Finsler metric topology coincides with Frölicher topology

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Abstract. In this paper, the weakest topology underlying a Frölicher space provided with a Finsler metric is shown to coincide with the induced metric topology.

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1 Introduction

The smooth spaces studied by Alfred Frölicher in several works in the 1980s (see among others [12], [13], [14], [15], [16]) were called *Frölicher spaces* by P. Michor and A. Kriegl ([17]), as well as in the basic paper *Frölicher versus Differential Spaces: A Prelude to Cosmology* by P. Cherenack ([9]), and subsequent ones by the same author.

The differential geometry on these spaces is the latest generalization of the well-known geometry of smooth manifolds. It is built on a smooth structure that originates from monoids. The most usual monoid is $C^\infty(\mathbb{R},\mathbb{R})$, even though one can consider more generally $C^\infty(S,R)$ with S and R being arbitrary topological spaces. We recall this construction as follows.

Let $\mathcal{P}(M^{\mathbb{R}}) := \mathbf{C}_c$, $\mathcal{P}(\mathbb{R}^M) := \mathbf{C_f}$ be small categories and $C^\infty(\mathbb{R},\mathbb{R})$, where M is a nonempty set. A *Frölicher structure* on M is a pair $(\mathcal{C}_M, \mathcal{F}_M)$, where $\mathcal{C}_M \in \mathcal{P}(M^{\mathbb{R}})$ and $\mathcal{F}_M \in \mathcal{P}(\mathbb{R}^M)$. The duality conditions $\Gamma\mathcal{F}_M = \mathcal{C}_M$, $\Phi\mathcal{C}_M = \mathcal{F}_M$ holds, with inclusion reversing (contravariant) functors, at set level, Γ and Φ given by $\Gamma : \mathbf{C}_f \longrightarrow \mathbf{C}_c$ and $\Phi : \mathbf{C}_c \longrightarrow \mathbf{C}_f$ such that

$$\Gamma\mathcal{F} = \{c : \mathbb{R} \to M \mid f \circ c \in C^\infty(\mathbb{R}) := C^\infty(\mathbb{R},\mathbb{R}), \text{ for all } f \in \mathcal{F}\}$$

$$\Phi\mathcal{C} = \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}) := C^\infty(\mathbb{R},\mathbb{R}), \text{ for all } c \in \mathcal{C}\}.$$

This implies that any set $\mathcal{F}$ of scalar functions on M determines a set $\Gamma\mathcal{F}$ of curves. Similarly, any set $\mathcal{C}$ of contours determines a set $\Phi\mathcal{C}$ of functions and since $\mathcal{P}(M^{\mathbb{R}})$ and $\mathcal{P}(\mathbb{R}^M)$ have Galois connection property, one gets a unique pair $(\mathcal{C}_M, \mathcal{F}_M)$ for the smooth structure. As such, a Frölicher structure on a set turns out to be a diffeology

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with only 1-plots. For more details on these properties, we refer the reader to the literature [6, 18, 24] and [2, 23].

There are two topologies and geometries underlying a Frölicher structure, which might be different from one another. In ([5]) it is proved that Frölicher smooth mappings are continuous in both the topologies induced by structure curves as well as the one induced by structure functions, the fact which is rephrased in Lemma 1.1 below.

We recall that a *Frölicher space* is a triple $(M, \mathcal{C}_M, \mathcal{F}_M)$, where M is a nonempty set and $(\mathcal{C}_M, \mathcal{F}_M)$ is a Frölicher structure on M. A map φ between Frölicher spaces $(M, \mathcal{C}_M, \mathcal{F}_M)$ and $(N, \mathcal{C}_N, \mathcal{F}_N)$ is said to be smooth if $\varphi_*(\mathcal{C}) := \{\varphi \circ c;\ c \in \mathcal{C}_M\} \subset \mathcal{C}_N$, or equivalently, $\varphi^*(\mathcal{F}_N) := \{f \circ \varphi;\ f \in \mathcal{F}_M\}$. Consequently, it becomes easy to see that the maps $c \in \Gamma\mathcal{F}$ and $f \in \Phi\mathcal{C}$ are smooth in the sense of Frölicher on an arbitrary Frölicher space (see [5]).

Frölicher spaces and smooth maps between them form a category called the category of Frölicher spaces, which we denote by **Frl**. In their studies, Frölicher and Kriegl ([15]), Kriegl and Michor ([17]) as well as Cherenack ([9]) proved how interesting the properties of the category **Frl** are. That is, it is complete, cocomplete, Cartesian closed, and topological over sets in the sense that Frölicher spaces have a behavior similar to topological spaces (see [2]). In this work we will consider a Frölicher space as a topological construct provided with its weakest topology in which all structure curves, functions and smooth maps are seen to be continuous. Next, and for further purposes on the geometry of this type of smooth spaces, we will compare this topology with the one induced by a Finsler metric. More on the topology of Frölicher spaces can be found in [7] by Andreas Cap, and in [2], [3], [4], [5], [6] by the authors.

2 Frölicher topology

The topology of a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ is defined to be the initial topology generated by structure functions $f \in \mathcal{F}_M$, which we call Frölicher topology, is the collection of all subsets $\mathcal{O}$ that are pre-images $f^{-1}(V)$, for $f \in \mathcal{F}_M$, of open sets V in the standard topology $\tau_{\mathbb{R}}$ of $\mathbb{R}$. It is denoted by $\tau_{\mathcal{F}_M}$. Notice that a stronger topology on $(M, \mathcal{C}_M, \mathcal{F}_M)$, denoted by $\tau_{\mathcal{C}_M}$, is obtained by taking the collection of all subsets $\mathcal{U} \subset M$ such that $c^{-1}(\mathcal{U}) \in \tau_{\mathbb{R}}$ for all curves $c \in \mathcal{C}_M$. In effect, it is clear that $\tau_{\mathcal{F}_M} \subseteq \tau_{\mathcal{C}_M}$ but for most Frölicher spaces this inclusion is just an equality, in which case they are called balanced spaces (see [7]). The natural examples of balanced Frölicher spaces are smooth manifolds and Euclidean spaces.

Lemma 2.1. *If $O \subseteq \mathbb{R}^n$ such that $c^{-1}(O)$ is an open set in the standard topology of $\mathbb{R}$ for all curves $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$, then O is an open set in the standard topology of $\mathbb{R}^n$.*

Proof. Boman's theorem on $\mathbb{R}^n$ states that $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ iff $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, for all C^∞ curves $c : \mathbb{R} \longrightarrow \mathbb{R}^n$. This turns $\mathbb{R}^n$ into a Frölicher space with the standard smooth structure being $(\mathcal{C}, \mathcal{F})$, where $\mathcal{C} = C^\infty(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{F} = C^\infty(\mathbb{R}^n, \mathbb{R})$. It follows that $\mathbb{R}^n$ carries the standard topology which we denote by $\tau_{\mathbb{R}^n}$, as well as the two topologies resulting from the Frölicher structure. These are

$$\tau_{\mathcal{F}} = \{U \subseteq \mathbb{R}^n /\ U = f^{-1}(I),\ I \in \tau_{\mathbb{R}}\}$$

generated by the structure functions $f \in \mathcal{F}$, and

$$\tau_{\mathcal{C}} = \{O \subseteq \mathbb{R}^n / \ J = c^{-1}(O) \in \tau_{\mathbb{R}}\},$$

generated by the structure curves $c \in \mathcal{C}$. From the latter, if $O \subseteq \mathbb{R}^n$ such that $c^{-1}(O) \in \tau_{\mathbb{R}}$ for all smooth curve c, then $O \in \tau_{\mathcal{C}}.(*)$

Now, we recall that $c \in \mathcal{C}$ in the standard Frölicher structure on $\mathbb{R}^n$ if and only if $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ by Boman's Theorem. And as such, it is continuous in the standard topology on $\mathbb{R}^n$. But c is smooth in the Frölicher structure $(\mathcal{C}, \mathcal{F})$. It follows from the Lemma above that c is continuous in both the topologies $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{F}}$ underlying the Frölicher structure.

Next, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ is therefore continuous for $\tau_{\mathbb{R}}$, so that one can assume that $(f \circ c)^{-1}(I) = c^{-1}(f^{-1}(I)) = c^{-1}(O)$ for some $I \in \tau_{\mathbb{R}}$ and $f \in \mathcal{F}$. Again, $f \in \mathcal{F}$ if and only if $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ by Boman's Theorem. So, we have both $f^{-1}(I) \in \tau_{\mathcal{F}}$ and $f^{-1}(I) \in \tau_{\mathbb{R}^n}$. But $\tau_{\mathcal{F}} \subseteq \tau_{\mathcal{C}}$ generally. Thus, $f^{-1}(I) \in \tau_{\mathcal{C}}$. It follows (see $(*)$) that $\tau_{\mathcal{C}} \subseteq \tau_{\mathbb{R}^n}$. In other words, we have proved that an open set in $\tau_{\mathcal{C}}$ is open in $\tau_{\mathbb{R}^n}$. That is, O is open in the standard topology of $\mathbb{R}^n$. □

Lemma 2.2. *The functor τ :* **Frl** $\longrightarrow$ **Top** *leaves* **Frl***-morphisms unchanged on the level of sets in the sense that for any* **Frl***-morphism φ of M into N, $\tau(\varphi)$ is a* **Top***-morphism. That is, $\tau(\varphi)$ is continuous in the underlying topology.*

Proof. Let φ be a morphism between Frölicher spaces $(M, \mathcal{C}_M, \mathcal{F}_M)$ and $(N, \mathcal{C}_N, \mathcal{F}_N)$ provided each with its pair of underlying topologies $(\tau_{\mathcal{C}_M}, \tau_{\mathcal{F}_M})$ and $(\tau_{\mathcal{C}_N}, \tau_{\mathcal{F}_N})$. We need to show that φ is both $(\tau_{\mathcal{C}_M}, \tau_{\mathcal{C}_N})$ and $(\tau_{\mathcal{C}_M}, \tau_{\mathcal{C}_N})$-continuous.

First, we know that for all $c \in \mathcal{C}_M$, $\varphi \circ c \in \mathcal{C}_N$. Now let $U \in \tau_{\mathcal{C}_N}$. Hence, $(\varphi \circ c)^{-1}(U) \in \tau_{\mathbb{R}}$. But $(\varphi \circ c)^{-1}(U) = c^{-1}(\varphi^{-1}(U))$, which is $c^{-1}(V) \in \tau_{\mathbb{R}}$, where $V = \varphi^{-1}(U)$. Therefore, $\varphi^{-1}(U) \in \tau_{\mathcal{C}_M}$. Thus, φ is $(\tau_{\mathcal{C}_M}, \tau_{\mathcal{C}_N})$-continuous.

Next, let $O \in \tau_{\mathcal{F}_N}$. Recall that $f \circ \varphi \in \mathcal{F}_M$ for all $f \in \mathcal{F}_N$. Now assume that $O = f^{-1}(I)$. We have $\varphi^{-1}(f^{-1}(I)) = (f \circ \varphi)^{-1}(I) \in \tau_{\mathcal{F}_M}$. But $f^{-1}(I) \in \tau_{\mathcal{F}_N}$ as by definition of $\tau_{\mathcal{F}_N}$. Thus, φ is $(\tau_{\mathcal{F}_M}, \tau_{\mathcal{F}_N})$-continuous. □

A **Frl**-object M is called a base space when it is a compact Hausdorff balanced $\mathbb{F}$-space (see [7]). Note that M is Hausdorff if $\tau_{\mathcal{F}_M}$ and $\tau_{\mathcal{C}_M}$ are both Hausdorff. From the inclusion above, the topology of a Frölicher space M shall be its weakest topology $\tau_{\mathcal{F}_M}$ induced by structure functions, unless otherwise specified. Therefore, $\tau_{\mathcal{C}_M}$ is Hausdorff if $\tau_{\mathcal{F}_M}$ is Hausdorff.

3 Finsler metric topology

3.1 Finsler structure on a Frölicher space

In what follows we assume that the reader is familiar with the concept of tangent spaces and bundles described in the relevant literature (see [2, 9, 20, 21, 22]) which are similar to those of n-dimensional smooth manifolds (see [20]) for the Frölicher of constant dimension, which we are interested in here.

Definition 3.1. Let M be an n-$\mathbb{F}$-space and p is running through M. Let the set denoted by $TM := \coprod_{p \in M} \{p\} \times T_pM = M \times (\coprod_{p \in M} T_pM) = \{(p, v_p) \mid p \in M, \ v_p \in T_pM\}$. That is, $TM \subseteq M \times Der(M) \subseteq M \times C^\infty(\mathcal{F}_M, \mathbb{R})$. Let $T^*M = \{(p, \theta_p) \mid p \in M, \theta_p \in T_p^*M\} = \coprod_{p\in M} \{p\} \times T_p^*M = M \times (\coprod_{p\in M} T_p^*M)$. Then TM is called the operational tangent bundle on M, and T^*M is called the operational cotangent bundle on M.

Remark 3.2. There exist natural projections defined as follows: $\pi : TM \to M, (p, v_p) \mapsto p$ and $\tau : T^*M \to M$, $(p, \theta_p) \mapsto p$. The final structure, on the coproduct spaces given above, are here generated by the families $(\iota_p)_{p\in M}$ and $(\tilde{\iota}_p)_{p\in M}$ of canonical inclusion maps $\iota_p : T_pM \hookrightarrow TM$ and $\tilde{\iota}_p : T_p^*M \hookrightarrow T^*M$. At each point $p \in M$, $d : \mathcal{F}_M \to \mathcal{F}_M$ induces a map $d_p : \mathcal{F}_M \to \mathbb{R}$ such that, for all $f \in \mathcal{F}_M$, $d_p(f) = (df)_p = ev_p(df) = (ev_p \circ d)(f)$ with ev_p the evaluation map at p. It follows that $d_p = ev_p \circ d$ is a smooth linear map and a derivation. As $(df)_p$ is defined for each $p \in M$, it determines globally a smooth map $df : TM \to \mathbb{R}$ such that $(df)_{|T_pM} = (df)_p = d_p(f)$. Also, $\pi^{-1}(p) = \{v \in TM \mid \pi(v) = p\} = T_pM$ is the fiber of TM at p and $\tau^{-1}(p) = T_p^*M$ is the fiber of T^*M at p.
Let M be an n-$\mathbb{F}$-space and $p \in M$, and $(\mathcal{U}, \varphi)$ a local chart at p. The sets T_pM and T_p^*M are linear n-$\mathbb{F}$-spaces diffeomorphic to $\mathbb{R}^n$ with respective bases $\{\frac{\partial}{\partial x_i}\}$ and $\{dx_i\}$, where (x_i) are local coordinates of $p \in \mathcal{U} \subset M$ such that $\varphi(p) = (x_1, \dots, x_n)$. The pair $(x, v) \in TM$ is given in local coordinates by $(x_i, \frac{\partial}{\partial x_i})$, and $(x, \theta) \in T^*M$ is given by (x_i, dx_i). Thus, TM and T^*M are both $2n$-$\mathbb{F}$-spaces. The $\mathbb{F}$-structure on TM is generated by the set of functions $\mathcal{F}_o = \{df \mid f \in \mathcal{F}_M\} \cup \{f \circ \pi \mid f \in \mathcal{F}_M\}$. Thus, $(TM, \Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o) := (TM, T\mathcal{C}_M, T\mathcal{F}_M)$. Now, the cotangent bundle T^*M on M has the natural structure generated by the set of functions $G_o = \{\chi^* \mid \chi \in \mathfrak{X}(M)\} \cup \{f \circ \tau \mid f \in \tau_M\}$, where $\mathfrak{X}(M)$ is the set of all smooth vector fields on M. Let $\varphi : M \to N$ be a diffeomorphism of $\mathbb{F}$-spaces. Thus, $T^*\varphi := \big((\varphi)^{-1}\big)^* = \big(\varphi^*\big)^{-1}$ and $\varphi^* = (T^*\varphi)^{-1}$ such that $\varphi^*(\theta) := \theta \circ \varphi_* = \alpha$ if, and only if $(\varphi^*)^{-1}(\alpha) := \alpha \circ \varphi_*^{-1} = \theta$.

Obviously, from Definition 3.1 and Remark 3.2, (TM, π, M) and (T^*M, τ, M) are $\mathbb{F}$-bundles.

Definition 3.3. Let M be an **Frl**-object. A set $S = \{f_1, \dots, f_n\}$ of structure functions on M is called functionally independent in the neighbourhood of a point $p \in M$ if $\{(df_1)(p), \dots, (df_n)(p)\}$ is a linearly independent set in the tangent space T_pM to M at p.

Lemma 3.1. [2, 20] Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be a Frölicher space. Let $f_1, \dots, f_n$ be some smooth functions defined in an open neighbourhood $\mathcal{U}$ of $p \in M$ such that one of them is injective. Then the map $\psi := (f_1, \dots, f_n)$ is a diffeomorphism of $(\mathcal{U}, \mathcal{C}_\mathcal{U}, \mathcal{F}_\mathcal{U})$ onto $(\psi(\mathcal{U}), \mathcal{C}_{\psi(\mathcal{U})}, \mathcal{F}_{\psi(\mathcal{U})})$

Remark 3.4. [2, 20] Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an n-$\mathbb{F}$-space. Let $\{f_1, \dots, f_n\}$ be a generating set of $\mathbb{F}$-structure on M such that the map given by $\psi(p) = (f_1(p), \dots, f_n(p))$ for all $p \in M$ is one-to-one. Then the associated tangent map $\psi_{*p} : T_pM \to T_{\psi(p)}\psi(M)$ is an isomorphism of linear spaces. It is known from linear algebra that the map $\varphi : T_pM \to \mathbb{R}^n$ defined by $\varphi(v) := (v_1, \dots, v_n)$, where the v_i are the coordinates of $v \in T_pM$, is an isomorphism. Recall that the canonical $\mathbb{F}$-structure on $\mathbb{R}^n$ is generated by $\{\pi_1, \dots, \pi_n\}$ and $\psi(M)$ is an n-$\mathbb{F}$-subspace of $\mathbb{R}^n$, generated by the restrictions $\hat{\pi}_i = \pi_{i_{|\psi(M)}}$ while each $f_i = \pi_{i_{|\psi(M)}} \circ \psi$ for $i = 1, \dots, n$. Thus, $n = dim\,\mathbb{R}^n = dim\,\psi(M) = dim\,T_pM = dim\,T_{\psi(p)}\psi(M) = dim\,\mathfrak{X}(\mathcal{U})$, where $\mathcal{U}$ is an open neighbourhood of p.

Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an n-$\mathbb{F}$-space and $p \in M$. Assume that $\mathbb{F}$-structure on M is generated by the set $\{f_1, \cdots, f_n\} \subset \mathcal{F}_M$, such that $\varphi(p) := (f_1(p), \cdots, f_n(p))$ is an $\mathbb{F}$-diffeomorphism on a neighbourhood of p onto an $\mathbb{F}$-subspace of $\mathbb{R}^n$ endowed with the canonical $\mathbb{F}$-structure.

Definition 3.5. Let M be an $\mathbb{F}$-space. M is said to be of constant dimension n if either

1. $dim\, T_pM = dim\, T_qM = n$ for any $p, q \in M$, with $p \neq q$ and for all $v \in T_pM$, there exists $\chi \in \mathfrak{X}(M)$ such that $\chi(p) = v$; or
2. for each $p \in M$, there exists an open neighbourhood $\mathcal{U}$ of p in M and a local basis of vector fields over $\mathcal{U}$ making $\mathfrak{X}(\mathcal{U})$ a free module on $\mathcal{F}_M$.

We may have different dimensions at different points of an $\mathbb{F}$-space. The rest of this paper is devoted to $\mathbb{F}$-spaces of constant dimension, which we will call "$\mathbb{F}$-spaces of dimension n" or indiscriminately "n-$\mathbb{F}$-spaces". An n-$\mathbb{F}$-space looks like $\mathbb{R}^n$ at both the $\mathbb{F}$-structure and the $\mathbb{F}$-topology points of view. Therefore, $(\mathbb{R}^n, C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is a natural model of $\mathbb{F}$-spaces of constant dimension, as well as a n-dimensional smooth manifold. [2]

Definition 3.6. [1, 8, 19] Let M be an n-$\mathbb{F}$-space. The set denoted by $TM_o = \{(p, y) \in TM \mid p \in M, y \in T_pM, \, y \neq 0\}$ is called a slit tangent bundle over M.

Since $T_pM_o = T_pM - \{0\} \subset T_pM \subset TM$ and $TM = \sqcup_{p\in M}\{p\} \times T_pM$, then TM is a balanced space and the coproduct topology is equal to the underlying $\mathbb{F}$-topologies, since the coproduct of Frölicher spaces is a final object [6]. It follows that T_pM_o is an open set in T_pM. Thus, $dim\, T_pM_o = n$, $TM_o = \sqcup_{p\in M}\{p\} \times T_pM_o$ is an open set in TM, and so $dim\, TM_o = 2n$. That is, T_pM_o and TM_o are respectively n-$\mathbb{F}$-space and $2n$-$\mathbb{F}$-space.

Let $(x, y), (\bar{x}, \bar{y}) \in TM_o$. We define a relation on TM_o by $(x, y) \sim (\bar{x}, \bar{y})$ if, and only if there exists a real $\lambda > 0$ such that $x = \bar{x}$ and $y = \lambda\bar{y}$. The relation $\sim$ is an equivalence relation on TM_o. The equivalence class of $(x, y) \in TM_o$ is of the form $(x, [y]) := \{(x, \lambda y) \mid \lambda > 0, \, (x, y) \in TM_o\}$, where $[y] = \{\bar{y} = \lambda y \mid \lambda > 0\}$. It is called a ray or a direction. The quotient $\mathbb{F}$-space $TM_{o/\sim}$ is called the projective sphere bundle denoted by $TM_{o/\sim} := SM = \{(x, [y]) \mid (x, y) \in TM_o\}$. Note that each T_xM is partitioned by the equivalence classes. SM is a $(2n-1)$-$\mathbb{F}$-subspace of TM. The fibers at $(x, [y]) \in SM$, denoted by $S_xM := \tau^{-1}(x)$ and $S^*_xM := \Gamma^{-1}(x)$, where $\Gamma : S^*M \to M$ is the canonical projection, are diffeomorphic to $(n-1)$-$\mathbb{F}$-subspaces in T_xM and T^*_xM respectively. Thus, S_xM and S^*_xM are diffeomorphic to S^{n-1}, and are called projective spheres at x.

Now, we assume that the reader is familiar with the basic concepts of Minkowski spaces (see [1, 11, 19]). Recall that each Euclidean space is a special Minkowski space. Also, let $(\mathbb{R}^n, F)$ be a Minkowski space. The set $\{y \in \mathbb{R}^n \mid F(y) = 1\}$ is a closed hypersurface strictly convex around $0 \in \mathbb{R}^n$, but never passing through the origin. It is called the indicatrix of F and is diffeomorphic to the standard sphere $S^{n-1} \subset \mathbb{R}^n$.

Definition 3.7. Let M be an n-$\mathbb{F}$-space. A function $F : TM \to [0, +\infty)$ is called a Finsler structure (or a Finsler metric) on M if it has the following properties: F is $\mathbb{F}$-smooth on TM_o, and $F_x(y) := F(x, y)$ is a Minkowski norm on T_xM for any $x \in M$, where $F_x : T_xM \to [0, +\infty)$. The function F is also called a fundamental function while the pair (M, F) is called a Finsler $\mathbb{F}$-space (or Finsler space, for short).

We can define the Finsler structure using local coordinates system (x^i, y^i) of TM by setting $F(x,y) = F(y^i \frac{\partial}{\partial x^i}|_x)$ in the following way:

Definition 3.8. Let M be an n-$\mathbb{F}$-space. A function $F : TM \to [0, +\infty)$ is called a Finsler structure on M if it satisfies the following: F is $\mathbb{F}$-smooth on TM_o; $F(x, ay) = aF(x,y)$ for $a > 0$, that is, F is homogeneous in y of degree 1; and the $n \times n$ Hessian matrix $(g_{ij}) := ([\frac{1}{2}F^2]_{y^iy^j})$ is positive-definite at every point $(x,y) \in TM_o$. That is, for any $y \in T_xM_o = T_xM - \{0\}$, the bilinear symmetric form $g_y : T_xM \times T_xM \to \mathbb{R}$, defined by $g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(x, y + su + tv)]_{|_{s=t=0}} = g_{ij}u^iv^j$, is an inner product on T_xM.

From [11], we note that the matrix (g_{ij}) is of constant rank $n-1$ in every point $(x,y) \in TM_o$. When $n = 2$, $g_y(u,v) = g_{ij}u^iv^j$ is understood as

$$(u^1, u^2) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

and is equal to $g_{11}u^1v^1 + g_{12}u^1v^2 + g_{21}u^2v^1 + g_{22}u^2v^2$. In the literature, sometimes $g_{ij}(x,y)dx^i \otimes dx^j$ stands for $g := g_{ij}(x,y)y^iy^j$, where (x^i) is a local coordinates system on M. The fundamental tensor of F denoted by $g := g_{ij}(x,y)dx^i \otimes dx^j$, where $g_{ij} := \frac{1}{2}F^2_{y^iy^j} = FF_{y^iy^j} + F_{y^i}F_{y^j}$, is the symmetric covariant 2-tensor defined on TM, also called a smooth tensor field of type $(0,2)$ on M.

Definition 3.9. Let (M, F) be a Finsler $\mathbb{F}$-space. The Finsler function F on M is said to be Riemannian if its restriction $F_x(y) = F(x,y)$ to T_xM, for any $x \in M$, is a Euclidean norm. That is, $F^2_x(y) = \langle x, y \rangle_x = g_{ij}(x)y^iy^j$, where $g_{ij}(x) = g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x^j})$ has no y dependence but is a smooth function of x only.

From [19], the Finsler n-$\mathbb{F}$-space (M, F) is locally Minkowski n-$\mathbb{F}$-space if for an arbitrary chart (U, x^i) of M, the fundamental tensor satisfies $g_{ij}(x,y) = g_{ij}(y)$. That is, F (also g) has no dependence on the x^i. Such a coordinate system (x^i) is called adapted. Furthermore, (M, F) is globally Minkowski n-$\mathbb{F}$-space if M is a vector space. The difference between the foundations of Riemannian geometry and Finsler geometry lies in the following: $F^2 = ds^2 = g_{ij}(u)du^idu^j$ (Riemannian quadratic form) and $F(u, du) = ds$ (Finsler non-quadratic restriction). Note that Finsler geometry is the study of geometric properties of Finsler metrics on the underlying structure. It is the Riemannian geometry without the quadratic restriction, since, according to Definition 3.9, the particular Finsler metric F where the restriction $F_x(y)$ of F obeys the relation $F^2_x(y) = \langle x, y \rangle_x$. A Riemannian structure on a $\mathbb{F}$-space extends the construction of the Euclidean norm on $\mathbb{R}^n$. Thus, a Riemannian structure F is a reversible Finsler structure. Recall that each T_xM is a linear space $\mathbb{F}$-diffeomorphic to $\mathbb{R}^n$. Thus all properties of F and g on $\mathbb{R}^n$ extend to T_xM.

3.2 Finsler metric topology

Below, we introduce the concept of distance on a Finsler $\mathbb{F}$-space as well as we define the topology induced by it, so-called the Finsler metric topology on M in order to compare the latter to the Frölicher topology.

Definition 3.10. Let B and B' be two bases in M, a topological space. B is equivalent to B' if the topologies they generate are equal.

Lemma 3.2. *Let B and B' be two bases in M with respective topology τ, τ'. The base B is said to be equivalent to B' if the following hold: For each $\mathcal{U}\in B$ and each $x\in\mathcal{U}$, there exists $\mathcal{U}'\in B'$ such that $x\in\mathcal{U}'\subset\mathcal{U}$, that is, $\tau\subset\tau'$; and, for each $\mathcal{U}'\in B'$ and each $x\in\mathcal{U}'$, there exists $\mathcal{U}\in B$ such that $x\in\mathcal{U}\subset\mathcal{U}'$, that is, $\tau'\subset\tau$.*

Example 3.11. In $\mathbb{R}^2$ each open ball can be inscribed in an open regular polygon. The converse also holds. The family B of all open balls forms a base for the usual topology on $\mathbb{R}^2$, say τ_B. But $\tau_B=\tau_{\mathcal{F}_{\mathbb{R}^2}}=\tau_{\mathcal{C}_{\mathbb{R}^2}}$. Thus, the base B is equivalent to the base $\{f^{-1}(0,+\infty)\mid f\in\mathcal{F}_{\mathbb{R}^2}\}$, since the topologies they generate are equal. Let P be the family of all open regular polygons of the same kind (that is equilateral triangles, or squares, or pentagons, or hexagons, etc.). Thus, P is a subbase generating a certain topology τ_P. Hence, P is a base for the topology τ_P. Now, since each $\mathcal{U}\in\tau_B$ is a union of some open balls, it follows that for each $x\in\mathcal{U}$, there exists $V\in P$ such that $x\in V\subset\mathcal{U}$. That is, x also belongs to an open ball centered at x, which contains the polygon V and is one of the factors of the union which yields $\mathcal{U}$. Therefore, $\tau_B\subset\tau_P$. From the first statement in this example, we can say that $\tau_P\subset\tau_B$. Therefore, $\tau_B=\tau_P=\tau_{\mathcal{F}_{\mathbb{R}^2}}=\tau_{\mathcal{C}_{\mathbb{R}^2}}$. Hence, P, B and $\{f^{-1}(0,+\infty)\mid f\in\mathcal{F}_{\mathbb{R}^2}\}$ are equivalent bases. These concepts can be generalized, for any n to $\mathbb{R}^n$, with open balls and open regular polytopes.

Definition 3.12. Let F be a Finsler structure on an n-$\mathbb{F}$-space M. Let C be a smooth curve parametrized by a map $c:=c(t)$ arising from p to q in M, with $c(a)=p$ and $c(b)=q$, that is, $c:[a,b]\longrightarrow M$. The length of C is defined by: $l_F(C):=\int_a^b F(c(t),\dot{c}(t))dt$, where a, b, $c(a)=p$ and $c(b)=q$ are given such that $a<b$. That is, $c=c(t)$ is a map parametrizing the curve C.

The curve C can be represented by another map $x:[\tilde{a},\tilde{b}]\longrightarrow M$ such that, given a transformation $\varphi:[a,b]\longrightarrow[\tilde{a},\tilde{b}]$ defined by $\varphi(t):=\tilde{t}>0$, with $\varphi(a)=\tilde{a}$ and $\varphi(b)=\tilde{b}$, then $x=x(\tilde{t})$, $x(\tilde{a})=p$ and $x(\tilde{b})=q$. So, $c=x\circ\varphi$. Therefore, $d\varphi(t)=\dot{\varphi}(t)dt$ and $\dot{x}(\tilde{t})=\dot{x}(\varphi(t))=\dot{x}(\varphi(t))\dot{\varphi}(t)$. In the sequel $l_F(C)=\int_a^b F(c(t),\dot{c}(t))dt=\int_{\tilde{a}}^{\tilde{b}} F(c(t)),\dot{c}(t)dt$. We have shown that $l_F(C)$ is well defined, that is, it is independent of the choice of a parametrization.

Definition 3.13. Let F be a Finsler structure on an n-$\mathbb{F}$-space M. Let C be a smooth curve passing through the pair p, q in M. Let $d_F:M\times M\longrightarrow\mathbb{R}$ be the function defined by: $d_F(p,q):=\inf\limits_C\ l_F(C)$, where the infimum is taken over all arcs ensuing from p to q. The function d_F is called a Finsler distance function. Furthermore, (M,d_F) is called a Finsler metric space ($\mathbb{F}$-space).

Given a Finsler structure F on an n-$\mathbb{F}$-space M, we have constructed a distance function d_F using the length of a smooth curve C on M. Conversely, d_F determines uniquely the Finsler structure on M by $F(p,y):=\lim\limits_{\epsilon\to 0^+}\dfrac{d_F(p,q)}{\epsilon}$, where $p\in M$, $y\in T_pM$, and $C(t)$ is an arbitrary smooth curve such that $C(0)=p$ and $\dot{C}(0)=y$. F is reversible if, and only if d_F is reversible. That is, $F(p,-y)=F(p,y)$ if, and only if $d_F(p,q)=d_F(q,p)$ for all pairs of points $p,q\in M$. In $[1,p161-163]$ the smoothness of F is studied for manifolds, and the study is based on the relationship between F and d_F given in Definitions 3.12 and 3.13 above. Thus, we can extend this method to the $\mathbb{F}$-spaces setting. To avoid a cumbersome notation, we will note $d := d_F$ in the rest of this section. Thus, $(M, d) := (M, d_F)$.

Let (M, d) be a Finsler metric space and $A, B \subset M$, where A, B are nonempty. The identity $d(A, B) := \inf_{x\in A, y\in B} d(x, y)$ defines the distance between A and B. Furthermore,

$$d(\{x\}, B) := d(x, B) := \inf_{y\in B} d(x, y),$$

where $x \in M$ and also, $d(A, \{y\}) := d(A, y) := \inf_{x\in A} d(x, y)$, where $y \in M$. Thus, it is obvious that if $A \cap B = \varnothing$ then $d(A, B) = 0$. The converse of this implication is not necessarily true, since if $d(A, B) = a$ there does not necessairily exist a pair of $x \in A$, $y \in B$ such that $d(x, y) = a$.

Lemma 3.3. *Let (M, d) be a Finsler metric space. Let an open ball be in M denoted by $B(a, r) = \{x \in M \mid d(a, x) < r\}$. The following statements hold:*

1. *Let $x \notin B(a, r)$. Then $d(a, x) - r \leq d(B(a, r), x) = \inf_{y\in B(a,r)} d(y, x)$.*
2. *Let d be symmetric. Then $d(a, x) - r \leq \inf_{y\in B(a,r)} d(x, y)$.*
3. *Let $x \in B(a, r)$, that is, $r - d(a, x) > 0$. Let $d(x, y) < r - d(a, x) = \epsilon$, where $y \in M$. Then $B(a, r) \supset B(x, \epsilon)$.*

Proof. 1. The given assumption $x \notin B(a, r)$ is equivalent to $d(a, r) \geq r$ if, and only if $d(a, x) - r \geq 0$. Assume that $y \in B(a, r)$, that is, $d(a, y) < r$. Now, we can write $0 \leq d(a, x) - r \leq d(a, x) - d(a, y)$. From the triangle inequality one gets $d(a, x) \leq d(a, y) + d(y, x)$. Thus, $d(a, x) - d(a, y) \leq d(y, x)$. Hence, $d(a, x) - r \leq d(y, x)$ for any $y \in B(a, r)$. Therefore, $d(a, x) - r \leq \inf_{y\in B(a,r)} d(y, x) = d(B(a, r), x)$.

2. d symmetric implies $d(x, y) = d(y, x)$. Then 1. above becomes $d(a, x) - r \leq \inf_{y\in B(a,r)} d(x, y)$.

3. Let $y \in B(x, \epsilon)$, that is, $d(x, y) < \epsilon$. The triangle inequality yields $d(a, y) \leq d(a, x) + d(x, y) < d(a, x) + \epsilon = d(a, x) + r - d(a, x) = r$. Thus, $d(a, y) < r$ if, and only if $y \in B(a, r)$. Hence, $B(a, r) \supset B(x, \epsilon)$.

□

M has its underlying Frölicher topology $\tau_{\mathcal{F}_M}$. But d induces the metric topology denoted by τ_d.

Now, we will define some objects in the latter topology. Also, we shall recall their similar objects in $\mathbb{R}^n$ and T_xM by means of the identification $\mathbb{R}^n = T_xM$.

On $\mathbb{R}^n$, we define:

$$\begin{aligned} S(0, r) &= \{v \in \mathbb{R}^n \mid \|v\| = r\} \\ &= \{v \in \mathbb{R}^n \mid v = \vec{oa} \text{ and } \mathrm{d(o, a) = r}\} \\ &= S^{n-1} \subset \mathbb{R}^n \end{aligned}$$

and

$$\begin{aligned} B(0,r) &= \{v\in\mathbb{R}^n \mid \|v\| < r\} \\ &= \{v\in\mathbb{R}^n \mid v=\vec{oa} \text{ and } d(o,a)<r\}, \end{aligned}$$

the open ball with centre 0 and radius r, and $\bar{B}(o,r)=\{v\in\mathbb{R}^n \mid \|v\| \le r\}=\{v\in\mathbb{R}^n \mid v=\vec{oa} \text{ and } d(o,a)\le r\}$, the closed ball with centre o and radius r. Note that each open ball is a basic open set for the metric (Cartesian, which is also Euclidean) topology. There is a one-to-one correspondence between open balls and boxes (cubical) as shown in Example 3.11.

On T_xM, we define: the set $S_x(o,r):=\{y\in T_xM \mid F_x(y)=r\}$ called the tangent sphere of radius r and centre o (centered at o); the set $B_x(o,r):=\{y\in T_xM \mid F_x(y)<r\}$ called the open tangent ball of radius r and centre o; and also the set $\bar{B}_x(o,r):=\{y\in T_xM \mid F_x(y)\le r\}$ called the closed tangent ball of radius r and centre o. Each open ball is a basic open set for the metric topology.

On M, we have the counterpart concepts denoted and defined as follows: The set $S^+(p,r):=\{x\in M \mid d(p,x)=r\}$ called the forward metric sphere of radius r and centre p (centered at p); the set $B^+(p,r):=\{x\in \mid d(p,x)<r\}$ called the forward metric open ball of radius r and centre p; also the set $\bar{B}^+(p,r):=\{x\in M \mid d(p,x)\le r\}$ called the forward metric closed ball of radius r and centre p. Each forward metric open ball is a basic open set for the metric topology.

The above Finsler distance function d is always taken from p to each x. Nevertheless, when we decide to take d from each x to p we are defining the dual concepts of backward metric sphere, named the open ball and closed ball. In the Finsler setting d is not supposed to be reversible. A natural relationship arises between these three topologies on $\mathbb{R}^n$, T_xM and M, since M is locally diffeomorphic to $\mathbb{R}^n$, while $\mathbb{R}^n$ is isometric to T_xM by use of $\theta:=\varphi$ constructed in Remark 3.4 for the latter, and $\varphi:\mathcal{U}\subset M\to\varphi(\mathcal{U})\subset\mathbb{R}^n$ the local $\mathbb{F}$-diffeomorphism for the former. Thus, we can define local diffeomorphism from T_xM to M, using θ locally and φ, that is,

$$B_p(o_p,r) \xrightarrow[\sim]{\theta} B(o,r) \xrightarrow[\sim]{\varphi^{-1}} B^+(p,r);$$

where $v_p:=v^i\frac{\partial}{\partial x^i}\longmapsto(v^1,\ldots,v^n)\longmapsto x$, such that $F_p(v_p)<r$, $\|v_p\|_F<r$, and $d(p,x)<r$, with $v_p\in B_p(o_p,r)\subset T_xM$, $(v^1,\ldots,v^n)\in B(o,r)\subset\mathbb{R}^n$, and $x\in B^+(p,r)\subset M$. From Definition 3.5 and Example 3.11, we set $\varphi(p)=o$ and $\varphi^{-1}(o)=p$, with $p\in M$, $o\in\mathbb{R}^n$; $\theta(o_p)=o$ and $\theta^{-1}(o)=o_p$, with $o_p\in T_xM$.

Lemma 3.4. [1] Let (M,F) be a Finsler n-$\mathbb{F}$-space and $\mathcal{U}$ an open neighbourhood at an arbitrary point $p\in M$. Given any pair $x_0,x_1\in\mathcal{U}$, then,

1. The closure of $\mathcal{U}$, that is $\bar{\mathcal{U}}$, is compact, $\varphi(p)=o$ and φ maps $\mathcal{U}$ diffeomorphically onto an open ball $\varphi(\mathcal{U})\subset\mathbb{R}^n$.

2. For all $y=y^i\frac{\partial}{\partial x^i}\in T_xM$ and $x\in\bar{\mathcal{U}}$, there exists a constant $c>1$ such that $\frac{1}{c}\|y\|\le F(y)\le c\|y\|$ and $F(-y)\le c^2F(y)$; $\|y\|:=\sqrt{\delta_{ij}y^iy^j}=\sqrt{(y^1)^2+\ldots+(y^n)^2}$.

3. $\frac{1}{c}\|\varphi(x_1)-\varphi(x_0)\|\le d(x_0,x_1)\le c\|\varphi(x_1)-\varphi(x_0)\|$, where $\varphi(x_i)\in\varphi(\mathcal{U})$.

4. $\frac{1}{c^2}d(x_1,x_0)\le dx_0,x_1)\le c^2 d(x_1,x_0)$

Proof. The proof is similar to that which is given in [1, p.146, Lemma 6.2.1]. It is also a straightforward consequence of the equivalence of norms on a linear space and of distance functions on a metric space. Recall that in a Finsler n-$\mathbb{F}$-space the distance is not reversible in general, that is, $d(x_1,x_0)\neq d(x_0,x_1)$. □

Now, we can show that the topologies $\tau_{\mathcal{F}_M}$ and τ_d are equal on M. That is, their natural bases are equivalent with respect to Definition 3.10 and the characterization in Lemma 3.2 of the equivalence of bases. To do so, we need two steps given below. First, for every forward metric open ball (basic open set) $B^+(x,s)\in\tau_d$ and for every $p\in B^+(x,s)$, there exists $f^{-1}(0,+\infty)\in\tau_{\mathcal{F}_M}$ with $p\in f^{-1}(0,+\infty)\subset B^+(x,s)$, that is, $\tau_d\subset\tau_{\mathcal{F}_M}$. Next, for every $f^{-1}(0,+\infty)$, a basic open set in $\tau_{\mathcal{F}_M}$, and for every $p\in f^{-1}(0,+\infty)$, there exists $B^+(x,s)$ a basic open set in τ_d with $p\in B^+(x,s)\subset f^{-1}(0,+\infty)$, that is, $\tau_{\mathcal{F}_M}\subset\tau_d$.

4 The main result

Theorem 4.1. *Let (M,F) be a Finsler n-$\mathbb{F}$-space with its induced Finsler distance function d. The $\mathbb{F}$-topology $\tau_{\mathcal{F}_M}$ is equal to the Finsler metric topology τ_d generated by the forward metric open balls $B^+(x,s)$.*

Proof. Firstly, for any $B^+(x,s)$ and for every $p\in B^+(x,s)=\{p\in M \mid d(x,p)<s\}$, there exists $\varepsilon>0$ such that $d(x,p)+\varepsilon=s$. Equivalently, $\varepsilon=s-d(x,p)$. Therefore, $B^+(x,s)\supset B^+(p,\varepsilon)$ by Lemma 3.3. Let $\mathcal{U}_p$ be an open neighbourhood at $p\in M$, thus $\mathcal{U}_p=\cup_{f\in\mathcal{F}_M}f^{-1}(0,+\infty)$, for some $f\in\mathcal{F}_M$. At least one of the factors of the union contains p. Now we refer to Lemma 3.4, (1) and (2). Let $B(o,r):=\varphi(\mathcal{U}_p)\subset\mathbb{R}^n$ and $\frac{\varepsilon}{c}\le r$, where $c>1$ and $\varphi(p)=o$. Thus, $B(o,\frac{\varepsilon}{c})\subset B(o,r)$. Furthermore, $B(o,r)=\cup_{\frac{\varepsilon}{c}\le r}B(o,\frac{\varepsilon}{c})=\varphi(\mathcal{U}_p)=\cup_{f\in\mathcal{F}_M}\varphi(f^{-1}(0,+\infty))$ implies that $\varphi(p)=o\in B(o,\frac{\varepsilon}{c})$ for a fixed ε and $p\in f^{-1}(0,+\infty)\subset\varphi^{-1}(B(o,\frac{\varepsilon}{c}))$. For any $q\in f^{-1}(0,+\infty)$, $\varphi(q)\in B(o,\frac{\varepsilon}{c})$. Hence, from Lemma 3.4, (3), where $\varphi(p)=o$ and $q\in\mathcal{U}_p$,
we have $\frac{1}{c}\|\varphi(q)\|\le d(p,q)\le c\|\varphi(q)\|<c\frac{\varepsilon}{c}=\varepsilon$. Therefore, $d(p,q)\le\varepsilon$ implies that $q\in f^{-1}(0,+\infty)\subset B^+(p,\varepsilon)\subset B^+(x,s)$. Hence, $q\in f^{-1}(0,+\infty)\subset B^+(x,s)$. That is, $\tau_d\subset\tau_{\mathcal{F}_M}$.
Conversely, for any $f^{-1}(0,+\infty)\in\tau_{\mathcal{F}_M}$ and for every $p\in f^{-1}(0,+\infty)$, there exists $\mathcal{U}_p$ an open neighbourhood at $p\in M$ such that $\varphi(\mathcal{U}_p):=B(o,r)$, where $\varphi(p)=o$, $\mathcal{U}_p=\cup_{f\in\mathcal{F}_M}f^{-1}(0,+\infty)$, and φ a local diffeomorphism. By shrinking r, we yield $\varepsilon(r)\le r$ such that $B(o,\varepsilon)\subset B(o,r)$ and $\varphi^{-1}(B(o,\varepsilon))\subset f^{-1}(0,+\infty)\subset\mathcal{U}_p$. Now, for any $q\in B^+(p,\frac{\varepsilon}{c})$, with $\frac{\varepsilon}{c}\le\varepsilon\le r$, $c>1$, c depending on p and r. Thus, $d(p,q)<\frac{\varepsilon}{c}$. Since $q\in\mathcal{U}_p$ then Lemma 3.4, (3) yields $\frac{1}{c}\|\varphi(q)\|\le d(p,q)\le c\|\varphi(q)\|$. Therefore, $\frac{1}{c}\|\varphi(q)\|\le d(p,q)<\frac{\varepsilon}{c}$ implies that $\|\varphi(q)\|<\varepsilon$. Hence, $\varphi(q)\in B(o,\varepsilon)$. It follows that $q\in\varphi^{-1}(B(o,\varepsilon))\subset f^{-1}(0,+\infty)\subset\mathcal{U}_p$. Thus, $q\in B^+(p,\frac{\varepsilon}{c})\subset f^{-1}(0,+\infty)$. That is, $\tau_{\mathcal{F}_M}\subset\tau_d$. Finally, $\tau_{\mathcal{F}_M}=\tau_d$ because of the double inclusion proved above. □

Corollary 4.2. *Every forward metric open ball is an open set in* $\tau_{\mathcal{F}_M}$*, that is, it is expressible as a union of basic open sets of* $\tau_{\mathcal{F}_M}$*. Also, every basic open set of* $\tau_{\mathcal{F}_M}$ *is an open set in* τ_d*, that is, it is expressible as a union of forward metric open balls in* τ_d*. Moreover, every* $\tau_{\mathcal{F}_M}$*-open set is a* τ_d*-open set, and conversely.*

References

[1] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Spring-Verlag, New York, Inc, 2000.

[2] T.A. Batubenge, *Symplectic Frölicher Spaces of Constant Dimension*, Ph.D. Thesis, University of Cape Town, http://hdl.handle.net/11427/4949, 2004.

[3] T.A. Batubenge, *A Survey on Frölicher Spaces*, Quaest. Math., 38, 6 (2015), 869-884.

[4] T.A. Batubenge, M.H. Tshilombo, *Notes on pre-Frölicher spaces*, Quest. Math., 39, 8 (2016), 1115-1129.

[5] T.A. Batubenge, M.H. Tshilombo, *Topologies and smooth maps on initial and final objects in the category of Frölicher spaces*, Dem. Math., 42, 3 (2009), 641-655.

[6] T.A. Batubenge, M.H. Tshilombo, *Topologies on product and coproduct Frölicher spaces*, Dem. Math. 47, 4 (2014), 1012-1024.

[7] A. Cap, *K-Theory for Convenient Algebras*, Ph.D. Thesis, 1993.

[8] S.-S. Chern, Z. Shen, *Riemann-Finsler Geometry*, World Scientific, Singapore, 2005.

[9] P. Cherenack, *Frölicher versus differential spaces: a prelude to Cosmology*, G. Brümmer and C. Gilmour (eds), Papers in Honour of Bernhard Banaschewski, Kluwer Academic Publishers, Printed in the Netherlands (2000), 391-413.

[10] P. Cherenack, *Applications of Frölicher spaces to cosmology*, Annals Univ. Sci. Budapest, 41(1998), 63-91.

[11] C.T.J. Dodson, *A short review of Landsberg spaces*, Workshop on Finsler and semi-Riemannian geometry, Universidad Autónoma de San Luis Potosí, Mexico, 2006.

[12] A. Frölicher, *Catégories cartésiennement fermées engendrées par des monoides*, Cah. Topol. Géom. Differ. Catég., 21, 4(1980), 367-375.

[13] A. Frölicher, *Smooth Structures. Category Theory (Gummersbach,* 1981, Lect. Notes in Math. 962, Springer, Berlin-New York 1982; 69-81.

[14] A. Frölicher, *Cartesian Closed Categories and Analysis of Smooth Maps*, Proceedings of the Workshop on Categories and Foundation of Continuum Physics, Buffalo, May 17-21 (1982), vol. 1174, U.S.A., 43-51.

[15] A. Frölicher, A. Kriegl, *Linear Spaces and Differentiation Theory*, Wiley-Interscience, 1988.

[16] A. Frölicher and A. Kriegl, *Topology and Differentiation Theory*, Recent Progress in General Topology, Elsevier, 1992.

[17] A. Kriegl, P.W. Michor, *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs, AMS, 53, 1997.

[18] M. Laubinger, *Differential Geometry in Cartesian Closed Categories of Smooth Spaces*, PhD Thesis, Louisiana State University, Department of Mathematics, 2008.

[19] X. Mo, *An Introduction to Finsler Geometry*, World Scientific, Singapore, 2006.

[20] P.P. Ntumba, T.A. Batubenge, *On the way to Frölicher Lie groups*, Quest. Math., 28(2005), 73-74.

[21] P.P. Ntumba, T.A. Batubenge, *Characterization of vector fields on the Frölicher standard n-simplex, imbedded in $\mathbb{R}^n$, and Hamiltonian formalism on differential spaces*, Quaest. Math., 32 (2009), 71-90.

[22] B. W. Toko, *Bundles in the Category of Frölicher Spaces and Symplectic Structure*, M.Sc. Dissertation, University of the Witwatersrand, Johannesburg, 2006.

[23] M.H. Tshilombo, *Symplectic Reduction on Pseudomanifolds*, MSc Dissertation, University of The Witwatersrand, Faculty of Science, School of Mathematics, Johannesburg, 2009.

[24] M.H. Tshilombo, *Cohomologies on Symplectic Quotients of Locally Euclidean Frölicher Spaces*, Ph.D. Thesis, University of South Africa (UNISA), Department of Mathematical Sciences, Pretoria, 2015.

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