A lower bound for the Dirichlet energy of moving frames on a torus immersed in $\mathbb{H}^n$

Teodor Ţurcanu, Constantin Udrişte

Abstract. We study the boundedness of the Dirichlet energy of moving frames associated to immersions of torus in the Poincaré upper half-space. The frame energy functional is closely related to the Willmore energy and also to the conformal structure on the underlying torus. A lower bound for the frame energy is established.

M.S.C. 2010: 30F45, 53A30, 53A35, 53C42.
Key words: Moving frames; Dirichlet energy; Poincaré metric; Willmore conjecture.

1 Introduction

Our aim in this paper is to study the boundedness of the Dirichlet energy of moving frames on torus immersed in $\mathbb{H}^n, n \geq 3$, where $\mathbb{H}^n$ is the upper half−space endowed with the Poincaré metric (this space is maximally symmetric, simply connected, Riemannian manifold with constant sectional curvature -1.) Bounds on analogous energies were established for immersed tori in $\mathbb{R}^n$ [10], and in S^n [12] respectively. The theory of moving frames is a powerful and elegant tool of classical differential geometry, which is particularly useful in the study of immersed surfaces (see for instance Cartan [2], Chern [3], Darboux [5], Willmore [17], etc.). More recently, moving frames were successfully employed in the study of harmonic maps (see for instance Hélein [8]). There is a close relation between moving frames on the immersed surface (the image) and conformal structures of its underlying abstract surface (the domain) [7, Chapter 5]. The question of selecting *the best moving frame* in surface theory is similar to the question of selecting *an optimal gauge* in physical problems.

2 Poincaré metric and frame energy

Let $\mathbb{H}^n$ denote the upper half-space $\{(x^1, \cdots, x^n) \in \mathbb{R}^n | x^n > 0\}$ together with the *Poincaré metric*

$$g_{\mathbb{H}^n} = \frac{1}{(x^n)^2} \delta_{ij} dx^i dx^j, \quad i, j = 1, \cdots, n$$

Balkan Journal of Geometry and Its Applications, Vol.20, No.2, 2015, pp. 84-0.

defined with respect to the coordinate functions (we adopt the Einstein's summation convention). Let $\mathbb{T}^2$ be an abstract torus (i.e. the unique smooth orientable 2-dimensional manifold of genus one) and let $\phi : \mathbb{T}^2 \hookrightarrow \mathbb{H}^n, n \geq 3$ be a smooth immersion.

Definition 2.1. A *moving frame* on the immersed torus $M := \phi(\mathbb{T}^2)$ is a pair $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) \in \Gamma(TM) \times \Gamma(TM)$, where TM is the tangent bundle of M, such that $\forall u = (u^1, u^2) \in \mathbb{T}^2$, $\{\mathbf{e}_1(u), \mathbf{e}_2(u)\}$ is a positively oriented orthonormal (with respect to $g_{\mathbb{H}^n}$) basis of the tangent space $T_{\phi(u)}M$.

By positive orientation we mean that the orientation of the above basis and that of the immersed torus (which is fixed beforehand) do agree.

Definition 2.2. Let $h := \phi^* g_{\mathbb{H}^n}$ be the pullback metric induced by the smooth immersion ϕ. With the above notations and hypotheses, we define the *frame energy* of the couple $(\phi, \mathbf{e})$, to be the functional

$$\mathcal{F}(\phi, \mathbf{e}) = \frac{1}{4} \int_{\mathbb{T}^2} |d\mathbf{e}|^2 d\mu_h, \tag{2.1}$$

where d is the differential of the frame, $|d\mathbf{e}|^2$ is the square length of the differential of the frame, and $d\mu_h$ is the volume form associated to the pullback metric.

Observe that

$$|d\mathbf{e}|^2 = \sum_{\gamma=1}^{2} |d\mathbf{e}_\gamma|^2 = \sum_{\gamma=1}^{2} h^{\alpha\beta} \langle \partial_{u^\alpha} \mathbf{e}_\gamma, \partial_{u^\beta} \mathbf{e}_\gamma \rangle, \quad \alpha, \beta \in \{1, 2\},$$

where the scalar product "$\langle ., . \rangle$" is realized by the metric g.

Denote by $\pi_T : \mathbb{H}^n \to TM$ and $\pi_N : \mathbb{H}^n \to NM$ the orthogonal projections on the tangent and on the normal space respectively. Notice that we can make the following decomposition

$$\begin{aligned} d\mathbf{e}_1 &= \pi_T(d\mathbf{e}_1) + \pi_N(d\mathbf{e}_1) = \langle d\mathbf{e}_1, \mathbf{e}_2 \rangle \mathbf{e}_2 + \pi_N(d\mathbf{e}_1), \\ d\mathbf{e}_2 &= \pi_T(d\mathbf{e}_2) + \pi_N(d\mathbf{e}_2) = \langle d\mathbf{e}_2, \mathbf{e}_1 \rangle \mathbf{e}_1 + \pi_N(d\mathbf{e}_2). \end{aligned}$$

Using the fact that $\langle d\mathbf{e}_1, \mathbf{e}_2 \rangle + \langle \mathbf{e}_1, d\mathbf{e}_2 \rangle = 0$, we can split the frame energy into its tangential and normal part respectively

$$\mathcal{F}(\phi, \mathbf{e}) = \frac{1}{2} \int_{\mathbb{T}^2} |\langle \mathbf{e}_1, d\mathbf{e}_2 \rangle|^2 d\mu_h + \frac{1}{4} \int_{\mathbb{T}^2} |A|^2 d\mu_h = \mathcal{F}_T(\phi, \mathbf{e}) + \mathcal{W}_H(\phi), \tag{2.2}$$

where $\mathcal{F}_T(\phi, \mathbf{e})$ is the tangential part of the frame energy, whereas $\mathcal{W}_H(\phi)$ can be seen as the Willmore functional (A is the second fundamental form) with respect to the Poincaré metric (after applying the Gauss-Bonnet theorem).

3 A lower bound of the frame energy

In this section we state and prove our main result concerning the lower bound of the frame energy (2.1). But first, let us make some remarks which will enable us, without any loss of generality, to reduce the problem to the case of coordinate moving frames associated to smooth conformal immersions of tori belonging to the moduli space of conformal structures.

Remark 3.1. There exists ([7, Lemma 4.1.3]) a moving frame which minimizes the tangential part of the frame energy

$$\mathcal{F}_T(\mathbf{e},\phi) := \int_{\mathbb{T}^2} |\langle \mathbf{e}_1, d\mathbf{e}_2\rangle|^2 d\mu_h$$

or, equivalently, which satisfies the *Coulomb condition*

$$d^{*g}\langle \mathbf{e}_1, d\mathbf{e}_2\rangle = 0, \tag{3.1}$$

written in isothermal coordinates as $\operatorname{div}\langle \mathbf{e}_1, \nabla \mathbf{e}_2\rangle = 0$. Since we are interested in minimizing the frame energy, we may assume further that the moving frame is Coulomb (i.e. satisfies (3.1)).

Remark 3.2. It is well known that any isothermal (conformal) chart generates a Coulomb frame and, according to S. S. Chern [4], the converse is also true. Indeed, given a Coulomb frame, using Chern's moving frame method, we can cover the torus $\mathbb{T}^2$ by finitely many balls $\{B_k\}$ such that $\forall k = 1, \cdots, N$ there is a difeomorphism $f_k : B_k \to B_k$, and such that $\phi \circ f_k$ is a smooth conformal immersion of B_k in $\mathbb{R}^n$ (and hence in $\mathbb{H}^n$) and

$$\mathbf{e}_\alpha = \frac{\partial_{u^\alpha}(\phi \circ f_k)}{|\partial_{u^\alpha}(\phi \circ f_k)|}, \quad \alpha = 1, 2.$$

Hence, we may assume that *the frame* $\mathbf{e}$ *is the coordinate moving frame* asociated to the smooth conformal immersion ϕ.

Remark 3.3. We have a conformal smooth structure on $\mathbb{T}^2$ induced by the local conformal coordinates. By the Uniformization Theorem (see for instance [8, Section 4.4]), $\mathbb{T}^2$ is conformally equivalent to a flat torus Σ (i.e. the quotient of $\mathbb{R}^2$ modulo a $\mathbb{Z}^2$ lattice) via a diffeomorphism ψ. Thus, we may assume that $f_k^{-1} \circ \psi$ is a conformal diffeomorphism. This implies that $\phi \circ \psi = \phi \circ f_k \circ f_k^{-1} \circ \psi$ is a smooth conformal immersion in $\mathbb{H}^n$, to which we can associate a natural Coulomb moving frame $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)$ i.e.

$$\mathbf{f}_\alpha = \frac{\partial_{v^\alpha}(\phi \circ \psi)}{|\partial_{v^\alpha}(\phi \circ \psi)|}, \ \alpha = 1, 2, \quad \langle \mathbf{f}_1, d\mathbf{f}_2\rangle = *d\lambda, \tag{3.2}$$

where (v^1, v^2) are the flat coordinates on Σ and $\lambda = \log|\partial_{v^\alpha}(\phi \circ \psi)|,\ \alpha = 1, 2$ is the conformal factor (the last equality in (3.2) expresses the connection between the moving frame on the image and the conformal structure on the domain).

Notice that $\mathbf{e} \circ \psi$ is a Coulomb moving frame on the immersed flat torus. We can relate the frames $\mathbf{f}$ and $\mathbf{e}$ locally by a rotation φ in the tangent space

$$\mathbf{e}_1 + i\mathbf{e}_2 = e^{i\varphi}(\mathbf{f}_1 + i\mathbf{f}_2),$$

where $\varphi : \Sigma \to S^1$ is a smooth function. Also notice (see for instance [10]) that

$$\mathcal{F}_T(\phi \circ \psi, \mathbf{e}) \geq \mathcal{F}_T(\phi \circ \psi, \mathbf{f}), \tag{3.3}$$

for the flat torus Σ.

Thus, we may also assume that *the given torus is flat.*

Remark 3.4. We also need to make some compactness assumption in order to avoid the situation in which the conformal classes of Σ degenerate (this leads to a irreversible loss of energy as well as topology). We know (see for instance [8, Section 2.7]) that up to composition with a linear transformation which preserves the orientation, the conformal structure of Σ is equivalent to that of a flat torus described by the lattice generated by $\{(1,0),(a,b)\}$, with $(a,b)\in M$, where

$$M=\{(a,b)\in\mathbb{R}^2|-\frac{1}{2}\leq a\leq\frac{1}{2},\ \sqrt{1-a^2}\leq b\}.$$

Hence we assume that $\Sigma=\mathbb{R}^2/(\mathbb{Z}\times\mathbb{Z}(a,b))$ *belongs to the moduli space of conformal structures* and will denote $\theta:=\arccos a\in\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$.

Now we are ready to state and prove the main result of this paper. By the above remarks it is enough to consider just moving frames associated to smooth conformal immersions of flat tori lying in the moduli space of conformal structures.

Theorem 3.1. *Let $\phi:\Sigma\hookrightarrow\mathbb{H}^n, n\geq 3$ be a smooth conformal immersion, let $\mathbf{e}$ be the coordinate frame associated to the immersion ϕ. Then the following inequality holds true*

$$\mathcal{F}(\phi,\mathbf{e})=\frac{1}{4}\int_\Sigma|d\mathbf{e}|^2d\mu_h>\pi^2\left(b+\frac{1}{b}\right)\frac{\sin^2\theta}{\sin^2\theta+\cos^4\theta}. \tag{3.4}$$

Proof. The main ingredient we are going to use in this proof is an analogue of the classical Fenchel Theorem ([6], [1]), due to Kuiper [13], which states that the *the total absolute curvature of a closed curve immersed in a complete, simply connected, Riemannian manifold with negative sectional curvature is greater than 2π.* Denote by (x,y) the flat coordinates on Σ. Let $c_x:[0,b]\to\Sigma$ and $c_y:[0,1]\to\Sigma$ be the curves (which are in fact straight lines) along the vectors generating the lattice of Σ given, for every $x\in[0,1],\ y\in[0,b]$, by

$$c_x(t)=(x+t\cot\theta,t),\quad c_y(t)=(y\cot\theta+t,y) \tag{3.5}$$

and consider the curves $\gamma_x=\phi(c_x(\cdot)),\gamma_y=\phi(c_y(\cdot))$. The immersion ϕ being conformal, we denote the conformal factor with $\lambda=\log|\partial_x\phi|=\log|\partial_y\phi|$, and we have $\partial_x\phi=e^\lambda\mathbf{e_1},\partial_y\phi=e^\lambda\mathbf{e_2}$. By Kuiper's theorem and taking into account that ϕ is conformal, we obtain

$$2\pi<\int_{\gamma_y}|k|ds=\int_0^{l(\gamma_y)}|\gamma_y''|ds=\int_0^{l(\gamma_y)}|d_{\mathbf{e}_1}\mathbf{e}_1|ds=\int_0^1|d_{\mathbf{e}_1}\mathbf{e}_1|\frac{1}{x^n}e^\lambda dx,$$

where k is the curvature of γ_y, s is the natural parameter and $l(\gamma_y)$ is the length of the curve γ_y. By squaring the above inequality, using Cauchy- Schwarz inequality and integrating with respect to $y\in[0,b]$, we find

$$4\pi^2b<\int_0^b\int_0^1|d_{\mathbf{e}_1}\mathbf{e}_1|^2\frac{1}{(x^n)^2}e^{2\lambda}dxdy=\int_\Sigma|d_{\mathbf{e}_1}\mathbf{e}_1|^2d\mu_h. \tag{3.6}$$

Now we do the same for the curve γ_x. We have

$$\frac{\gamma_x''}{|\gamma_x''|} = \cos\theta \mathbf{e_1} + \sin\theta \mathbf{e_2} =: \mathbf{e}_2^\theta,$$

and similar arguments as above yield

$$2\pi < \int_{\gamma_x} |k| ds = \int_0^{l(\gamma_x)} |\gamma_x''| ds = \int_0^{l(\gamma_x)} |d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta| ds = \frac{1}{\sin\theta} \int_0^b |d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta| \frac{1}{x^n} e^\lambda dy. \tag{3.7}$$

Again, by squaring the inequality, using Cauchy-Schwarz inequality and integrating with respect to $x \in [0, 1]$, we get

$$\frac{4\pi^2}{b} < \frac{1}{\sin^2\theta} \int_0^1 \int_0^b |d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta|^2 \frac{1}{(x^n)^2} e^{2\lambda} dx dy = \frac{1}{\sin^2\theta} \int_\Sigma |d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta|^2 d\mu_h. \tag{3.8}$$

Using the definition of $\mathbf{e}_2^\theta$, and decomposing into normal and tangential parts respectively, we obtain

$$\begin{aligned} &|d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta|^2 = |\pi_N (d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta)|^2 + \langle d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta, \mathbf{e}_1 \rangle^2 + \langle d_{\mathbf{e}_2^\theta} \mathbf{e}_2^\theta, \mathbf{e}_2 \rangle^2 = \\ &= e^{-4\lambda} \left[\cos^4\theta A_{11}^2 + 4 \sin^2\theta \cos^2\theta A_{12}^2 + \sin^4\theta A_{22}^2 \right] + \\ &+ \cos^2\theta \langle d_{\mathbf{e}_1} \mathbf{e}_1, \mathbf{e}_2 \rangle^2 + \sin^2\theta \langle d_{\mathbf{e}_2} \mathbf{e}_2, \mathbf{e}_1 \rangle^2, \end{aligned} \tag{3.9}$$

where (A_{ij}) is the second fundamental form (with respect to the Poincaré metric). By substituting the equality (3.9) in the inequality (3.8), we find

$$\begin{aligned} &\int_\Sigma e^{-4\lambda} \left[\frac{\cos^4\theta}{\sin^2\theta} A_{11}^2 + 4\cos^2\theta A_{12}^2 + \sin^2\theta A_{22}^2 \right] + \\ &+ \left[\cot^2\theta \langle d_{\mathbf{e}_1} \mathbf{e}_1, \mathbf{e}_2 \rangle^2 + \langle d_{\mathbf{e}_2} \mathbf{e}_2, \mathbf{e}_1 \rangle^2 \right] d\mu_h > \frac{4\pi^2}{b}. \end{aligned} \tag{3.10}$$

Computing as above $|d_{\mathbf{e}_1} \mathbf{e}_1|^2 = e^{-4\lambda} A_{11}^2 + \langle d_{\mathbf{e}_1} \mathbf{e}_1, \mathbf{e}_2 \rangle^2$, adding (3.6) and (3.10), yields

$$\begin{aligned} &\int_\Sigma e^{-4\lambda} \left[\left(1 + \frac{\cos^4\theta}{\sin^2\theta} \right) A_{11}^2 + 4\cos^2\theta A_{12}^2 + \sin^2\theta A_{22}^2 \right] + \\ &+ \left[\left(1 + \cot^2\theta \right) \langle d_{\mathbf{e}_1} \mathbf{e}_1, \mathbf{e}_2 \rangle^2 + \langle d_{\mathbf{e}_2} \mathbf{e}_2, \mathbf{e}_1 \rangle^2 \right] d\mu_h > 4\pi^2 \left(b + \frac{1}{b} \right). \end{aligned} \tag{3.11}$$

Multiplying both sides of the last inequality by $\dfrac{\sin^2\theta}{\sin^2\theta + \cos^4\theta}$, we obtain

$$\begin{aligned} &\int_\Sigma e^{-4\lambda} \left[A_{11}^2 + \frac{4\cos^2\theta \sin^2\theta}{\sin^2\theta + \cos^4\theta} A_{12}^2 + \frac{\sin^4\theta}{\sin^2\theta + \cos^4\theta} A_{22}^2 \right] + \\ &+ \left[\frac{1}{\sin^2\theta + \cos^4\theta} \langle d_{\mathbf{e}_1} \mathbf{e}_1, \mathbf{e}_2 \rangle^2 + \frac{\sin^2\theta}{\sin^2\theta + \cos^4\theta} \langle d_{\mathbf{e}_2} \mathbf{e}_2, \mathbf{e}_1 \rangle^2 \right] d\mu_h > \\ &> 4\pi^2 \left(b + \frac{1}{b} \right) \frac{\sin^2\theta}{\sin^2\theta + \cos^4\theta}. \end{aligned} \tag{3.12}$$

Now, notice that

$$|d\mathbf{e}|^2 = \sum_{i,j=1}^{2} |d_{\mathbf{e}_i}\mathbf{e}_j|^2 = e^{-4\lambda}\left[A_{11}^2 + 2A_{12}^2 + A_{22}^2\right] + \\ +\langle d_{\mathbf{e}_1}\mathbf{e}_1, \mathbf{e}_2\rangle^2 + \langle d_{\mathbf{e}_2}\mathbf{e}_2, \mathbf{e}_1\rangle^2. \tag{3.13}$$

We easily get the following estimates

$$2 > \frac{1}{\sin^2\theta + \cos^4\theta} \geq \frac{\sin^2\theta}{\sin^2\theta + \cos^4\theta},$$

$$2 \geq \frac{4\cos^2\theta\sin^2\theta}{\sin^4\theta + \cos^4\theta} \geq \frac{4\cos^2\theta\sin^2\theta}{\sin^2\theta + \cos^4\theta}, \quad 1 \geq \frac{\sin^4\theta}{\sin^2\theta + \cos^4\theta},$$

which imply (3.4). □

Our next goal is to prove, using the result we just obtained, that the frame energy is bounded below (strictly) by $2\pi^2$, namely, we have the following

Corollary 3.2. *For the frame energy defined as above, the following lower bound holds:*

$$\mathcal{F}(\phi, \mathbf{e}) = \frac{1}{4}\int_\Sigma |d\mathbf{e}|^2 d\mu_h > 2\pi^2. \tag{3.14}$$

Proof. First of all, notice that (6) is symmetric with respect to a, so we may consider only the case when $(a, b) \in M^+ = M \cap \{a > 0\}$.

Now, consider the function

$$f : M^+ \to \mathbb{R}, \quad f(b, \theta) = \left(b + \frac{1}{b}\right)\frac{\sin^2\theta}{\sin^2\theta + \cos^4\theta}. \tag{3.15}$$

Observe that the function is not bounded below by 2, for example we have $f(\sin\theta, \theta) < 2, \forall\theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \setminus \{\frac{\pi}{2}\}$. Consider the set

$$\Omega = \{(a, b) | \left(a - \frac{1}{2}\right)^2 + (b - 1)^2 \leq 4\} \cap M^+$$

and observe that $f|_{\partial\Omega} \geq 2$, with equality if and only if $b = 1$ and $\theta = \frac{\pi}{2}$. Also notice that the function $b \mapsto f(b, \theta)$ is monotone strictly increasing, which implies that

$$f|_{M^+\setminus\Omega} \geq 2. \tag{3.16}$$

Now, let us prove our claim for $(a, b) \in \Omega$. By the results of [9] and [11] we know that the Willmore conjecture holds true for $(a, b) \in \Omega$ i.e. $\mathcal{W}(\phi) \geq 2\pi^2$ for $(a, b) \in \Omega$, where $\mathcal{W}(\phi)$ is the standard Willmore energy. By a result of Weiner [16] the integral

$$\int_\Sigma (H^2 + K)d\mu_h,$$

is a conformal invariant (i.e. is invariant under any conformal change of the metric in the ambient space), K being the constant sectional curvature (in our case $K = -1$) and H being the mean curvature.

Thus, observing that $\mathcal{F}_T(\phi, \mathbf{e})$ in (2.2) is nonnegative, we find

$$2\pi^2 \leq \mathcal{W}(\phi) = \int_\Sigma (H^2 - 1) d\mu_h < \int_\Sigma H^2 d\mu_h = \frac{1}{4}\int_\Sigma |A|^2 d\mu_h \leq \mathcal{F}(\phi, \mathbf{e}), \tag{3.17}$$

where we have used again the Gauss-Bonnet theorem. So we conclude that (3.14) holds true. □

Acknowledgments. The paper was presented in The VIII-th International Conference of Differential Geometry and Dynamical Systems (DGDS-2014), 1 - 4 September 2014, Mangalia, Romania. The work was partially supported by University Politehnica of Bucharest and by Academy of Romanian Scientists.

References

[1] K. Borsuk, *Sur la courbure totale des courbes fermées*, Ann. Soc. Pol. Math. 20, (1948), 251-265.

[2] E. Cartan, *Les Systèmes Extèrieurs et leurs Applications Gèometriquè*, Hermann 1945.

[3] S. S. Chern, *Moving frames*, Astérisque, horsérie, Société Mathématique de France (1985), 67-77.

[4] S. S. Chern,*An elementary proof of the existence of isothermal parameters on a surface*, Proc. Amer. Math. Soc. 1955, 771-782.

[5] G. Darboux, *Leçons sur la Théorie Générale des Surfaces*, Chelsea 1972.

[6] W. Fenchel, *Über Krüummung und Windung geschlossener Raumskurven*, Math. Ann. 101 (1929), 238-252.

[7] J. Jost, *Compact Riemann Surfaces. An introduction to contemporary mathematics*, Third edition, Universitext, Springer-Verlag, Berlin 2006.

[8] F. Hélein, *Harmonic maps, conservation laws and moving frames*, Cambridge Tracts in Math. 150, Cambridge University Press 2002.

[9] P. Li, S. T. Yau,*A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. 69, 2 (1982), 269-291.

[10] A. Mondino, T. Rivière,*A frame energy for immersed tori and applications to regular homotopy classes*, arXiv:1307.6884v1.

[11] S. Montiel, A. Ros,*Minimal immersions of surfaces by the first eigenfunctions and conformal area*, Invent. Math. 83 (1986), 153-166.

[12] P. Topping, *Towards the Willmore conjecture*, Calc. Var. and P.D.E. 11 (2000), 361-393.

[13] Y. Tsukamoto, *On the total absolute curvature of closed curves in manifolds of negative curvature*, Math. Ann. 210 (1974), 313-319.

[14] C. Udrişte, I. Ţevy, *Sturm-Liouville operator controlled by sectional curvature on Riemannian manifolds*, Balkan J. Geom. Appl. 17, 2 (2012), 129-140.

[15] C. Udrişte, *Minimal submanifolds and harmonic maps through multitime maximum principle*, Balkan J. Geom. Appl. 18, 2 (2013), 69-82.

[16] J. L. Weiner, *On a problem of Chen, Willmore, et al.*, Indiana Univ. Math. J. 27, 1 (1978), 19-35.

[17] T. J. Willmore, *Riemannian Geometry*, Oxford Science Publications, Oxford University Press 1993.

Authors' addresses:

Teodor Ţurcanu, Constantin Udrişte
University Politechnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics-Informatics, Splaiul Independentei 313,
Bucharest 060042, Romania.
E-mail: deimosted@yahoo.com , udriste@mathem.pub.ro