

On the existence of convex functions on Finsler manifolds

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Abstract. We show that a non-compact (forward) complete Finsler manifold whose Holmes-Thompson volume is finite admits no non-trivial convex functions. We apply this result to some Finsler manifolds whose Busemann function is convex.

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1 Introduction

Finsler manifolds are a natural generalization of Riemannian ones in the sense that the metric depends not only on the point, but on the direction as well. This generalization implies the non-reversibility of geodesics, the difficulty of defining angles and many other particular features that distinguish them from Riemannian manifolds. Even though classical Finsler geometry was mainly concerned with the local aspects of the theory, recently a great deal of effort was made to obtain global results in the geometry of Finsler manifolds ([3], [13], [15], [17] and many others).

In a previous paper [16], by extending the results in [9], we have studied the geometry and topology of Finsler manifolds that admit convex functions, showing that such manifolds are subject to some topological restrictions. We recall that a function $f : (M, F) \rightarrow \mathbb{R}$, defined on a (forward) complete Finsler manifold (M, F) , is called *convex* if and only if along every geodesic $\gamma : [a, b] \rightarrow M$, the composed function $\varphi := f \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex, that is

$$(1.1) \quad f \circ \gamma[(1 - \lambda)a + \lambda b] \leq (1 - \lambda)f \circ \gamma(a) + \lambda f \circ \gamma(b), \quad 0 \leq \lambda \leq 1.$$

If the above inequality is strict for all geodesics γ , the function f is called *strictly convex*, and if the equality holds good for all geodesics γ , then f is called *linear*. A function $f : M \rightarrow \mathbb{R}$ is called *locally non-constant* if it is non-constant on any open subset U of M , and *locally constant* otherwise. We are interested in locally non-constant convex functions on M .

It can be easily seen that any non-compact smooth manifold M always admits a complete Riemannian or Finsler metric and a non-trivial smooth function which is convex with respect to this metric (see [9] for the Riemannian case and [16] for the Finsler case).

On the other hand, it was shown by Yau (see [20]) that in the case of a non-compact manifold M , endowed with an *a priori* given complete Riemannian metric g , there is no non-trivial continuous convex function on (M, g) if the Riemannian volume of M is finite.

In the present paper, we are going to generalize Yau's result to the case of Finsler manifolds, namely, *if the non-compact manifold M is endowed with an a priori given (forward) complete Finsler metric, what are the conditions on (M, F) for the existence of non-trivial convex functions.*

Recall that in the case of a Finsler manifold (we do not assume our Finsler norms to be absolute homogeneous), the induced volume is not unique as in the Riemannian case and hence several choices are available (see Section 3). The Busemann-Hausdorff and Holmes-Thompson volumes are the most well known ones.

Here is our main result.

Theorem 1.1. *Let (M, F) be a (forward) complete non-compact Finsler manifold with finite Holmes-Thompson volume. Then any convex function $f : (M, F) \rightarrow \mathbb{R}$ must be constant.*

Since all volume forms are bi-Lipschitz equivalent in the absolute homogeneous case (see for instance [4]), then the result above holds good for any Finslerian volume, that is we have

Corollary 1.2. *Let (M, F) be an absolute homogeneous complete non-compact Finsler manifold endowed with a Finslerian volume measure.*

If the Finsler volume of (M, F) is finite, then any convex function $f : (M, F) \rightarrow \mathbb{R}$ must be constant.

Our present results show that there are many topological restrictions on (forward) complete non-compact Finsler manifolds with infinite Holmes-Thompson volume. Indeed, the topology of Finsler manifolds admitting convex functions was studied in detail in [16], hence the topological structure stated in the main three theorems in [16] hold good for (forward) complete non-compact Finsler manifolds with infinite Holmes-Thompson volume.

Here is the structure of the paper.

In Section 2 we recall the basic setting of Finsler manifolds (M, F) . In special, we present here the properties of the Riemannian volume of the indicatrix SM and the invariance of this volume under the geodesic flow of F .

In Section 3 we introduce the Busemann-Hausdorff and the Holmes-Thompson volumes of a Finsler manifold (M, F) , respectively, and point out the relation with the volume of the indicatrix. In particular, if the Holmes-Thompson volume of (M, F) is finite, then the total measure of the indicatrix SM is also finite (Proposition 3.2).

Section 4 is where we prove Theorem 1.1 by making use of Lemmas 4.1, 4.2, 4.3. In the proof of Lemma 4.3 we use the Poincaré recurrence theorem ([14]).

Finally, in Section 5, we apply Theorem 1.1 to the case of complete Berwald spaces of non-negative flag curvature and obtain that these kind of spaces must have infinite Holmes-Thompson volume (Corollary 5.1). More generally, a (forward) complete

Finsler manifold of non-negative flag curvature whose Finsler-Minkowski norm F_x is 2-uniformly smooth, at each point $x \in M$, must also have infinite Holmes-Thompson volume (Corollary 5.2).

2 Finsler manifolds

Let (M, F) be a (connected) n -dimensional Finsler manifold (see [3] for basics of Finsler geometry).

The fundamental function F of a Finsler structure (M, F) determines and it is determined by the (tangent) *indicatrix*, or the total space of the unit tangent bundle of F , namely

$$SM := \{u \in TM : F(u) = 1\} = \cup_{x \in M} S_x M$$

which is a smooth hypersurface of the tangent space TM . At each $x \in M$ we also have the *indicatrix at x*

$$S_x M := \{v \in T_x M \mid F(x, v) = 1\} = \Sigma_F \cap T_x M$$

which is a smooth, closed, strictly convex hypersurface in $T_x M$.

To give a Finsler structure (M, F) is therefore equivalent to giving a smooth hypersurface $SM \subset TM$ for which the canonical projection $\pi : SM \rightarrow M$ is a surjective submersion and having the property that for each $x \in M$, the π -fiber $S_x M = \pi^{-1}(x)$ is strictly convex including the origin $O_x \in T_x M$.

Recall that the geodesic spray of (M, F) is the vector field S , on the tangent space TM , given by

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where $G^i : TM \rightarrow \mathbb{R}$ are the spray coefficients of (M, F) . For any $u = (x, y) \in TM$, the geodesic flow of (M, F) is the one parameter group of S , i.e.

$$\phi : (-\varepsilon, \varepsilon) \times U \rightarrow TM, \quad u \mapsto \phi_t(u).$$

The following result is well known.

Lemma 2.1 ([18]). *We have*

1.

$$\frac{d}{dt} F(\phi_t(y)) = dF(S_{\phi_t(y)}) = 0,$$

that is $F(\phi_t(y))$ is constant.

2. For any t , we have

$$\frac{d}{dt} [(\phi_t^*)\omega] = \frac{1}{2} d[(\phi_t^*)F^2],$$

where $\omega = g_{ij}(x, y)y^j dx^i$ is the Hilbert form of (M, F) .

It follows

$$(\phi_t^*)d\omega = d\omega.$$

3 Finslerian volumes

In order to fix notations, we recall that the *Euclidean volume form* in \mathbb{R}^n , with the coordinates (x^1, x^2, \dots, x^n) , is the n -form

$$dV_{\mathbb{R}^n} := dx^1 dx^2 \dots dx^n,$$

and the *Euclidean volume* of a bounded open set $\Omega \subset \mathbb{R}^n$ is given by

$$(3.1) \quad \text{Vol}(\Omega) = \text{Vol}_{\mathbb{R}^n}(\Omega) = \int_{\Omega} dV_{\mathbb{R}^n} = \int_{\Omega} dx^1 dx^2 \dots dx^n.$$

More generally, let us consider a Riemannian manifold (M, g) with the *Riemannian volume form*

$$dV_g := \sqrt{g} dx^1 dx^2 \dots dx^n,$$

and hence the *Riemannian volume* of (M, g) can be computed as

$$\text{Vol}(M, g) = \int_M dV_g = \int_M \sqrt{g} dx^1 dx^2 \dots dx^n = \int_M \theta^1 \theta^2 \dots \theta^n,$$

where $\{\theta^1, \theta^2, \dots, \theta^n\}$ is a g -orthonormal co-frame on M .

We remark that this Riemannian volume is uniquely determined by the following two properties:

1. The Riemannian volume in \mathbb{R}^n is the standard Euclidean volume (3.1).
2. The volume is monotone with the metric.

On the other hand, in the Finslerian case, this is not true anymore. Indeed, even if we ask for the Finslerian volume to satisfy the same two properties above, the volume is not uniquely defined, but depends on the choice of a positive function on M . More precisely, a *volume form* $d\mu$ on an n -dimensional Finsler manifold (M, F) is a global defined, non-degenerate n -form on M written in the local coordinates (x^1, x^2, \dots, x^n) of M as

$$(3.2) \quad d\mu = \sigma(x) dx^1 \wedge \dots \wedge dx^n,$$

where σ is a positive function on M (see [4] for details in the absolute homogeneous case).

Depending on the choice of σ several different volume forms are known: the Busemann volume, the Holmes-Thompson volume, etc.

The *Busemann-Hausdorff* volume form is defined as

$$(3.3) \quad dV_{BH} := \sigma_{BH}(x) dx^1 \wedge \dots \wedge dx^n,$$

where

$$(3.4) \quad \sigma_{BH}(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}(\mathcal{B}_x^n M)},$$

here $\mathbb{B}^n(1)$ is the Euclidean unit n -ball, $\mathcal{B}_x^n M = \{y : F(x, y) = 1\}$ is the Finslerian ball and Vol the canonical Euclidean volume.

The *Busemann-Hausdorff* volume of the Finsler manifold (M, F) is defined by

$$\text{vol}_{BH}(M, F) = \int_M dV_{BH}.$$

Using the Brunn-Minkowski theory, Busemann showed in [5] that the Busemann-Hausdorff volume of an n -dimensional normed space equals its n -dimensional Hausdorff volume, hence the naming.

However, we point out that except for the case of absolute homogeneous Finsler manifolds, the Busemann-Hausdorff volume does not have the expected geometrical properties, and hence it is not suitable for the study of Finsler manifolds (see [1] for a description of these properties and the main issues that appear; see also [7] for the Berwald case when the Busemann-Hausdorff volume has some special properties).

Remark 3.1. Observe that the n -ball Euclidean volume is

$$\text{Vol}(\mathbb{B}^n(1)) = \frac{1}{n} \text{Vol}(\mathbb{S}^{n-1}) = \frac{1}{n} \text{Vol}(\mathbb{S}^{n-2}) \int_0^\pi \sin^{n-2}(t) dt.$$

Another volume form naturally associated to a Finsler structure is the *Holmes-Thompson* volume defined by

$$(3.5) \quad \sigma_{HT}(x) := \frac{\text{Vol}(\mathcal{B}_x^n M, g_x)}{\text{Vol}(\mathbb{B}^n(1))} = \frac{1}{\text{Vol}(\mathbb{B}^n(1))} \int_{\mathcal{B}_x^n M} \det g_{ij}(x, y) dy,$$

and the *Holmes-Thompson* volume of the Finsler manifold (M, F) is defined as

$$\text{vol}_{HT}(M, F) = \int_M dV_{HT}.$$

This volume was introduced by Holmes and Thompson in [10] from geometrical reasons as the dual functor of Busemann-Hausdorff volume. It has better geometrical properties than the Busemann-Hausdorff volume and hence we consider it appropriate for the study of Finsler manifolds.

Remark 3.2. 1. If (M, F) is an absolute homogeneous Finsler manifold, then the Busemann-Hausdorff volume is a Hausdorff measure of M , and we have

$$\text{vol}_{HT}(M, F) \leq \text{vol}_{BH}(M, F).$$

(see [8]).

2. If (M, F) is not absolute homogeneous, then the inequality above is not true anymore. Indeed, for instance let $(M, F = \alpha + \beta)$ be a Randers space. Then, one can easily see that

$$\text{vol}_{BH}(M, F) = \int_M (1 - b^2(x)) dV_\alpha \leq \text{vol}(M, \alpha) = \text{vol}_{HT}(M, F),$$

where $b^2(x) = a_{ij}(x)b^i b^j$, and $\text{vol}(M, \alpha)$ is the Riemannian volume of M (see [18]).

In the case of a smooth surface endowed with a positive defined slope metric $(M, F = \frac{\alpha^2}{\alpha - \beta})$, we have

$$\text{vol}_{BH}(M, F) < \text{vol}_{HT}(M, F) < \text{vol}(M, \alpha),$$

where α and β are the same as above (see [6]).

More generally, in the case of an (α, β) , one can compute explicitly the Finslerian volume in terms of the Riemannian volume (see [2]). Indeed, if $(M, F(\alpha, \beta))$ is an (α, β) metric on an n -dimensional manifold M , one denotes

$$(3.6) \quad \begin{aligned} f(b) &:= \frac{\int_0^\pi \sin^{n-2}(t) dt}{\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b \cos(t))^n} dt} \\ g(b) &:= \frac{\int_0^\pi \sin^{n-2}(t) T(b \cos t) dt}{\int_0^\pi \sin^{n-2}(t) dt}, \end{aligned}$$

where $F = \alpha\phi(s)$, $s = \beta/\alpha$, and

$$T(s) := \phi(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi''].$$

Then the Busemann-Hausdorff and Holmes-Thompson volume forms are given by

$$dV_{BH} = f(b)dV_\alpha, \text{ and } dV_{HT} = g(b)dV_\alpha,$$

respectively, where f and g are given by (3.6).

It is remarkable that if the function $T(s) - 1$ is an odd function of s , then $dV_{HT} = dV_\alpha$. This is the case of Randers metrics.

We will consider now the volume induced by the Hilbert form

$$\omega := g_{ij}(x, y)y^j dx^i = p_i dx^i$$

of the Finsler manifold (M, F) .

It follows

$$d\omega = \frac{\partial g_{jk}}{\partial x^i} y^k dx^i \wedge dx^j - g_{ij} dx^i \wedge dy^j,$$

and hence, we have

$$(d\omega)^n = d\omega \wedge \dots \wedge d\omega = (-1)^{\frac{n(n+1)}{2}} n! \det g_{ij}(x, y) dx^1 \wedge \dots \wedge dx^n \wedge dy^1 \wedge \dots \wedge dy^n.$$

The Hilbert form ω induces a volume form on $TM \setminus \{0\}$ defined by

$$dV_\omega := (-1)^{\frac{n(n+1)}{2}} \frac{1}{n!} (d\omega)^n = \det |g(x, y)| dx \wedge dy,$$

where $\det |g(x, y)|$ is the determinant of the matrix $g_{ij}(x, y)$.

Observe that the volume of (M, F) defined as

$$\text{vol}_\omega(M, F) := \frac{1}{\text{Vol}(\mathbb{B}^n(1))} \int_{BM} dV_\omega = \text{vol}_{HT}(M, F),$$

where $BM := \{(x, y) \in TM : F(x, y) < 1\} \subset TM$, is in fact the same as the HT-volume of the Finsler manifold (M, F) .

The following lemma is elementary.

Lemma 3.1. *The following formula holds good*

$$(3.7) \quad \text{vol}_{HT}(M, F) = \frac{1}{(2n-1)\text{Vol}(\mathbb{B}^n(1))} \int_{SM} dV_\omega.$$

Indeed, it is useful to observe first that

$$(3.8) \quad \int_{B_x M} dV_\omega = \frac{1}{(2n-1)} \int_{S_x M} dV_\omega.$$

To see this, it is easy to see that, due to homogeneity, we can identify $T_x M \setminus \{0\}$ with $(0, \infty) \times S_x M$, by

$$y \mapsto \left(F(y), \frac{y}{F(y)} \right).$$

It follows that

$$G = (dt)^2 \oplus t^2 \hat{G},$$

where $t \in (0, \infty)$, G is the Riemannian metric of $T_x M \setminus \{0\}$, that is the Sasakian metric, and \hat{G} is the restriction of G to $S_x M$.

Then

$$\det |G| = t^{2n-2} \det |\hat{G}|,$$

and hence

$$\int_{B_x M} dV_\omega = \int_{B_x M} \det |G| dy = \int_0^1 t^{2n-2} dt \int_{S_x M} dV_\omega,$$

therefore (3.8) follows. By integrating this formula over M we get the formula in Lemma 3.1.

From Lemma 3.1 we obtain

Proposition 3.2. *Let (M, F) be a Finsler metric whose Holmes-Thompson volume is finite. Then the symplectic volume $\text{vol}_\omega(SM) = \int_{SM} dV_\omega$ of SM is also finite.*

We recall for later use the following Liouville-type theorem.

Theorem 3.3. *The volume form dV_ω is invariant under the geodesic flow of (M, G) .*

The proof is trivial taking into account Lemma 2.1.

4 The proof of Theorem 1.1

In the following, let (M, F) be a non-compact (forward) complete Finsler manifold with bounded Holmes-Thompson volume, and let $f : (M, F) \rightarrow \mathbb{R}$ be a convex function on M . We denote again by ϕ the geodesic flow of F on SM .

Taking into account that a convex function cannot be bounded, from the convexity of f it is elementary to see that

Lemma 4.1. *If $\gamma : [0, \infty) \rightarrow M$ is any F -geodesic on M such that $\lim_{i \rightarrow \infty} \gamma(t_i) = \gamma(0)$ for some divergent numerical sequence $\{t_i\}$, $\lim_{i \rightarrow \infty} t_i = \infty$, then $f \circ \gamma : [0, \infty) \rightarrow \mathbb{R}$ must be constant.*

Moreover, we have

Lemma 4.2. *For any open set $U \subset SM$, there is an infinite sequence t_i , $\lim_{i \rightarrow \infty} t_i = \infty$ such that*

$$\phi_{t_i}(U) \cap U \neq \emptyset, \quad \text{for all } t_i,$$

where ϕ_t is the one parameter group generated by the geodesic flow of (M, g) .

Indeed, if we assume the contrary, then there are infinitely many pairwise disjoint open sets with equal measure, which contradicts the fact that SM has finite symplectic volume.

Lemma 4.3. *The set of points*

$$L := \{u \in SM : \lim_{t_i \rightarrow \infty} \phi_{t_i}(u) = u, \text{ for some sequence } t_i \rightarrow \infty\}$$

is dense in SM .

Proof. The result follows from the more general Poincare recurrence theorem ([14]), that is, *the set of recurrent points, of a measure preserving flow on a measure space with bounded measure, is a full measure set.*

Finally, observe that, being of full measure, the set of recurrent vectors must be in fact dense subset of SM . The proof is complete. \square

Remark 4.1. In the proof above we have used the fact that a full measure subset X , of a space E with measure, is dense in E . Observe that the inverse is not true because one can easily construct examples of dense subsets that are not of full measure.

Now the main theorem can be proved.

Proof of the Theorem 1.1. Consider any point $u = (p, v) \in SM$. Since L is dense (see Lemma 4.3), there always exists a sequence of points $u_i \in SM$ converging to u , i.e. $\lim_{i \rightarrow \infty} u_i = u$.

Let γ_u and γ_{u_i} be the geodesics on (M, g) determined by u and u_i .

Observe that Lemma 4.1 implies that $f \circ \gamma_{u_i}$ must be constant for any i . By continuity it follows that $f \circ \gamma$ must also be constant.

Therefore, f is locally constant, thus must be constant on M . \square

5 Corollaries

Recall that a function $f : (M, F) \rightarrow \mathbb{R}$ defined on a non-compact (forward) complete Finsler manifold is called convex if $f \circ \gamma : [0, 1] \rightarrow \mathbb{R}$ is a convex function in the usual sense, for any Finsler geodesic $\gamma : [0, 1] \rightarrow M$. To be non-compact is a necessary condition for the existence of non-trivial convex functions. Indeed, it is trivial to see that if M is compact, then f must be bounded and hence constant.

Let (M, F) be a forward complete boundaryless Finsler manifold. A unit speed globally minimizing geodesic $\gamma : [0, \infty) \rightarrow M$ is called a (*forward*) *ray*. A ray γ is called *maximal* if it is not a proper sub-ray of another ray, i.e. for any $\varepsilon > 0$ its extension to $[-\varepsilon, \infty)$ is not a ray anymore. Moreover, let us assume that (M, F) is bi-complete, i.e. forward and backward complete. A Finslerian unit speed globally minimizing geodesic $\gamma : \mathbb{R} \rightarrow M$ is called a *straight line*. We point out that, even though for

defining rays and straight lines we do not need any completeness hypothesis, without completeness, introducing rays and straight lines would be meaningless.

Let (M, F) be a forward complete boundaryless non-compact Finsler manifold (see [3], [18] for details on the completeness of Finsler manifolds). In Riemannian geometry, the forward and backward completeness are equivalent, hence the words “forward” and “backward” are superfluous, but in Finsler geometry these are not equivalent anymore.

Definition 5.1. If $\gamma : [0, \infty) \rightarrow M$ is a ray in a forward complete boundaryless non-compact Finsler manifold (M, F) , then the function

$$(5.1) \quad \mathbf{b}_\gamma : M \rightarrow \mathbb{R}, \quad \mathbf{b}_\gamma(x) := \lim_{t \rightarrow \infty} \{t - d(x, \gamma(t))\}$$

is called *the Busemann function with respect to γ* , where d is the Finsler distance function.

See [13] and [15] for basic results on Busemann function for Finsler manifolds.

It is known that the Busemann function of a non-compact complete Riemannian manifold of non-negative sectional curvature is convex. However, in the Finslerian case, due to the different behaviour of geodesics and the dependence of the metric on direction, bounded conditions on the flag curvature are not enough to assure the convexity of the Busemann function \mathbf{b}_γ .

The case of Berwald spaces is well understood. Indeed, the Busemann function of any Berwald space of non-negative flag curvature is convex (see [12], [13], [11]).

From our Main Theorem it follows

Corollary 5.1. *The Holmes-Thompson volume of a Berwald space of non-negative flag curvature is infinite.*

Remark 5.2. If (M, F) is a Berwald space of non-negative flag curvature, then Corollary 5.1 can be also proved exactly as in the Riemannian case (see [19] for an elementary proof of the Riemannian case). Indeed, the specific features of Berwald spaces, like the reversibility of geodesics, the vanishing of the tangent curvature and the formula for the second variation of the arc length (see [3] or [18]), make the Riemannian arguments working.

Remark 5.3. Let us also observe that in the Berwald case, the volume conditions obtained above also holds good for the Busemann-Hausdorff volume. Even though we have pointed out that the Busemann-Hausdorff volume is not quite suitable for the study of arbitrary Finsler manifolds, in the Berwald case it has some special properties that make it more useful than in the general case. Indeed, if (M, F) is a Berwald space, then by averaging over the indicatrices, one can obtain a Riemannian metric (actually several Riemannian metrics depending on the averaging formula, see [7]) whose volume is proportional with the Busemann-Hausdorff volume. The details follow easily.

Observe that the papers [11], [12], [13] link the notion of uniform smoothness with the convexity of Busemann function. Indeed, the essential result is that if (M, F) is a non-compact connected (forward) complete Finsler manifold such that

1. it is of non-negative flag curvature,

2. for all $x \in M$, the Finsler-Minkowski norms F_x are 2-uniformly smooth,

then for any reversible ray $\gamma : [0, \infty) \rightarrow M$, the Busemann function \mathbf{b}_γ is convex (see [11] Lemma 3.11, Corollary 3.12).

By combining this result with Theorem 1.1 it results

Corollary 5.2. *The Finsler manifolds with the properties 1, 2 above must have infinite Holmes-Thompson volume.*

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References

- [1] J. C. Alvarez Paiva, G. Berck, *What is wrong with the Hausdorff measure in Finsler spaces*, Advances in Mathematics 204 (2006), 647–663.
- [2] S. Bacso, X. Cheng, Z. Shen, *Curvature properties of (α, β) -metrics*, Advanced Studies in Pure Mathematics **48**, 2007, Finsler Geometry, Sapporo 2005 - In Memory of Makoto Matsumoto, 73–110.
- [3] D. Bao, S. S. Chern, Z. Shen, *An Introduction to Riemann Finsler Geometry*, Springer, GTM **200**, 2000.
- [4] D. Burago, Y. Burago, S. Ivanov, *A Course in Metric Geometry*, GTM, vol. 33, AMS 2001.
- [5] H. Busemann, *Intrinsic Area*, Ann. Math. **48** (1947), 234–267.
- [6] P. Chansri, P. Chansangiam, S. V. Sabau, *The geometry on the slope of a mountain*, preprint 2018.
- [7] M. Crampin, *On the construction of Riemannian metrics for Berwald spaces by averaging*, Houston J. Math., Vol. 40, No. 3 (2014), 737–750.
- [8] C. Duran, *A Volume Comparison Theorem for Finsler manifolds*, Proc. of AMS, Vol. 126, No. 10 (1998), 3079–3082.
- [9] R. Greene, K. Shiohama, *Convex functions on complete noncompact manifolds. Topological structure*, Invent. Math. **63** (1981), 129–157.
- [10] R. D. Holmes, A. C. Thompson, *N -dimensional area and content in Minkowski spaces*, Pacific J. Math. **85** (1979), 77–110.
- [11] M. Kell, *Sectional curvature-type conditions on metric spaces*, J Geom Anal (2018). <https://doi.org/10.1007/s12220-018-0013-7>
- [12] S. Ohta, *Uniform convexity and smoothness, and their applications in Finsler geometry*, Math. Ann. (2009), 343: 669–699.
- [13] S. Ohta, *Splitting theorems for Finsler manifolds of nonnegative Ricci curvature*, J. Reine Angew. Math. 700 (2015), 155–174.
- [14] H. Poincaré, *Sur le problème des trois corps et les équations de la dynamique*, Acta Math. 13 (1890), 1–270. Oeuvres VII, 262–490 (theorem 1 section 8).
- [15] S. V. Sabau, *The co-points are cut points of level sets for Busemann functions*, SIGMA 12, (2016) 036, 12 pages.
- [16] S. V. Sabau, K. Shiohama, *Topology of complete Finsler manifolds admitting convex functions*, 2015, Pacific Journal of Mathematics, Vol. 276 (2015), No. 2, 459–481.

- [17] S. V. Sabau, M. Tanaka, *The cut locus and distance function from a closed subset of a Finsler manifold*, Houston Jour. of Math., Vol. 42, No. 4 (2016), 1157–1197.
- [18] Z. Shen, *Lectures on Finsler Geometry*, World Scientific, 2001.
- [19] H. Wu, *An Elementary method in the study of non-negative curvature*, Acta Math. **142** (1979), 57–78.
- [20] S. T. Yau, *Non-existence of Continuous Convex Functions on Certain Riemannian manifolds*, Math. Ann. **207** (1974), 269–270.

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