p-Laplacian first eigenvalue controls on Finsler manifolds

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Abstract. Given a Finsler manifold (M, F), it is proved that the first eigenvalue of the Finslerian p-Laplacian is bounded above by a constant depending on p, the dimension of M, the Busemann-Hausdorff volume and the reversibility constant of (M, F).

For a Randers manifold $(M, F := \sqrt{g} + \beta)$, where g is a Riemannian metric on M and β an appropriate 1-form on M, it is shown that the first eigenvalue $\lambda_{1,p}(M, F)$ of the Finslerian p-Laplacian defined by the Finsler metric F is controled by the first eigenvalue $\lambda_{1,p}(M, g)$ of the Riemannian p-Laplacian defined on (M, g).

Finally, the Cheeger's inequality for Finsler Laplacian is extended for p-Laplacian, with $p > 1$.

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1 Introduction

The study of the p-Laplace operator − and in particular of its first eigenvalue − is a classical and important problem in Riemannian geometry. In [8, 9], the author studies the first eigenvalue of the p-Laplacian Δ_p on a compact Riemannian manifold M as a functional on the space of Riemannian metrics on M. He proved that on any compact manifold of dimension $n \geq 3$, there is a Riemannian metric of volume one such that the first eigenvalue of the p-Laplacian can be taken arbitrary large and that the eigenvalue functional restricted to the conformal class is bounded above for $1 < p \leq n$.

In Finsler geometry, there is no canonical way to introduce the Laplacian. Hence, several authors proposed different extensions of the standard Riemannian Laplacian to the Finsler setting like Antonelli and Zastawniak [1], Bao and Lackey [2], Barthelmé [3], Centoré [4] and Shen [14]. In the last decade, the non-linear Shen's Finsler-Laplacian received a particular attention and Q. He and S-T Yin use it to introduce

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the p-Laplacian on Finsler manifolds [6, 7]. They established some inequalities related to the first eigenvalue and obtained a regularity theorem of its associated functions.

Eigenfunctions of the p-Laplacian have weaker regularities in the Finslerian setting than the Riemannian one, due to the non-linearity of the Finsler Laplacian.

In [10], the author shows that a canonical smooth Riemannian metric can be associated to any Finsler metric F. This Riemannian metric is called Binet-Legendre metric and is bi-lipschitz equivalent to F with lipschitz constant depending only on the dimension of the manifold and on the reversibility constant of F (see Section 2.3). It allows us to control the first eigenvalue of the Finsler p-Laplacian and to prove our main result:

Theorem 1.1. *Let (M, F) be a compact Finsler n-dimensional manifold. Then, for any $p \in (1, n]$, there exists a constant $C := C(n, p, \kappa_F, [F])$ depending only on the dimension n, p, the reversibility constant κ_F and the conformal class $[F]$ of F such that,*

$$\lambda_{1,p}(M, F)Vol(M, F)^{\frac{p}{n}} \leq C(n, p, \kappa_F, [F]).$$

Randers metrics are an important class of Finsler metrics. They are Finsler metrics of the form $F := \sqrt{g} + \beta$ where g is a Riemannian metric and β a 1-form which norm with respect to the metric g is smaller than one. It is interesting to know the relations between geometric quantities related to F and g respectively. We prove the following

Theorem 1.2. *If $(M, F := \sqrt{g} + \beta)$ is a Randers manifold endowed with the Holmes-Thompson volume form $d\mu_{HT}$ then, for any $p > 1$, we have*

$$\frac{1}{\kappa_F^p}\lambda_{1,p}(M, g) \leq \lambda_{1,p}(M, F) \leq \kappa_F^p \lambda_{1,p}(M, g),$$

where $\lambda_{1,p}(M, g)$ is the first eigenvalue of the p-Laplacian on the Riemannian manifold (M, g) and κ_F, the reversibility constant of (M, F).

In [5], Cheeger introduced for a closed Riemannian manifold (M, g) an geometric invariant $\mathbf{h}(M)$ called Cheeger invariant, and he proved that $4\lambda_{1,2}(M) \geq \mathbf{h}^2(M)$. The authors in [18] generalize this inequality for the Finslerian Laplacian. In this paper we extend their result to the Finslerian p-Laplacian for $p > 1$.

The content of the paper is organized as follows. In section 2 , we recall some fundamental notions which are necessary and important for this article. Section 3 and 4 are devoted to the proofs of Theorem 1.1 and Theorem 1.2 respectively. We prove the Cheeger's type inequaliy in the last section.

2 Preliminaries

Let M be a connected, n-dimensional smooth manifold without boundary. Given a local coordinates system $(x^i)_{i=1}^n$ on an open set U of M, we will use the coordinates $(x^i, v^i)_{i=1}^n$ of TU such that for all $v \in T_xM$, $x \in U$,

$$v := v^i \left.\frac{\partial}{\partial x^i}\right|_x.$$

2.1 Finsler geometry

Definition 2.1. A Finsler metric on M is a nonnegative function $F : TM \to [0, \infty)$ satisfying:

1. (Regularity) F is C^∞ on $TM \backslash O$, where O stands for the zero section,
2. (Positive 1-homogeneity) It holds $F(cv) = cF(v)$ for all $v \in TM$ and $c \geq 0$,
3. (Strong convexity) The $n \times n$ matrix

$$(g_{ij}(v))_{1 \leq i,j \leq n} := (\frac{1}{2} \frac{\partial^2 (F^2)}{\partial v^i \partial v^j}(v))_{1 \leq i,j \leq n} \tag{2.1}$$

is positive-definite for all $v \in T_x M \backslash \{0\}$.

Remark that for each $v \in T_x M \backslash \{0\}$, the positive-definite matrix $(g_{ij}(v))_{1 \leq i,j \leq n}$ in the Definition 2.1 defines the Riemannian structure g_v of $T_x M$ via

$$g_v \left(\sum_{i=1}^n a_i \frac{\partial}{\partial x^i}, \sum_{j=1}^n b_j \frac{\partial}{\partial x^j} \right) := \sum_{i,j=1}^n g_{ij}(v) a_i b_j.$$

The *reversibility constant* of (M, F) is defined by

$$\kappa_F := \sup_{x \in M} \sup_{v \in T_x M \backslash \{0\}} \frac{F(v)}{F(-v)} \in [1, \infty].$$

F is said to be reversible if $\kappa_F = 1$, that is $F(v) = F(-v), \ \forall \ x \in T_x M$.

The dual metric $F^* : T^*M \to [0, \infty)$ of F on M is defined for any $\alpha \in T^*M$ by

$$F^*(\alpha) := \sup_{v \in T_x M, F(v) \leq 1} \alpha(v) = \sup_{v \in T_x M, F(v) = 1} \alpha(v).$$

One also define the 2-uniform concavity constant as

$$\sigma_F := \sup_{x \in M} \sup_{v, w \in T_x M \backslash \{0\}} \frac{g_v(w, w)}{F(w)^2} = \sup_{x \in M} \sup_{\alpha, \beta \in T_x^* M \backslash \{0\}} \frac{F^*(\beta)^2}{g^*_\alpha(\beta, \beta)} \in [1, \infty].$$

F is Riemannian if and only if $\sigma_F = 1$ (see [13]).

Given a vector field $X := X^i \frac{\partial}{\partial x^i}$, the covariant derivate of X by $v \in T_x M$ with the reference $w \in T_x M \backslash \{0\}$ is defined by

$$D_v^w X(x) := \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w) v^j X^k(x) \right\} \frac{\partial}{\partial x^i},$$

where $\Gamma^i_{jk}(w)$ are the coefficients of the Chern connection.

The flag curvature of the plane spanned by two linearly independent vector V and W of $T_x M \backslash \{0\}$ is given by

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V) g_V(W, W) - g_V(V, W)^2},$$

where R^V is the Chern curvature:

$$R^V(X,Y)Z := D^V_X D^V_Y Z + D^V_Y D^V_X Z - D^V_{[X,Y]} Z.$$

The Ricci curvature of (M,F) is defined by

$$Ric(V) := \sum_{i=1}^{n-1} K(V, e_i),$$

where $\{e_1, e_2, \dots, e_n = \frac{V}{F(V)}\}$ is an orthonormal basis of T_xM with respect to g_V.

2.2 Finsler p-laplacian

Denote by $J^* : T^*M \to TM$ the Legendre transform which assigns to each $\alpha \in T^*_xM$ the unique maximizer of the function $v \mapsto \alpha(v) - \frac{1}{2}F^2(x,v)$ on T_xM. The quantity $J^*(x,\alpha)$ is characterized as the unique vector $v \in T_xM$ with $F(x,v) = F^*(x,\alpha)$ and $\alpha(v) = F^*(x,\alpha)F(x,v)$.

For a differentiable function $f : M \to \mathbb{R}$, the gradient vector of f at x is defined as the Legendre transform of the derivative of f: $\nabla f(x) := J^*(x, df(x))$. In coordinates, we have

$$\nabla f(x) = \begin{cases} g^{ij}(x, df(x)) \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}, & \text{if } df(x) \neq 0 \\ 0, & \text{if } df(x) = 0 \end{cases}$$

where $g^{ij}(x,\alpha) := \frac{1}{2} \frac{\partial^2 F^*(x,\alpha)^2}{\partial \alpha^i \partial \alpha^j}$. Remark that $(g^{ij}(x,\alpha))_{ij}$ is the inverse matrix of $(g_{ij}(x, J^*(x,\alpha)))_{ij}$.

We fix an arbitrary positive C^∞-measure $\mathfrak{m}$ on M as our base measure. In a local coordinates system, the measure element is given by $d\mathfrak{m} := e^\Phi dx^1 \dots dx^n$. Usually, the Busemann-Hausdorff volume form $d\mathfrak{m}_{BH}$ and the Holmes-Thompson volume form $d\mathfrak{m}_{HT}$ are used. They are defined by

$$d\mathfrak{m}_{BH} := \frac{\omega_n}{Vol(B_xM)} dx^1 \wedge \dots \wedge dx^n,$$

and

$$d\mathfrak{m}_{HT} := \left(\frac{1}{\omega_n} \int_{B_xM} det g_{ij}(x,v) dv^1 \wedge \dots \wedge dv^n \right) dx^1 \wedge \dots \wedge dx^n,$$

where $B_xM := \{v \in T_xM : F(x,v) < 1\}$ and ω_n denotes the volume of the n-dimensional Euclidean ball.

The divergence of a differentiable vector field V on M with respect to $\mathfrak{m}$ is defined by

$$div_{\mathfrak{m}} V := \sum_{i=1}^{n} \left(\frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \Phi}{\partial x^i} \right).$$

Denote by $W^{1,p}(M)$ the completion of $C^\infty(M)$. For a function $f \in W^{1,p}(M)$, its Finsler p-Laplacian ($p > 1$) is defined as

$$\Delta_p(f) := div_{\mathfrak{m}}(F(\nabla f)^{p-2} \nabla f) := div_{\mathfrak{m}}(|\nabla f|^{p-2} \nabla f),$$

where the equality is in the distibutional sense.

For $p = 2$, we obtain the non-linear Shen's Finsler Laplacian:

$$\Delta_2(f) := \Delta(f) = div_{\mathfrak{m}}(\nabla f).$$

This operator is naturally associated to the canonical energy functional E defined on $W^{1,p}(M)\backslash\{0\}$ by

$$E(f) := \frac{\int_M |\nabla f|^p \ d\mathfrak{m}}{\int_M |f|^p \ d\mathfrak{m}}.$$

The first (closed) eigenvalue of the Finsler p-Laplacian is defined by

$$\lambda_{1,p}(M,F) := \inf_{f\in\mathcal{H}_0^p} E(f),$$

where $\mathcal{H}_0^p := \{f \in W^{1,p}(M)\backslash\{0\} : \int_M |f|^{p-2}f \ d\mathfrak{m} = 0\}$. An eigenfunction related to the first eigenvalue is a function $f \in W^{1,p}(M)$ satisfying $\Delta_p f + \lambda_{1,p}(M)|f|^{p-2}f = 0$. We have the following characterization: for all $\varphi \in W^{1,p}(M)$,

$$\int_M |\nabla f|^{p-2} d\varphi(\nabla f) \ d\mathfrak{m} = \lambda_{1,p}(M) \int_M |f|^{p-2} f\varphi \ d\mathfrak{m}.$$

Now, we will recall the construction of a canonical Riemannian metric associated to the Finsler manifold (M, F). See [10, 11] for more details.

2.3 Binet-Legendre metric

In this part, $d\mathfrak{m}_F$ will always denote the Busemann-Hausdorff measure induced by the metric F on M.

Let define a scalar product on the cotangent spaces T_x^*M, $(x \in M)$ by

$$g_F^*(\alpha,\beta) := \frac{n+2}{\lambda(B_xM)} \int_{B_xM} \alpha(v).\beta(v) \ d\lambda(v),$$

where λ is a Lebesgue measure on T_xM.

The Binet-Legendre metric g_F associated to the Finsler metric F is the Riemannian metric dual to the scalar product g_F^* .

Proposition 2.1. *[11] Let (M, F) be a n-dimensional Finsler manifold with finite reversibility constant κ_F and g_F its associated Binet-Legendre metric. Then*

(i) The metric g_F is as smooth as F;

(ii) We have

$$(\kappa_F\sqrt{2n})^{-n-1}\sqrt{g_F} \le F \le (\kappa_F\sqrt{2n})^{n+1}\sqrt{g_F};$$

(iii) If dV_{g_F} denotes the Riemannian volume density of g_F, there is a constant k such that

$$\omega_n k^{-n} dV_{g_F} \le d\mathfrak{m}_F \le \omega_n k^n dV_{g_F},$$

where ω_n denotes the volume of the standard n-dimensional Euclidean ball. In particular, $dV_{g_F} \le d\mathfrak{m}_F$.

Proposition 2.2. *Let (M, F) be a closed n-dimensional Finsler manifold with reversibility constant κ_F and g_F its associated Binet-Legendre metric. Then*

$$\frac{1}{(\kappa_F\sqrt{2n})^{p(n+1)}k^{2n}} \leq \frac{\lambda_{1,p}(M,F)}{\lambda_{1,p}(M,g_F)} \leq (\kappa_F\sqrt{2n})^{p(n+1)}k^{2n},$$

for some constant $k \geq 1$.

Proof. Let f be the eigenfunction relative to the first eigenvalue $\lambda_{1,p}(M, F)$. Then, we have

$$\lambda_{1,p}(M,F) = \frac{\int_M F^*(df)^p \ d\mathfrak{m}_F}{\int_M |f|^p \ d\mathfrak{m}_F}, \tag{2.2}$$

and

$$\int_M |f|^{p-2} f \ d\mathfrak{m}_F = 0. \tag{2.3}$$

Equation (2.3) implies that

$$\int_M |f|^p \ d\mathfrak{m}_F = \max_{s\in\mathbb{R}} \int_M |f+s|^p \ d\mathfrak{m}_F.$$

So, $\lambda_{1,p}(M, F) \leq \frac{\int_M F^*(d(f+s))^p \ d\mathfrak{m}_F}{\int_M |f+s|^p \ d\mathfrak{m}_F}$, $\forall s \in \mathbb{R}$.

In other hand, there exists a unique $s_0 \in \mathbb{R}$ such that

$$\int_M |f+s_0|^p \ dV_{g_F} = \max_{s\in\mathbb{R}} \int_M |f+s|^p \ dV_{g_F} \ \textit{and} \ \int_M |f+s_0|^{p-2}(f+s_0) \ dV_{g_F} = 0. \tag{2.4}$$

Therefore,

$$\begin{aligned} \lambda_{1,p}(M,F) &\leq \frac{\int_M F^*(d(f+s_0))^p \ d\mathfrak{m}_F}{\int_M |f+s_0|^p \ d\mathfrak{m}_F}, \\ &\leq k^{2n}(\kappa_F\sqrt{2n})^{p(n+1)} \frac{\int_M F_0^*(d(f+s_0))^p \ dV_{g_F}}{\int_M |f+s_0|^p \ dV_{g_F}}, \\ &\leq k^{2n}(\kappa_F\sqrt{2n})^{p(n+1)} \lambda_{1,p}(M,g_F), \end{aligned}$$

where we used $(\kappa_F\sqrt{2n})^{-(n+1)}F_0^* \leq F^* \leq (\kappa_F\sqrt{2n})^{n+1}F_0^*$ in the second line with $F_0 := \sqrt{g_F}$, and (2.4) in the last line.

An analogue argument provides the second inequality by exchanging F and F_0. □

Definition 2.2. Two Finsler metrics F_0 and F defined on a smooth manifold M are called bi-Lipschitz if there exists a constant $C > 1$ such that, for any $(x, v) \in TM$,

$$C^{-1}F_0(x,v) \leq F(x,v) \leq CF_0(x,v). \tag{2.5}$$

Example 2.3. Let (M, g) be a Riemannian manifold and β_1, β_2 two 1-form on M such that

$$0 \leq \sup_{x\in M} \|(\beta_1)_x\|_g := b_1 \leq b_2 := \sup_{x\in M} \|(\beta_2)_x\|_g < 1.$$

Then the Randers metrics $F_1 := \sqrt{g} + \beta_1$ and $F_2 := \sqrt{g} + \beta_2$ are bi-Lipschitz:

$$\frac{1 - b_2}{1 + b_1} \leq \frac{F_1}{F_2} \leq \frac{1 + b_1}{1 - b_2}.$$

Particulary, a Randers metric $F = \sqrt{g} + \beta$ and the associated Riemannian metric g are bi-Lipschitz.

Lemma 2.3. *[11] If F and F_0 are Finsler metrics on M satisfying (2.5) for some constant $C > 1$ then the Binet-Legendre metrics g_F and g_{F_0} associated to F and F_0 respectively satisfy*

$$C^{-n}\sqrt{g_{F_0}} \leq \sqrt{g_F} \leq C^n\sqrt{g_{F_0}}.$$

Theorem 2.4. *Let $F,\ F_0$ be two C-bi-Lipschitz Finsler metrics on a closed n- dimensional manifold M. Then, for any $p > 1$, there exists a constant $K(n, p, \kappa, \kappa_0) \geq 1$ depending on p, the dimension n and the reversibility constants κ and κ_0 of F and F_0 respectively such that,*

$$C^{-K} \leq \frac{\lambda_{1,p}(M, F)}{\lambda_{1,p}(M, F_0)} \leq C^K.$$

Proof. Applying Proposition 2.2 to (M, F) and (M, F_0), there are some constants k and k_0 such that

$$\begin{aligned} \frac{1}{(2n\kappa\kappa_0)^{p(n+1)}(kk_0)^{2n}} \frac{\lambda_{1,p}(M, g_F)}{\lambda_{1,p}(M, g_{F_0})} \quad &\leq \quad \frac{\lambda_{1,p}(M, F)}{\lambda_{1,p}(M, F_0)} \\ &\leq \quad (2n\kappa\kappa_0)^{p(n+1)}(kk_0)^{2n} \frac{\lambda_{1,p}(M, g_F)}{\lambda_{1,p}(M, g_{F_0})}. \end{aligned}$$

Furthermore, from Lemma 2.3, we have

$$\frac{1}{C^{n(p+2n)}} \leq \frac{\lambda_{1,p}(M, g_F)}{\lambda_{1,p}(M, g_{F_0})} \leq C^{n(p+2n)}.$$

Then

$$\frac{1}{(2n\kappa\kappa_0)^{p(n+1)}(kk_0)^{2n}C^{n(p+2n)}} \leq \frac{\lambda_{1,p}(M, F)}{\lambda_{1,p}(M, F_0)} \leq (2n\kappa\kappa_0)^{p(n+1)}(kk_0)^{2n}C^{n(p+2n)}.$$

Since $(2n\kappa\kappa_0)^{p(n+1)}(kk_0)^{2n} > 1$, there exists a positive constant $K'(n, p, \kappa, \kappa_0)$ depending on n, p, κ, κ_0 such that $(2n\kappa\kappa_0)^{p(n+1)}(kk_0)^{2n} \leq C^{K'}$. This completes the proof. □

Remark 2.4. One can prove this theorem directly following idea of the proof of Proposition 2.2.

3 Boundedness on conformal class

Let $\mathcal{F}(M)$ be the set of Finsler metrics F on a manifold M with $Vol(M, F) = 1$, where $Vol(M, F)$ denotes the volume of the Finsler manifold (M, F) with respect to the Busemann-Hausdorff measure induced by F. The following holds for the first eigenvalues of the p-Laplacians, $p > 1$:

$$\inf_{F\in\mathcal{F}(M)} \lambda_{1,p}(M,F) = 0.$$

In the Riemannian case the eigenvalues-functional is not generally bounded. For $p = 2$, it is shown that the functional $\lambda_{1,2}$ is bounded when the dimension $n = 2$ and is unbounded when $n \geq 3$, but $\lambda_{1,2}$ is uniformly bounded when restricted to any conformal class. Matei generalizes these results to any $p > 1$ (see [8, 9]). Using mainly Matei's works and Proposition 2.2, we have the following:

Theorem 3.1. *Let (M, F) be a closed Finsler n-dimensional manifold. Then, for any $p \in (1, n]$, there exists a constant $C := C(n, p, \kappa_F, [F])$ depending only on the dimension n, p, the reversibility constant κ_F and the conformal class $[F]$ of F such that,*

$$\lambda_{1,p}(M,F)Vol(M,F)^{\frac{p}{n}} \leq C(n,p,\kappa_F,[F]).$$

Before proving this theorem, let's remark that, in the Mathei's result used ([9]), the dependence on the conformal class of the Riemannian metric comes from the n-conformal volume of the compact Riemannian manifold (M, g) which is defined as

$$V_n^c(M,[g]) := \inf_{\phi\in I_n(M,[g])} \sup_{\gamma\in G_n} Vol(M,(\gamma\circ\phi)^*can),$$

where can denotes the canonical Riemannian metric on the n-dimensional sphere $\mathbb{S}^n$, $G_n := \{\gamma \in Diff(\mathbb{S}^n)|\ \gamma^*can \in [can]\}$ the group of conformal diffeomorphism of $(\mathbb{S}^n, can)$ and $I_n(M, [g]) := \{\phi : M \to \mathbb{S}^n|\ \phi^*can \in [g])\}$ the set of conformal immersion from (M, g) to $(\mathbb{S}^n, can)$. Using a nice property of the Binet-Legendre metric associated to the Finsler metric F, we can obtain a dependence on the conformal class of F.

Proof. From Proposition 2.2, there is a constant $C_1(n, p, \kappa_F)$ depending only on n, p and κ_F such that $\lambda_{1,p}(M, F) \leq C_1\lambda_{1,p}(M, g_F)$, where g_F is the Binet-Legendre metric associated with F.

Set $\alpha^{-1} := Vol(M, g_F)^{\frac{2}{n}}$ and $\tilde{g} := \alpha g_F$. Then, we have

$$Vol(M,\tilde{g}) = \alpha^{\frac{n}{2}} Vol(M,g_F) = 1$$

and

$$\lambda_{1,p}(M,g_F) = \alpha^{\frac{p}{2}}\lambda_{1,p}(M,\tilde{g}).$$

Furthermore, Matei proved in [9] that there exists a constant $C_2(n, p, [\tilde{g}])$[1] depending on n, p and the conformal class of the metric $\tilde{g}$ which satisfy $\lambda_{1,p}(M, \tilde{g}) \leq C_2$.

Hence, by Proposition 2.1, we obtain

$$\lambda_{1,p}(M,F)Vol(M,F)^{\frac{p}{n}} \leq C_1C_2\left(\frac{Vol(M,F)}{Vol(M,g_F)}\right)^{\frac{p}{n}} \leq C_1C_2(\omega_n k^n)^{\frac{p}{n}}.$$

It is known that when F_1 and F_2 are in the same conformal class, then the associated Binet-Legendre metrics g_{F_1} and g_{F_2} are also in the same conformal class. Hence,

[1] In [9], $C_2 = n^{\frac{p}{2}}(n+1)^{|p/2-1|}V_n^c(M,[\tilde{g}])$ where $V_n^c(M,[\tilde{g}])$ denote the conformal volume of $(M,\tilde{g})$

the constant $C_1C_2(\omega_n k^n)^{\frac{p}{n}}$ depends on n, p, κ_F and the conformal class $[F]$ of the metric F.

□

Particulary, for compact surface, we have the following:

Theorem 3.2. *Let (Σ, F) be a compact Finsler surface with genus δ and reversibility constant κ_F. Then, for any $1 < p \leq 2$, there exists a constant $K(p, \kappa_F)$ depending only on p and κ_F such that*

$$\lambda_{1,p}(\Sigma, F) Vol(\Sigma, F)^{\frac{p}{2}} \leq K(p, \kappa_F) \left(\frac{\delta + 3}{2}\right)^{\frac{p}{2}}.$$

Proof. From the proof of Theorem 3.1, there exists a constant $A_1(p, \kappa_F)$ depending on p and κ_F such that $\lambda_{1,p}(\Sigma, F) \leq A_1(p, \kappa_F)\alpha^{\frac{p}{2}}\lambda_{1,p}(\Sigma, \tilde{g})$ where $\tilde{g} := \alpha g_F$ and $\alpha := Vol(\Sigma, g_F)^{-\frac{2}{n}}$. By a result of Matei (see [9]), $\lambda_{1,p}(\Sigma, \tilde{g}) \leq C(p)\left(\frac{\delta+3}{2}\right)^{\frac{p}{2}}$ for some constant C depending only on p. Then, we have

$$\begin{aligned} \lambda_{1,p}(\Sigma, F) Vol(\Sigma, F)^{\frac{p}{2}} &\leq A_1 C \left(\frac{Vol(\Sigma, F)}{Vol(\Sigma, g_F)}\right)^{\frac{p}{2}} \left(\frac{\delta + 3}{2}\right)^{\frac{p}{2}} \\ &\leq A_1(p, \kappa_F) C(p) (\omega_2 k^2)^{\frac{p}{2}} \left(\frac{\delta + 3}{2}\right)^{\frac{p}{2}}. \qquad (3.1) \end{aligned}$$

This completes the proof.

□

Theorem 3.3. *Let (M, F) be a compact Finsler manifold of dimension n. Then for any $p > n$, there exists a conformal metric $\tilde{F} \in [F]$ such that the quantity $\lambda_{1,p}(M, \tilde{F}) Vol(M, \tilde{F})^{\frac{p}{n}}$ can be taken arbitrarily large.*

Proof. Let $K > 0$. From [9], there exists a metric $\tilde{g} := \varphi^2 g_F \in [g_F]$ satisfying

$$\lambda_{1,p}(M, \tilde{g}) Vol(M, \tilde{g})^{\frac{p}{n}} > \frac{K}{C_1},$$

for a fixed positive constant C_1. Consider the metric $\tilde{F} := \varphi F \in [F]$. Then the Binet-Legendre metric associated to $\tilde{F}$ is $\tilde{g}$ (see [10]). Hence, Proposition 2.2 implies $\lambda_{1,p}(M, \tilde{F}) \geq C(n, p, \kappa_{\tilde{F}})\lambda_{1,p}(M, \tilde{g})$ for some constant C and from Proposition 2.1, $Vol(M, \tilde{F}) \geq Vol(M, \tilde{g})$. This implies that $\lambda_{1,p}(M, \tilde{F}) Vol(M, \tilde{F})^{\frac{p}{n}} > K$ taking $C_1 = C(n, p, \kappa_{\tilde{F}})$.

□

4 Randers spaces

Consider a Randers metric $F := \sqrt{g} + \beta$. In local coordinates (x^i, v^i) on TM, we write

$$g(v, w) := g_{ij} v^i w^j, \ \beta(v) = b_i v^i, \ v = v^i \frac{\partial}{\partial x^i}, \ w = w^j \frac{\partial}{\partial x^j}.$$

Denote $\|\beta\|_x := \sqrt{g^{ij}(x)b_i(x)b_j(x)}$ and $\mathbf{b} = \sup_{x\in M} \|\beta\|_x$ where (g^{ij}) stands for the inverse matrix of (g_{ij}).

To prove theorem 1.2, we need the following lemmas:

Lemma 4.1. *[15] For any smooth function f on M, we have*

$$F(\nabla f) = F^*(df) = \frac{\sqrt{(1-\|\beta\|^2)|df|^2 + \langle \beta, df\rangle^2} - \langle \beta, df\rangle}{1-\|\beta\|^2},$$

where

$$|df|_x := \sqrt{g^{ij}(x)\frac{\partial f}{\partial x^i}(x)\frac{\partial f}{\partial x^j}(x)}, \text{ and } \langle \beta, df\rangle_x := g^{ij}(x)b_i(x)\frac{\partial f}{\partial x^j}(x).$$

Lemma 4.2. *[18] The reversibility constant and the 2-uniform concavity constant of the Randers space $(M, F := \sqrt{g} + \beta)$ are given by*

$$\sigma_F = \left(\frac{1+\mathbf{b}}{1-\mathbf{b}}\right)^2 = \kappa_F^2.$$

The first eigenvalue of (M, F) and (M, g) can be controlled by the reversibility constant as the next proposition showing. Note that a similar result is obtained in [12] using Bao-Lackey Laplacian.

Proposition 4.3. *Let $(M, F := \sqrt{g} + \beta, d\mathfrak{m}_{HT})$ be a Randers space, where $d\mathfrak{m}_{HT}$ is the Holmes-Thompson measure. Then we have*

$$\frac{1}{\kappa_F^p}\lambda_{1,p}(M, g) \leq \lambda_{1,p}(M, F) \leq \kappa_F^p \lambda_{1,p}(M, g),$$

where $\lambda_{1,p}(M, g)$ is the first eigenvalue of the Riemannian manifold (M, g).

Proof. Since $d\mathfrak{m}_{HT}$ denotes the Holmes-Thompson measure then it coincides with the Riemannian measure dV_g induced by g. Recall that the first eigenvalue on the Riemannian space (M, g) is defined by

$$\lambda_{1,p}(M, g) := \inf_{f\in\mathcal{H}_0^p} \frac{\int_M |df|^p \, dV_g}{\int_M |f|^p \, dV_g}.$$

Furthermore, from lemma 4.1, we have

$$\frac{1}{\kappa_F}|df| \leq F^*(df) \leq \kappa_F |df|.$$

Indeed, for all $x \in M$,

$1 - \mathbf{b} \leq 1 - \mathbf{b}^2 \leq 1 - \|\beta\|_x^2 \leq 1 + \mathbf{b}^2 \leq 1 + \mathbf{b}$ and

$$\begin{aligned} \sqrt{(1-\|\beta\|^2)|df|^2 + \langle \beta, df\rangle^2} - \langle \beta, df\rangle &\leq |df| + 2|\langle \beta, df\rangle| \\ &\leq (1+2\mathbf{b})|df|. \end{aligned}$$

Then

$$F^*(df) \leq \frac{1+2\mathbf{b}}{1-\mathbf{b}^2}|df| \leq \kappa_F |df|.$$

Also, we have $F^*(df) \geq (1-\mathbf{b})|df| \geq \kappa_F |df|$.

□

As a direct consequence, we have

Corollary 4.4. *Let (M, g) be a Riemannian manifold of dimension n and $(\beta_k)_k$ be a sequence of 1-forms, with $\|\beta_k\| < 1$ for all k, converging to the null 1-form in $\Lambda^1(M)$. Consider the corresponding sequence of Finsler metrics $(F_k)_k$ with $F_k := \sqrt{g} + \beta_k$. Then the real sequence of first eigenvalues $\mu_k = \lambda_{1,p}(M, F_k)$ converges to the first eigenvalue $\mu = \lambda_{1,p}(M, g)$.*

Proof. For all k, we have

$$\frac{1 - \mathbf{b}_k}{1 + \mathbf{b}_k} \leq \frac{\lambda_{1,p}(M, F_k)}{\lambda_{1,p}(M, g)} \leq \frac{1 + \mathbf{b}_k}{1 - \mathbf{b}_k}$$

Since $\beta_k \longrightarrow 0$ then $\mathbf{b}_k \longrightarrow 0$. Hence

$$\lim_{k\to\infty} \frac{\lambda_{1,p}(M, F_k)}{\lambda_{1,p}(M, g)} = 1.$$

□

Corollary 4.5. *Let $(M, F := \sqrt{g} + \beta)$ be a compact Randers manifold. For any $p, q \in \mathbb{R}$ such that $1 < p \leq q$, the positive eigenvalues $\lambda_{1,p}(M, F)$ and $\lambda_{1,p}(M, F)$ satisfy*

$$\frac{p \sqrt[p]{\lambda_{1,p}(M, F)}}{q \sqrt[q]{\lambda_{1,q}(M, F)}} \leq \sigma_F.$$

Proof. Let $1 < p < q$. By Proposition 4.3, we obtain

$$\frac{p \sqrt[p]{\lambda_{1,p}(M, F)}}{q \sqrt[q]{\lambda_{1,q}(M, F)}} \leq \kappa_F^2 \frac{p \sqrt[p]{\lambda_{1,p}(M, g)}}{q \sqrt[q]{\lambda_{1,q}(M, g)}}.$$

However, the map $t \mapsto t \sqrt[t]{\lambda_{1,t}(M, g)}$ is strictly increasing on $(1, \infty)$ (see [8]). Then,

$$\frac{p \sqrt[p]{\lambda_{1,p}(M, F)}}{q \sqrt[q]{\lambda_{1,q}(M, F)}} \leq \kappa_F^2 = \sigma_F.$$

□

5 Cheeger-type inequality

Definition 5.1. Let $(M, F, d\mathfrak{m})$ be a closed n-dimensional Finsler manifold. The Cheeger's constant is defined by

(5.1) $$\mathbf{h}(M) := \inf_{\Gamma} \frac{\min\{A_\pm(\Gamma)\}}{\min\{\mathfrak{m}(D_1), \mathfrak{m}(D_2)\}},$$

where Γ varies over $(n-1)$-dimensional submanifolds of M which divide M into disjoint open submanifolds D_1, D_2 of M with common boundary $\partial D_1 = \partial D_2 = \Gamma$. One denotes $A_\pm(\Gamma)$ the areas of Γ induced by the outward and inward normal vector field $\mathbf{n}_\pm$.

We have the following useful co-area formula:

Lemma 5.1. *[18] Let $(M, F, \mathfrak{m})$ be a Finsler measure space. Let ϕ be a piecewise C^1 function on M such that $\phi^{-1}(\{t\})$ is compact for all $t \in \mathbb{R}$. Then for any continuous function f on M, we have*

$$\int_M f F(\nabla\phi)\ d\mathfrak{m} = \int_{-\infty}^{\infty} \left(\int_{\phi^{-1}(t)} f\ dA_{\mathbf{n}} \right)\ dt,$$

where $\mathbf{n} := \nabla\phi / F(\nabla\phi)$.

Lemma 5.1 yields the following :

Lemma 5.2. *Given a positive function $f \in C^1(M)$. Then, we have*

$$\int_M F(\nabla f)\ d\mathfrak{m} \geq \mathbf{h}(M) \int_M f\ d\mathfrak{m}.$$

Proof. Let $f \in C^1(M)$. From Lemma 5.1, we have

$$\begin{aligned} \int_M F(\nabla f)\ d\mathfrak{m} &= \int_0^{\infty} \left(\int_{f^{-1}(t)} dA_n \right)\ dt \\ &= \int_0^{\infty} A_n(\{f = t\})\ dt \\ &= \int_0^{\infty} \frac{A_n(\{f = t\}}{\mathfrak{m}(\{f \geq t\})} . \mathfrak{m}(\{f \geq t\})\ dt \\ &\geq \inf_t \frac{A_n(\{f = t\}}{\mathfrak{m}(\{f \geq t\})} \int_0^{\infty} \mathfrak{m}(\{f \geq t\})\ dt \\ &\geq \mathbf{h}(M) \int_M f\ d\mathfrak{m}. \end{aligned}$$

□

We now state our Cheeger-type inequality:

Theorem 5.3. *Let $(M, F, \mathfrak{m})$ be a closed Finsler manifold such that the 2-uniform concavity constant $\sigma_F \leq \sigma$. Then*

$$\lambda_{1,p}(M) \geq \left(\frac{\mathbf{h}(M)}{\sigma p} \right)^p .$$

Proof. Let f be a smooth function on M. Let define the positive and the negative parts of f by $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$. Then

$$\begin{aligned}\mathbf{h}(M)\int_M |f|^p \, d\mathfrak{m} &= \mathbf{h}(M)\left(\int_M f_+^p \, d\mathfrak{m} + \int_M f_-^p \, d\mathfrak{m}\right) \\ &\leq \int_M F^*(Df_+^p) \, d\mathfrak{m} + \int_M F^*(Df_-^p) \, d\mathfrak{m} \\ &= p\left[\int_M f_+^{p-1} F^*(Df_+) \, d\mathfrak{m} + \int_M f_-^{p-1} F^*(Df_-) \, d\mathfrak{m}\right] \\ &\leq p\sigma_F \int_M |f|^{p-1} F^*(Df) d\mathfrak{m} \\ &\leq p\sigma \left(\int_M |f|^p \, d\mathfrak{m}\right)^{\frac{p-1}{p}} \left(\int_M F^*(Df)^p \, d\mathfrak{m}\right)^{\frac{1}{p}}.\end{aligned}$$

Hence,

$$\int_M F^*(Df)^p \, d\mathfrak{m} \geq \left(\frac{\mathbf{h}(M)}{p\sigma}\right)^p \int_M |f|^p \, d\mathfrak{m}.$$

Taking the infimum over $\mathcal{H}_0^p(M)$, the inequality follows.

□

In [17], Yau showed that on a n-dimensional compact Riemannian manifold without boundary whose Ricci curvature is bounded from below by $(n-1)K$, the first eigenvalue can be bounded from below in terms of the diameter, the volume of the manifold and the constant K. The authors of [18] gave a finslerian version of this result for the non-linear Shen's Laplacian. As in [18], we use the following Croke-type inequality to obtain the general case:

Proposition 5.4. *[16] Let $(M, F, d\mathfrak{m})$ be a closed Finsler n-dimensional manifold satisfying $Ric \geq (n-1)K$ for some constant K, where $d\mathfrak{m}$ denotes either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then*

$$\mathbf{h}(M) \geq \frac{(n-1)\mathfrak{m}(M)}{2Vol(\mathbb{S}^{n-2})\sigma_F^{4n+\frac{1}{2}} diam(M) \int_0^{diam(M)} \mathfrak{s}_K^{n-1}(t) \, dt},$$

where $diam(M)$ denotes the diameter of M and the function $\mathfrak{s}_K$ is defined by

$$\mathfrak{s}_K(t) := \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}t), & K > 0, \\ t, & K = 0, \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}t), & K < 0. \end{cases}$$

From Theorem 5.3 and Proposition 5.4, we obtain the following Yau-type estimate.

Proposition 5.5. *Let $(M, F, d\mathfrak{m})$ be a n-dimensional closed Finsler manifold whose Ricci curvature satisfies $Ric \geq (n-1)K$ for some real constant K, where $d\mathfrak{m}$ denotes either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then*

$$\lambda_{1,p}(M) \geq \left(\frac{(n-1)\mathfrak{m}(M)}{2pVol(\mathbb{S}^{n-2})\sigma_F^{4n+\frac{3}{2}} diam(M) \int_0^{diam(M)} \mathfrak{s}_K^{n-1}(t) \, dt}\right)^p.$$

Proof. By Proposition 5.4, we have

$$\frac{\mathbf{h}(M)}{p\sigma_F} \geq \frac{(n-1)\mathfrak{m}(M)}{2pVol(\mathbb{S}^{n-2})\sigma_F^{4n+\frac{3}{2}} diam(M) \int_0^{diam(M)} \mathfrak{s}_K^{n-1}(t)\ dt}.$$

A direct application of Theorem 5.3 completes the proof.

□

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