition, with specific coefficients.

References

[1] P. Borejko, *Inhomogeneous plane waves in a constrained elastic body*, The Quarterly Journal of Mechanics and Applied Mathematics 40 (1987), 71-87.

[2] Ph. Boulanger, M. Destrade and M. Hayes, *Inhomogeneous "longitudinal" circularly-polarized plane waves in anisotropic elastic crystals*, Acta Acustica, 92 (2006), 247-255.

[3] Ph. Boulanger and M. Hayes, *Electromagnetic plane waves in anisotropic media: an approach using bivectors*, Phil. Trans. R. Soc. Lond. A, 330 (1990), 335-393.

[4] Ph. Boulanger and M. Hayes, *Bivectors and Waves in Mechanics and Optics*, Chapman & Hall, New Delhi, 1993.

[5] S. Chiriţă, *Rayleigh waves on an exponentially graded poroelastic half space*, J. of Elasticity, 110 (2013), 185-199.

[6] S. Chiriţă and I. Ghiba, *Inhomogeneous plane waves in elastic materials with voids*, Wave Motion, 47 (2010), 333-342.

[7] M. Destrade, R. Ogden and G. Saccomandi, *Small amplitude waves and stability for pre-stressed viscoelastic solid*, ZAMP, 60 (2009), 511-528.

[8] J.W. Gibbs, *Elements of Vector Analysis*, in *Scientific papers* vol. 2, part 2, Dover Publications, New York, 1961.

[9] W.R. Hamilton, *Lectures on Quaternions*, Hodges and Smith, Dublin, 1853.

[10] M. Hayes, *Inhomogeneous plane waves*, Arch. Rational Mech. Anal., 85 (1984), 41-79.

[11] D. Royer and E. Dieulesaint, *Elastic Waves in Solids*, vol. I, Springer, Berlin, 2000.

[12] O. Simionescu-Panait, *Propagation of attenuated waves in isotropic solids subject to initial electro-mechanical fields*, in "Proc. Int. Conf. New Trends in Continuum Mechanics" (Ed.: M. Suliciu), Ed. Theta, Bucharest, (2005), 267-275.

[13] O. Simionescu-Panait, *Attenuated wave propagation on a face of a cubic crystal subject to initial electro-mechanical fields*, Int. J. of Applied Electromagnetics and Mechanics, 22 (2005), 111-120.

[14] O. Simionescu-Panait, *Propagation of attenuated waves along an edge of a cubic crystal subject to initial electro-mechanical fields*, Math. and Mech. of Solids, 12 (2007), 107-118.

[15] O. Simionescu-Panait, *Waves in strained/polarized media*, in "Special topics in the theory of piezoelectricity" (Ed.: J. Yang), Springer, New York, (2009), 169-246.

[16] O. Simionescu-Panait, *Geometry of electroelastic materials subject to biasing fields*, Balkan J. Geom. Appl. 14 (2009), 91-101.

[17] O. Simionescu-Panait, *Inhomogeneous plane waves in monoclinic crystals subject to a bias*, PAMM, 11 (2011), 637-638.

[18] O. Simionescu-Panait and I. Ana, *On the coupling of guided waves propagation in piezoelectric crystals subject to initial fields*, Math. and Mech. of Solids, 14 (2009), 502-513.

[19] O. Simionescu-Panait and I. Ana, *Guided wave propagation in an isotropic solid subject to a bias*, Math. and Mech. of Solids, 16 (2011), 208-216.

[20] J. L. Synge, *The Petrov Classification of Gravitational Fields*, Commun. Dublin Inst. Adv. Stud. A, vol. 15, 1964.

Author's address:

Olivian Simionescu-Panait, Livia Harabagiu
Department of Mathematics, University of Bucharest, Bucharest, Romania.
E-mail: o_ simionescu@yahoo.com , harabagiu_ livia@yahoo.com