

# Some types of recurrence in Finsler Geometry

A. Soleiman and Nabil L. Youssef

**Abstract.** The pullback approach to global Finsler geometry is adopted. Three classes of recurrence in Finsler geometry are introduced and investigated: simple recurrence, Ricci recurrence and concircular recurrence. Each of these classes consists of four types of recurrence. The interrelationships between the different types of recurrence are studied. The generalized concircular recurrence, as a new concept, is singled out.

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**Key words:** recurrence; generalized recurrent; Ricci recurrent; generalized Ricci recurrent; concircularly recurrent; generalized concircularly recurrent.

## 1 Introduction

Many types of recurrent Riemannian manifolds have been studied by many authors (e.g., [2, 3, 6, 7, 8]). On the other hand, some types of recurrent Finsler spaces have been also studied (e.g., [4, 5, 9]).

In this paper, we gather all known types of Finsler recurrence (related to Cartan connection), besides some new ones, in a single general setting. We study *intrinsically* three classes of recurrence: simple recurrence, Ricci recurrence and concircular recurrence. Each of these classes consists of four types of recurrence. The interrelationships between the different types of recurrence are investigated. A special emphasis is focused on the new concept of generalized concircular recurrence. At the end of the paper we provide a concise diagram presenting the relationships among the different types of Finsler recurrences treated.

## 2 Notation and preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1, 10, 11, 12]. We shall use the notations of [10].

In the following, we denote by  $\pi : \mathcal{T}M \rightarrow M$  the subbundle of nonzero vectors tangent to  $M$ ,  $\mathfrak{F}(TM)$  the algebra of  $C^\infty$  functions on  $TM$ ,  $\mathfrak{X}(\pi(M))$  the  $\mathfrak{F}(TM)$ -module of differentiable sections of the pullback bundle  $\pi^{-1}(TM)$ . The elements of

$\mathfrak{X}(\pi(M))$  will be called  $\pi$ -vector fields and will be denoted by barred letters  $\bar{X}$ . The tensor fields on  $\pi^{-1}(TM)$  will be called  $\pi$ -tensor fields. The fundamental  $\pi$ -vector field is the  $\pi$ -vector field  $\bar{\eta}$  defined by  $\bar{\eta}(u) = (u, u)$  for all  $u \in TM$ .

We have the following short exact sequence of vector bundles

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(TM) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

with the well known definitions of the bundle morphisms  $\rho$  and  $\gamma$ . The vector space  $V_u(TM) = \{X \in T_u(TM) : d\pi(X) = 0\}$  is the vertical space to  $M$  at  $u$ .

Let  $D$  be a linear connection on the pullback bundle  $\pi^{-1}(TM)$ . We associate with  $D$  the map  $K : TTM \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X \bar{\eta}$ , called the connection map of  $D$ . The vector space  $H_u(TM) = \{X \in T_u(TM) : K(X) = 0\}$  is called the horizontal space to  $M$  at  $u$ . The connection  $D$  is said to be regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM) \quad \forall u \in TM.$$

If  $M$  is endowed with a regular connection, then the vector bundle maps  $\gamma, \rho|_{H(TM)}$  and  $K|_{V(TM)}$  are vector bundle isomorphisms. The map  $\beta := (\rho|_{H(TM)})^{-1}$  will be called the horizontal map of the connection  $D$ .

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors of  $D$ , denoted by  $Q$  and  $T$  respectively, are defined by

$$Q(\bar{X}, \bar{Y}) = \mathbf{T}(\beta\bar{X}\beta\bar{Y}), \quad T(\bar{X}, \bar{Y}) = \mathbf{T}(\gamma\bar{X}, \beta\bar{Y}) \quad \forall \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)),$$

where  $\mathbf{T}$  is the (classical) torsion tensor field associated with  $D$ .

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of  $D$ , denoted by  $R, P$  and  $S$  respectively, are defined by

$$R(\bar{X}, \bar{Y})\bar{Z} = \mathbf{K}(\beta\bar{X}\beta\bar{Y})\bar{Z}, \quad P(\bar{X}, \bar{Y})\bar{Z} = \mathbf{K}(\beta\bar{X}, \gamma\bar{Y})\bar{Z}, \quad S(\bar{X}, \bar{Y})\bar{Z} = \mathbf{K}(\gamma\bar{X}, \gamma\bar{Y})\bar{Z},$$

where  $\mathbf{K}$  is the (classical) curvature tensor field associated with  $D$ .

The contracted curvature tensors of  $D$ , denoted by  $\hat{R}, \hat{P}$  and  $\hat{S}$  (known also as the (v)h-, (v)hv- and (v)v-torsion tensors respectively), are defined by

$$\hat{R}(\bar{X}, \bar{Y}) = R(\bar{X}, \bar{Y})\bar{\eta}, \quad \hat{P}(\bar{X}, \bar{Y}) = P(\bar{X}, \bar{Y})\bar{\eta}, \quad \hat{S}(\bar{X}, \bar{Y}) = S(\bar{X}, \bar{Y})\bar{\eta}.$$

**Theorem 2.1.** [11] *Let  $(M, L)$  be a Finsler manifold and  $g$  the Finsler metric defined by  $L$ . There exists a unique regular connection  $\nabla$  on  $\pi^{-1}(TM)$  such that*

- (a)  $\nabla$  is metric:  $\nabla g = 0$ ,
- (b) The (h)h-torsion of  $\nabla$  vanishes:  $Q = 0$ ,
- (c) The (h)hv-torsion  $T$  of  $\nabla$  satisfies:  $g(T(\bar{X}, \bar{Y}), \bar{Z}) = g(T(\bar{X}, \bar{Z}), \bar{Y})$ .

Such a connection is called the Cartan connection associated with the Finsler manifold  $(M, L)$ .

The only linear connection we deal with in this paper is the Cartan connection.

### 3 Three types of Finsler recurrence

In this section, we introduce three classes of recurrent Finsler spaces which will be the object of our investigation in the next sections. These notions are defined in Riemannian geometry [2, 3, 6, 7, 8]. We extend them to the Finslerian case.

For a Finsler manifold  $(M, L)$ , we set the following notations:

$$\begin{aligned} \overset{h}{\nabla} & : \text{ the } h\text{-covariant derivatives associated with Cartan connection,} \\ \text{Ric} & : \text{ the horizontal Ricci tensor of Cartan connection,} \\ r & : \text{ the horizontal scalar curvature of Cartan connection,} \\ G(\bar{X}, \bar{Y})\bar{Z} & := g(\bar{X}, \bar{Z})\bar{Y} - g(\bar{Y}, \bar{Z})\bar{X}, \\ C & := R - \frac{r}{n(n-1)}G : \text{ the concircular curvature tensor,} \\ \mathbf{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) & := g(R(X, Y)Z, W), \\ \mathbf{G}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) & := g(G(X, Y)Z, W), \\ \mathbf{C}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) & := g(C(X, Y)Z, W). \end{aligned}$$

A Finsler manifold is said to be horizontally integrable if its horizontal distribution is completely integrable (or, equivalently,  $\hat{R} = 0$ ).

**Definition 3.1.** Let  $(M, L)$  be a Finsler manifold of dimension  $n \geq 3$  with non-zero  $h$ -curvature tensor  $R$ . Then,  $(M, L)$  is said to be:

- (a) recurrent Finsler manifold ( $F_n$ ) if  $\overset{h}{\nabla} R = A \otimes R$ ,
- (b) 2-recurrent Finsler manifold ( $2F_n$ ) if  $\overset{h}{\nabla} \overset{h}{\nabla} R = \alpha \otimes R$ ,
- (c) generalized recurrent Finsler manifold ( $GF_n$ ) if  $\overset{h}{\nabla} R = A \otimes R + B \otimes G$ ,
- (d) generalized 2-recurrent Finsler manifold ( $G(2F_n)$ ) if  $\overset{h}{\nabla} \overset{h}{\nabla} R = \alpha \otimes R + \mu \otimes G$ ,

where  $A$  and  $B$  (resp.  $\alpha$  and  $\mu$ ) are non-zero scalar 1-forms (resp. 2-forms) on  $TM$ , and positively homogenous of degree zero in  $y$ , called the recurrence forms.

In particular, if  $\overset{h}{\nabla} R = 0$ , then  $(M, L)$  is called symmetric.

**Definition 3.2.** Let  $(M, L)$  be a Finsler manifold of dimension  $n \geq 3$  with non-zero horizontal Ricci tensor  $\text{Ric}$ . Then,  $(M, L)$  is said to be:

- (a) Ricci recurrent Finsler manifold ( $RF_n$ ) if  $\overset{h}{\nabla} \text{Ric} = A \otimes \text{Ric}$ ,
- (b) 2-Ricci recurrent Finsler manifold ( $2RF_n$ ) if  $\overset{h}{\nabla} \overset{h}{\nabla} \text{Ric} = \alpha \otimes \text{Ric}$ ,
- (c) generalized Ricci recurrent Finsler manifold ( $GRF_n$ ) if  $\overset{h}{\nabla} \text{Ric} = A \otimes \text{Ric} + B \otimes g$ ,
- (d) generalized 2-Ricci recurrent Finsler manifold ( $G(2RF_n)$ ) if

$$\overset{h}{\nabla} \overset{h}{\nabla} \text{Ric} = \alpha \otimes \text{Ric} + \mu \otimes g,$$

where  $A$  and  $B$  (resp.  $\alpha$  and  $\mu$ ) are as given in Definition 3.1.

In particular, if  $\overset{h}{\nabla} \text{Ric} = 0$ , then  $(M, L)$  is called Ricci symmetric.

**Definition 3.3.** Let  $(M, L)$  be a Finsler manifold of dimension  $n \geq 3$  with non-zero concircular curvature tensor  $C$ . Then,  $(M, L)$  is said to be:

- (a) concircularly recurrent Finsler manifold ( $CF_n$ ) if  $\overset{h}{\nabla} C = A \otimes C$ ,
- (b) 2-concircularly recurrent Finsler manifold ( $2CF_n$ ) if  $\overset{h}{\nabla} \overset{h}{\nabla} C = \alpha \otimes C$ ,
- (c) generalized concircularly recurrent Finsler manifold ( $GCF_n$ ) if

$$\overset{h}{\nabla} C = A \otimes C + B \otimes G,$$

- (d) generalized 2-concircularly recurrent Finsler manifold ( $G(2CF_n)$ ) if

$$\overset{h}{\nabla} \overset{h}{\nabla} C = \alpha \otimes C + \mu \otimes G,$$

where  $A$  and  $B$  (resp.  $\alpha$  and  $\mu$ ) are as given in Definition 3.1.

In particular, if  $\overset{h}{\nabla} C = 0$ , then  $(M, L)$  is called concircularly symmetric.

We quote the following two Lemmas from [9]; they are very useful in the sequel.

**Lemma 3.1.** For a horizontally integrable Finsler manifold, we have:

- (a)  $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{R(\overline{X}, \overline{Y})\overline{Z}\} = 0$ .<sup>1</sup>
- (b)  $\mathbf{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \mathbf{R}(\overline{Z}, \overline{W}, \overline{X}, \overline{Y})$ .
- (c)  $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{\overset{h}{\nabla} R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\} = 0$ .
- (d) The horizontal Ricci tensors  $\text{Ric}$  is symmetric.
- (e)  $\mathfrak{S}_{\overline{U}, \overline{V}; \overline{W}, \overline{X}; \overline{Y}, \overline{Z}} \{(R(\overline{U}, \overline{V})\mathbf{R})(\overline{W}, \overline{X}, \overline{Y}, \overline{Z})\} = 0$ .<sup>2</sup>
- (f)  $\mathfrak{S}_{\overline{U}, \overline{V}; \overline{W}, \overline{X}; \overline{Y}, \overline{Z}} \{(R(\overline{U}, \overline{V})\mathbf{C})(\overline{W}, \overline{X}, \overline{Y}, \overline{Z})\} = 0$ .
- (g)  $(\overset{h}{\nabla} \overset{h}{\nabla} \omega)(\overline{Y}, \overline{X}, \overline{Z}) - (\overset{h}{\nabla} \overset{h}{\nabla} \omega)(\overline{X}, \overline{Y}, \overline{Z}) = (R(\overline{X}, \overline{Y})\omega)(\overline{Z})$ ;  $\omega$  is a  $\pi(1)$ -form.

**Lemma 3.2.** Let  $(M, L)$  be a horizontally integrable Finsler manifold and let  $\omega$  be a  $\pi(2)$ -form. If any one of the following relations holds

$$\begin{aligned} \mathfrak{S}_{\overline{U}, \overline{V}; \overline{W}, \overline{X}; \overline{Y}, \overline{Z}} \{\omega(\overline{U}, \overline{V})\mathbf{R}(\overline{W}, \overline{X}, \overline{Y}, \overline{Z})\} &= 0, \\ \mathfrak{S}_{\overline{U}, \overline{V}; \overline{W}, \overline{X}; \overline{Y}, \overline{Z}} \{\omega(\overline{U}, \overline{V})\mathbf{C}(\overline{W}, \overline{X}, \overline{Y}, \overline{Z})\} &= 0, \\ \mathfrak{S}_{\overline{U}, \overline{V}; \overline{W}, \overline{X}; \overline{Y}, \overline{Z}} \{\omega(\overline{U}, \overline{V})\mathbf{G}(\overline{W}, \overline{X}, \overline{Y}, \overline{Z})\} &= 0, \end{aligned}$$

then  $\omega$  vanishes identically.

<sup>1</sup> $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}$  denotes the cyclic sum over  $\overline{X}, \overline{Y}, \overline{Z}$ .

<sup>2</sup> $\mathfrak{S}_{\overline{U}, \overline{V}; \overline{W}, \overline{X}; \overline{Y}, \overline{Z}}$  denotes the cyclic sum over the three pairs of arguments  $\overline{U}, \overline{V}$ ;  $\overline{W}, \overline{X}$ ;  $\overline{Y}, \overline{Z}$ .

## 4 Recurrence (2-recurrence)

**Proposition 4.1.** *Let  $(M, L)$  be a horizontally integrable Finsler manifold of dimension  $n \geq 3$ . If  $(M, L)$  is recurrent (resp. 2-recurrent) with recurrence form  $A$  ( resp.  $\alpha$ ), then we have:*

(a)  $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{(A \otimes R)(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\} = 0$

(b)  $\overset{h}{\nabla} A$  (resp.  $\alpha$ ) is symmetric,

(c)  $R(\overline{X}, \overline{Y})\mathbf{R} = 0$ .

*Proof.* The proof follows from Definition 3.1 together with Lemmas 3.1 and 3.2.  $\square$

**Theorem 4.2.** *If  $(M, L)$  is a recurrent Finsler manifold of dimension  $n \geq 3$  with recurrence form  $A$ , then*

(a)  $(M, L)$  is  $RF_n$ .

(b)  $(M, L)$  is  $CF_n$  provided that  $r \neq 0$ .

(c)  $(M, L)$  is  $2F_n$  ( resp.  $2RF_n$ ) provided that  $\overset{h}{\nabla} A + A \otimes A \neq 0$ .

*Proof.* (a) is clear from the definitions of recurrence and Ricci recurrence.

(b) As  $(M, L)$  is a recurrent Finsler manifold,  $\overset{h}{\nabla} R = A \otimes R$ , with  $A \neq 0$ . Hence,  $\overset{h}{\nabla} r = rA$ , with  $r \neq 0$  by assumption. Consequently,

$$\begin{aligned} \overset{h}{\nabla} C &= \overset{h}{\nabla} \left\{ R - \frac{r}{n(n-1)} G \right\} = \overset{h}{\nabla} R - \frac{\overset{h}{\nabla} r}{n(n-1)} \otimes G, \quad \text{since } \overset{h}{\nabla} G = 0 \\ &= A \otimes R - \frac{rA}{n(n-1)} \otimes G = A \otimes C. \end{aligned}$$

(c) Using  $\overset{h}{\nabla} R = A \otimes R$ , we have

$$\overset{h}{\nabla} \overset{h}{\nabla} R = \overset{h}{\nabla} A \otimes R + A \otimes \overset{h}{\nabla} R = (\overset{h}{\nabla} A + A \otimes A) \otimes R = \alpha \otimes R,$$

where  $\alpha := \overset{h}{\nabla} A + A \otimes A$ . Hence,  $(M, L)$  is  $2F_n$  provided that  $\alpha \neq 0$ . Similarly, one can show that  $(M, L)$  is  $2RF_n$ .  $\square$

**Remark 4.1.** One can easily show that, the sufficient condition for a Ricci recurrent Finsler manifold of dimension  $n \geq 3$  with recurrence form  $A$  to be a 2-Ricci recurrent Finsler manifold is that  $\overset{h}{\nabla} A + A \otimes A \neq 0$ .

## 5 Concircular recurrence (2-recurrence)

**Proposition 5.1.** *Let  $(M, L)$  be a horizontally integrable concircularly recurrent (resp. 2-concircularly recurrent) Finsler manifold of dimension  $n \geq 3$  with recurrence form  $A$  (resp.  $\alpha$ ), then we have:*

- (a)  $\overset{h}{\nabla} A$  (resp.  $\alpha$ ) is symmetric,
- (b)  $R(\overline{X}, \overline{Y})\mathbf{C} = 0$ ,
- (c)  $R(\overline{X}, \overline{Y})\mathbf{R} = 0$ .

*Proof.* The proof follows from Definition 3.3 together with Lemmas 3.1 and 3.2, after some calculations.  $\square$

**Theorem 5.2.** *If  $(M, L)$  is a concircularly recurrent Finsler manifold of dimension  $n \geq 3$  with recurrence form  $A$ , then*

- (a)  $(M, L)$  is  $2CF_n$  provided that  $\overset{h}{\nabla} A + A \otimes A \neq 0$ .
- (b)  $(M, L)$  is  $GF_n$  (resp.  $GRF_n$ ) provided that  $\overset{h}{\nabla} r - rA \neq 0$ .
- (c)  $(M, L)$  is  $F_n$  provided that  $\widehat{R} = 0$ .

*Proof.*

(a) Let  $(M, L)$  be concircularly recurrent, then  $\overset{h}{\nabla} C = A \otimes C$ , with  $A \neq 0$ . Consequently,

$$\overset{h}{\nabla} \overset{h}{\nabla} C = \overset{h}{\nabla} A \otimes C + A \otimes \overset{h}{\nabla} C = (\overset{h}{\nabla} A + A \otimes A) \otimes C = \alpha \otimes C,$$

where  $\alpha := \overset{h}{\nabla} A + A \otimes A$ . Hence,  $(M, L)$  is  $2CF_n$  if  $\alpha \neq 0$ .

(b) As  $\overset{h}{\nabla} C = A \otimes C$ ,  $C = R - \frac{r}{n(n-1)}G$  and  $\overset{h}{\nabla} G = 0$ , we get

$$(5.1) \quad \begin{aligned} \overset{h}{\nabla} R &= A \otimes R + \frac{1}{n(n-1)} \{ \overset{h}{\nabla} r - rA \} \otimes G \\ \overset{h}{\nabla} R &= A \otimes R + B \otimes G, \end{aligned}$$

where  $B = \frac{1}{n(n-1)}(\overset{h}{\nabla} r - rA)$ . Since  $A \neq 0$ , then,  $(M, L)$  is  $GF_n$  if  $B \neq 0$ .

Now, taking the trace of both sides of (5.1), one gets

$$\overset{h}{\nabla} Ric = A \otimes Ric + B_1 \otimes g,$$

where  $B_1 = \frac{1}{n} \{ \overset{h}{\nabla} r - rA \}$ . Hence,  $(M, L)$  is  $GRF_n$  if  $B_1 \neq 0$ .

(c) Follows from Theorem C of [9].  $\square$

## 6 Generalized recurrence (2-recurrence)

**Theorem 6.1.** *If  $(M, L)$  is a generalized recurrent Finsler manifold of dimension  $n \geq 3$  with recurrence forms  $A$  and  $B$ , then*

- (a)  $(M, L)$  is  $G(2F_n)$  provided that  $\overset{h}{\nabla} A + A \otimes A \neq 0$  and  $\overset{h}{\nabla} B + A \otimes B \neq 0$ .
- (b)  $(M, L)$  is  $CF_n$  provided that  $r \neq 0$ .
- (c)  $(M, L)$  is  $2CF_n$  provided that  $\overset{h}{\nabla} A + A \otimes A \neq 0$  and  $r \neq 0$ .
- (d)  $(M, L)$  is  $GRF_n$
- (e)  $(M, L)$  is  $F_n$  provided that  $\widehat{R} = 0$  and  $r \neq 0$ .

*Proof.*

(a) Let  $(M, L)$  be a generalized recurrent Finsler manifold, then  $\overset{h}{\nabla} R = A \otimes R + B \otimes G$ , with  $A \neq 0 \neq B$ . Consequently,

$$\begin{aligned}
 \overset{h}{\nabla} \overset{h}{\nabla} R &= (\overset{h}{\nabla} A \otimes R + A \otimes \overset{h}{\nabla} R) + \overset{h}{\nabla} B \otimes G, \text{ since } \overset{h}{\nabla} G = 0 \\
 &= (\overset{h}{\nabla} A \otimes R + A \otimes A \otimes R + A \otimes B \otimes G) + \overset{h}{\nabla} B \otimes G \\
 (6.1) \quad &= (\overset{h}{\nabla} A + A \otimes A) \otimes R + (A \otimes B + \overset{h}{\nabla} B) \otimes G \\
 &= \alpha \otimes R + \mu \otimes G,
 \end{aligned}$$

where  $\alpha := \overset{h}{\nabla} A + A \otimes A$  and  $\mu := \overset{h}{\nabla} B + A \otimes B$  are non-zero scalar 2-forms.

(b) By double contraction of  $\overset{h}{\nabla} R = A \otimes R + B \otimes G$ , we get

$$(6.2) \quad \overset{h}{\nabla} r = rA + n(n-1)B.$$

Consequently,

$$\begin{aligned}
 \overset{h}{\nabla} C &= \overset{h}{\nabla} \left\{ R - \frac{r}{n(n-1)} G \right\} = \overset{h}{\nabla} R - \frac{\overset{h}{\nabla} r}{n(n-1)} \otimes G \\
 (6.3) \quad &\stackrel{(6.2)}{=} A \otimes R - \frac{rA}{n(n-1)} \otimes G = A \otimes C.
 \end{aligned}$$

(c) follows from (b), (d) is trivial, (e) follows from (b) and Theorem 5.2.  $\square$

**Remark 6.1.** One can easily show that, the sufficient condition for a generalized Ricci recurrent Finsler manifold of dimension  $n \geq 3$ , with recurrence forms  $A$  and  $B$ , to be a generalized 2-Ricci recurrent Finsler manifold is that  $\overset{h}{\nabla} A + A \otimes A \neq 0$  and  $\overset{h}{\nabla} B + A \otimes B \neq 0$ .

**Proposition 6.2.** *Let  $(M, L)$  be a horizontally integrable generalized recurrent Finsler manifold of dimension  $n \geq 3$  with recurrence forms  $A, B$  and scalar curvature  $r$ , then we have:*

$$(a) \mathfrak{S}_{\bar{X}, \bar{Y}, \bar{Z}} \{(A \otimes R + B \otimes G)(\bar{X}, \bar{Y}, \bar{Z}, \bar{W})\} = 0.$$

(b)  $\bar{d}A$  and  $\bar{d}B + A \wedge B$  vanish identically.

$$(c) R(\bar{X}, \bar{Y})\mathbf{R} = 0,$$

$$\text{where } \bar{d}A(\bar{X}, \bar{Y}) := (\overset{h}{\nabla} A)(\bar{X}, \bar{Y}) - (\overset{h}{\nabla} A)(\bar{Y}, \bar{X}).$$

*Proof.* The proof of (a) is easy.

Now, we prove (b). Let  $(M, L)$  be horizontally integrable and generalized recurrent with recurrence forms  $A$  and  $B$ . Then, by Theorem 6.1,  $(M, L)$  is concircular recurrent, i.e.,  $\overset{h}{\nabla} C = A \otimes C$ , by (6.3). Consequently,  $\overset{h}{\nabla} \mathbf{C} = A \otimes \mathbf{C}$ . Hence,

$$\overset{h}{\nabla} \overset{h}{\nabla} \mathbf{C} = (\overset{h}{\nabla} A) \otimes \mathbf{C} + A \otimes \overset{h}{\nabla} \mathbf{C} = (\overset{h}{\nabla} A + A \otimes A) \otimes \mathbf{C}.$$

From which, taking into account Lemma 3.1(g), we obtain

$$R(\bar{U}, \bar{V})\mathbf{C} = -(\bar{d}A)(\bar{U}, \bar{V})\mathbf{C}.$$

Hence, using Lemma 3.1(f), it follows that

$$\mathfrak{S}_{\bar{U}, \bar{V}; \bar{W}, \bar{X}; \bar{Y}, \bar{Z}} \{\bar{d}A(\bar{U}, \bar{V})\mathbf{C}(\bar{W}, \bar{X}, \bar{Y}, \bar{Z})\} = 0.$$

From which, together with Lemma 3.2, we conclude that

$$(6.4) \quad \bar{d}A = 0$$

On the other hand, from (6.1), we obtain

$$\overset{h}{\nabla} \overset{h}{\nabla} \mathbf{R} = (\overset{h}{\nabla} A + A \otimes A) \otimes \mathbf{R} + (A \otimes B + \overset{h}{\nabla} B) \otimes \mathbf{G}.$$

From which, taking into account (6.4) and Lemma 3.1, we get

$$(6.5) \quad R(\bar{U}, \bar{V})\mathbf{R} = -(\bar{d}B + A \wedge B)(\bar{U}, \bar{V})\mathbf{G}.$$

Hence, from Lemma 3.1(f), we obtain

$$\mathfrak{S}_{\bar{U}, \bar{V}; \bar{W}, \bar{X}; \bar{Y}, \bar{Z}} \{(\bar{d}B + A \wedge B)(\bar{U}, \bar{V})\mathbf{G}(\bar{W}, \bar{X}, \bar{Y}, \bar{Z})\} = 0.$$

Therefore,  $\bar{d}B + A \wedge B$  vanishes identically.

Finally, the proof of (c) follows from (b) and (6.5).  $\square$

**Proposition 6.3.** *Let  $(M, L)$  be a horizontally integrable generalized 2-recurrent Finsler manifold of dimension  $n \geq 3$  with recurrence forms  $\alpha, \mu$  and non-zero constant scalar curvature  $r$ , then we have:*

(a)  $\alpha$  and  $\mu$  are symmetric scalar 2-forms.

$$(b) R(\bar{X}, \bar{Y})\mathbf{R} = 0.$$



## 7 Generalized concircular recurrence

In this section, we study a new type of Finsler recurrence, namely the generalized concircular recurrence, which generalizes the concircular recurrence investigated in [9] by the present authors.

**Theorem 7.1.** *Let  $(M, L)$  be a generalized concircularly recurrent Finsler manifold of dimension  $n \geq 3$  with recurrence forms  $A, B$  and scalar curvature  $r$ , then*

- (a)  $(M, L)$  is  $G(2CF_n)$  provided that  $\overset{h}{\nabla} A + A \otimes A \neq 0$  and  $\overset{h}{\nabla} B + A \otimes B \neq 0$ ,
- (b)  $(M, L)$  is  $GF_n$  (resp.  $GRF_n$ ) provided that  $B - \frac{rA}{n(n-1)} + \frac{\overset{h}{\nabla} r}{n(n-1)} \neq 0$ .

*Proof.*

- (a) Let  $(M, L)$  be generalized concircularly recurrent. Then,  $\overset{h}{\nabla} C = A \otimes C + B \otimes G$ , with  $A \neq 0 \neq B$ . Consequently,

$$\begin{aligned} \overset{h}{\nabla} \overset{h}{\nabla} C &= (\overset{h}{\nabla} A \otimes C + A \otimes \overset{h}{\nabla} C) + \overset{h}{\nabla} B \otimes G, \\ &= (\overset{h}{\nabla} A \otimes C + A \otimes A \otimes C + A \otimes B \otimes G) + \overset{h}{\nabla} B \otimes G \\ &= (\overset{h}{\nabla} A + A \otimes A) \otimes C + (\overset{h}{\nabla} B + A \otimes B) \otimes G \\ &= \alpha \otimes C + \mu \otimes G. \end{aligned}$$

where  $\alpha := \overset{h}{\nabla} A + A \otimes A$  and  $\mu := \overset{h}{\nabla} B + A \otimes B$ . If  $\alpha$  and  $\mu$  are none-zero, then  $(M, L)$  is  $G(2CF_n)$ .

- (b) As  $\overset{h}{\nabla} C = A \otimes C + B \otimes G$ , with  $A \neq 0 \neq B$ , then

$$\begin{aligned} \overset{h}{\nabla} (R - \frac{r}{n(n-1)} G) &= A \otimes (R - \frac{r}{n(n-1)} G) + B \otimes G \\ \overset{h}{\nabla} R - \frac{\overset{h}{\nabla} r}{n(n-1)} \otimes G &= A \otimes (R - \frac{r}{n(n-1)} G) + B \otimes G, \text{ since } \overset{h}{\nabla} G = 0 \\ (7.1) \quad \overset{h}{\nabla} R &= A \otimes R + (B - \frac{rA}{n(n-1)} + \frac{\overset{h}{\nabla} r}{n(n-1)}) \otimes G \\ \overset{h}{\nabla} R &= A \otimes R + B_1 \otimes G, \end{aligned}$$

where  $B_1 := B - \frac{rA}{n(n-1)} + \frac{\overset{h}{\nabla} r}{n(n-1)}$ . Since  $B_1 \neq 0$ , then  $(M, L)$  is  $GF_n$ .

On the other hand, from (7.1), we obtain

$$\begin{aligned} (7.2) \quad \overset{h}{\nabla} \text{Ric} &= A \otimes \text{Ric} + \frac{1}{n}(n(n-1)B - rA + \overset{h}{\nabla} r) \otimes g \\ &= A \otimes \text{Ric} + B_2 \otimes g, \end{aligned}$$

where  $B_2 = (n-1)B_1$ . This completes the proof.  $\square$

**Lemma 7.2.** *Let  $(M, L)$  be a horizontally integrable generalized concircularly recurrent Finsler manifold with recurrence forms  $A$  and  $B$ . The scalar curvature  $r$  of  $(M, L)$  is horizontally parallel if and only if  $2rA = 2nA \circ \text{Ric}_o - n(n-1)(n-2)B$ , where  $\text{Ric}_o$  is defined by  $g(\text{Ric}_o \bar{X}, \bar{Y}) := \text{Ric}(\bar{X}, \bar{Y})$ .*

*Proof.* By (7.1) and Lemma 3.1(c), we obtain

$$\begin{aligned} & A(\bar{W})R(\bar{X}, \bar{Y})\bar{Z} + A(\bar{X})R(\bar{Y}, \bar{W})\bar{Z} + A(\bar{Y})R(\bar{W}, \bar{X})\bar{Z} \\ & + \frac{1}{n(n-1)}\{n(n-1)B(\bar{W}) - rA(\bar{W}) + \overset{h}{\nabla} r(\bar{W})\}\{g(\bar{X}, \bar{Z})\bar{Y} - g(\bar{Y}, \bar{Z})\bar{X}\} \\ & + \frac{1}{n(n-1)}\{n(n-1)B(\bar{X}) - rA(\bar{X}) + \overset{h}{\nabla} r(\bar{X})\}\{g(\bar{Y}, \bar{Z})\bar{W} - g(\bar{W}, \bar{Z})\bar{Y}\} \\ & + \frac{1}{n(n-1)}\{n(n-1)B(\bar{Y}) - rA(\bar{Y}) + \overset{h}{\nabla} r(\bar{Y})\}\{g(\bar{W}, \bar{Z})\bar{X} - g(\bar{X}, \bar{Z})\bar{W}\} = 0. \end{aligned}$$

Contracting the above relation with respect to  $\bar{Y}$ , given that  $g(\bar{X}, \bar{\sigma}) := A(\bar{X})$ , we get

$$\begin{aligned} & A(\bar{W})\text{Ric}(\bar{X}, \bar{Z}) - A(\bar{X})\text{Ric}(\bar{W}, \bar{Z}) + \mathbf{R}(\bar{W}, \bar{X}, \bar{Z}, \bar{\sigma}) \\ & + \frac{1}{n}\{n(n-1)B(\bar{W}) - rA(\bar{W}) + \overset{h}{\nabla} r(\bar{W})\}g(\bar{X}, \bar{Z}) \\ & - \frac{1}{n}\{n(n-1)B(\bar{X}) - rA(\bar{X}) + \overset{h}{\nabla} r(\bar{X})\}g(\bar{Z}, \bar{W}) \\ & + \frac{1}{n(n-1)}\{n(n-1)B(\bar{X}) - rA(\bar{X}) + \overset{h}{\nabla} r(\bar{X})\}g(\bar{W}, \bar{Z}) \\ & - \frac{1}{n(n-1)}\{n(n-1)B(\bar{W}) - rA(\bar{W}) + \overset{h}{\nabla} r(\bar{W})\}g(\bar{X}, \bar{Z}) = 0. \end{aligned}$$

This Relation reduces to

$$\begin{aligned} & A(\bar{W})\text{Ric}(\bar{X}, \bar{Z}) - A(\bar{X})\text{Ric}(\bar{W}, \bar{Z}) + \mathbf{R}(\bar{W}, \bar{X}, \bar{Z}, \bar{\sigma}) \\ & + \frac{n-2}{n(n-1)}\{n(n-1)B(\bar{W}) - rA(\bar{W}) + \overset{h}{\nabla} r(\bar{W})\}g(\bar{X}, \bar{Z}) \\ & - \frac{n-2}{n(n-1)}\{n(n-1)B(\bar{X}) - rA(\bar{X}) + \overset{h}{\nabla} r(\bar{X})\}g(\bar{Z}, \bar{W}) = 0. \end{aligned}$$

Contracting the above relation with respect to  $\bar{X}$  and  $\bar{Z}$ , we obtain

$$2rA(\bar{W}) - 2nA(\text{Ric}_o \bar{W}) + n(n-1)(n-2)B(\bar{W}) + (n-2)\overset{h}{\nabla} r(\bar{W}) = 0.$$

Hence, the result follows.  $\square$

**Theorem 7.3.** *Let  $(M, L)$  be a horizontally integrable generalized concircularly recurrent Finsler manifold with recurrence forms  $A$  and  $B$ . If the scalar curvature  $r$  of  $(M, L)$  is constant, then  $(M, L)$  is  $\text{GRF}_n$ .*

*Proof.* If the scalar curvature  $r$  of  $(M, L)$  is constant, then  $\overset{h}{\nabla} r = 0$ . Hence, in view of Lemma 7.2, we get

$$(7.3) \quad rA = nA \circ \text{Ric}_o - \frac{n(n-1)(n-2)}{2}B.$$

From which, together (7.2), we obtain

$$(7.4) \quad \begin{aligned} \frac{h}{\nabla} \text{Ric} &= A \otimes \text{Ric} + \left( \frac{n(n-1)}{2} B - A \circ \text{Ric}_o \right) \otimes g \\ &= A \otimes \text{Ric} + D \otimes g, \end{aligned}$$

where  $D := \frac{n(n-1)}{2} B - A \circ \text{Ric}_o$ .

Now, we show that  $D \neq 0$ . Assume the contrary, then

$$A \circ \text{Ric}_o = \frac{n(n-1)}{2} B.$$

Substituting into (7.3), we obtain

$$(7.5) \quad rA = n(n-1)B.$$

From which together with (7.1), noting that the scalar curvature  $r$  is constant, we get  $rA = 0$ . Hence, again by (7.5),  $B = 0$ . This is a contradiction.

Therefore, by (7.4),  $(M, L)$  is  $GRF_n$  (as  $D \neq 0$ ).  $\square$

**Remark 7.1.** Both Theorem 7.1(b) and Theorem 7.3 state roughly that, under certain conditions, a  $GCF_n$  manifold is a  $GRF_n$  manifold. The difference between the two results is that in Theorem 7.1(b) the condition is posed on the recurrence forms  $A$  and  $B$ , whereas in Theorem 7.3 the condition is posed on the geometric structure of the underlying manifold ( $r$  is constant,  $\hat{R} = 0$ ).

## 8 Concluding remarks

Three classes of recurrence in Finsler geometry are introduced and investigated. The interrelationships among these classes of recurrence are studied. The following diagram presents concisely the most important results of the paper, where an arrow means "if ... then". Here are some comments on this diagram:

- Continuous arrows represent results (theorems) of the paper. Dashed arrows represent examples of results that can be deduced from continuous arrows.
- Conditions posed on the recurrence forms are not written in the diagram. The written conditions are those posed on the geometric structure of the underlying manifold.
- One can deduce the following result from the diagram:

$$F_n \xrightarrow{r \neq 0, \hat{R} = 0} C F_n$$

This is one of the main result of [9].

- Among other new important results that can be deduced from the diagram, we set:

$$\begin{aligned} G F_n &\longleftrightarrow C F_n \\ F_n &\xrightarrow{r \neq 0, \hat{R} = 0} G F_n \\ G C F_n &\xrightarrow{r \neq 0} C F_n \end{aligned}$$

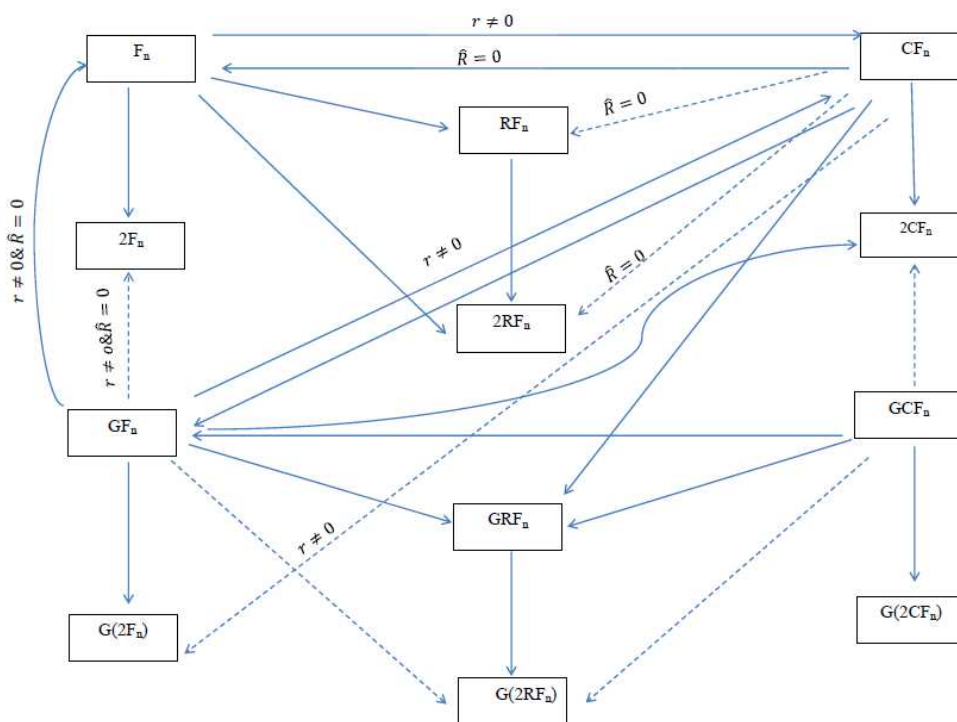


Figure 1. Relationships among different types of recurrent Finsler manifolds

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*Authors' addresses:*

Amr Hassan Soleiman  
Department of Mathematics, Faculty of Sciences,  
Benha University, Egypt.  
E-mail: amrsoleiman@yahoo.com

Nabil L. Youssef  
Department of Mathematics, Faculty of Sciences,  
Cairo University, Egypt.  
E-mail: nlyoussef@sci.cu.edu.eg, nlyoussef2003@yahoo.fr