On hyper-generalized recurrent Finsler spaces

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Abstract. The aim of the present paper is to investigate new types of recurrence in Finsler geometry, namely, hyper-generalized recurrence and generalized conharmonic recurrence. The properties of such recurrences and their relations to other Finsler recurrences are studied.

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1 Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1, 12, 14, 15, 16, 17]. We shall use the notations of [14].

In what follows, we denote by $\pi : \mathcal{TM} \longrightarrow M$ the subbundle of nonzero vectors tangent to M, $\mathfrak{F}(TM)$ the algebra of C^∞ functions on TM, $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\bar{X}$. The tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\bar{\eta}$ defined by $\bar{\eta}(u) = (u, u)$ for all $u \in \mathcal{TM}$.

We have the following short exact sequence of vector bundles

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{TM}) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

with the well known definitions of the bundle morphisms ρ and γ. The vector space $V_u(\mathcal{TM}) = \{X \in T_u(\mathcal{TM}) : d\pi(X) = 0\}$ is the vertical space to M at u.

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map $K : T\mathcal{TM} \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\bar{\eta}$, called the connection map of D. The vector space $H_u(\mathcal{TM}) = \{X \in T_u(\mathcal{TM}) : K(X) = 0\}$ is called the horizontal space to M at u . The connection D is said to be regular if

$$T_u(\mathcal{TM}) = V_u(\mathcal{TM}) \oplus H_u(\mathcal{TM}) \;\; \forall\, u \in \mathcal{TM}.$$

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If M is endowed with a regular connection, then the vector bundle maps γ, $\rho|_{H(\mathcal{TM})}$ and $K|_{V(\mathcal{TM})}$ are vector bundle isomorphisms. The map $\beta := (\rho|_{H(\mathcal{TM})})^{-1}$ will be called the horizontal map of the connection D.

The horizontal ($(h)h-$) and mixed ($(h)hv-$) torsion tensors of D, denoted by Q and T respectively, are defined by

$$Q(\bar{X},\bar{Y}) = \mathbf{T}(\beta\bar{X},\beta\bar{Y}), \quad T(\bar{X},\bar{Y}) = \mathbf{T}(\gamma\bar{X},\beta\bar{Y}) \quad \forall\, \bar{X},\bar{Y} \in \mathfrak{X}(\pi(M)),$$

where $\mathbf{T}$ is the (classical) torsion tensor field associated with D.

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of D, denoted by R, P and S respectively, are defined by

$$R(\bar{X},\bar{Y})\bar{Z} = \mathbf{K}(\beta\bar{X},\beta\bar{Y})\bar{Z}, \quad P(\bar{X},\bar{Y})\bar{Z} = \mathbf{K}(\beta\bar{X},\gamma\bar{Y})\bar{Z}, \quad S(\bar{X},\bar{Y})\bar{Z} = \mathbf{K}(\gamma\bar{X},\gamma\bar{Y})\bar{Z},$$

where $\mathbf{K}$ is the (classical) curvature tensor field associated with D.

The contracted curvature tensors of D, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ (known also as the (v)h-, (v)hv- and (v)v-torsion tensors respectively), are defined by

$$\widehat{R}(\bar{X},\bar{Y}) = R(\bar{X},\bar{Y})\bar{\eta}, \quad \widehat{P}(\bar{X},\bar{Y}) = P(\bar{X},\bar{Y})\bar{\eta}, \quad \widehat{S}(\bar{X},\bar{Y}) = S(\bar{X},\bar{Y})\bar{\eta}.$$

Theorem 1.1. [16] *Let (M,L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

(a) *∇ is metric: $\nabla g = 0$,*

(b) *The (h)h-torsion of ∇ vanishes: $Q = 0$,*

(c) *The (h)hv-torsion T of ∇ satisfies: $g(T(\bar{X},\bar{Y}),\bar{Z}) = g(T(\bar{X},\bar{Z}),\bar{Y})$.*

Such a connection is called the Cartan connection associated with the Finsler manifold (M,L).

The only linear connection we deal with in this paper is the Cartan connection.

2 Hyper-generalized recurrence

In this section, we introduce and study a new special Finsler space called hyper generalized recurrent Finsler spaces. The properties of such spaces are investigated. Some relations between such recurrence and other Finsler recurrences are obtained.

For a Finsler manifold (M,L), we set the following notations:

$$\begin{aligned}
\overset{h}{\nabla} \quad &: \quad \text{the } h\text{-covariant derivatives associated with Cartan connection } \nabla,\\
\mathrm{Ric} \quad &: \quad \text{the horizontal Ricci curvature tensor of Cartan connection},\\
\mathrm{Ric}_o \quad &: \quad \text{the horizontal Ricci vector form defined by}\\
&\qquad g(\mathrm{Ric}_o\bar{X},\bar{Y}) = \mathrm{Ric}(\bar{X},\bar{Y}),\\
r \quad &: \quad \text{the horizontal scalar curvature of Cartan connection},\\
G(\bar{X},\bar{Y})\bar{Z} \quad &:= \quad g(\bar{X},\bar{Z})\bar{Y} - g(\bar{Y},\bar{Z})\bar{X},\\
C \quad &:= \quad R - \frac{r}{n(n-1)}\,G: \ \text{the concircular curvature tensor},\\
\mathbf{G}(\bar{X},\bar{Y},\bar{Z},\bar{W}) \quad &:= \quad g(G(\bar{X},\bar{Y})\bar{Z},\bar{W}),\\
\mathbf{C}(\bar{X},\bar{Y},\bar{Z},\bar{W}) \quad &:= \quad g(C(\bar{X},\bar{Y})\bar{Z},\bar{W}),\\
\mathbf{R}(\bar{X},\bar{Y},\bar{Z},\bar{W}) \quad &:= \quad g(R(\bar{X},\bar{Y})\bar{Z},\bar{W}).
\end{aligned}$$

A Finsler manifold is said to be horizontally integrable if its horizonal distribution is completely integrable or, equivalently, if $\widehat{R} = 0$.

Firstly, for a Finsler manifold of dimension $n \geq 3$ with non-zero h-curvature tensor R, we define the Kulkarni-Nomizu product $g \wedge \mathrm{Ric}$ of the Finsler metric g and the Ricci curvature tensor Ric [7]:

$$\begin{aligned}(g \wedge \mathrm{Ric})(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &:= g(\bar{X}, \bar{Z})\mathrm{Ric}(\bar{Y}, \bar{W}) + g(\bar{Y}, \bar{W})\mathrm{Ric}(\bar{X}, \bar{Z}) \\ &\quad -g(\bar{X}, \bar{W})\mathrm{Ric}(\bar{Y}, \bar{Z}) - g(\bar{Y}, \bar{Z})\mathrm{Ric}(\bar{X}, \bar{W}).\end{aligned} \tag{2.1}$$

Remark 2.1. One can show that $g \wedge g = 2\mathbf{G}$.

Definition 2.2. Let (M, L) be a Finsler manifold of dimension $n \geq 3$ with non-zero h-curvature tensor R. Then, (M, L) is called hyper-generalized recurrent Finsler manifold, denoted by HGF_n, if

$$\overset{h}{\nabla} \mathbf{R} = A \otimes \mathbf{R} + B \otimes (g \wedge \mathrm{Ric}),$$

where A and B are nonzero scalar 1-forms on TM, called the recurrence forms.

The following result gives some properties for the hyper-generalized recurrent Finsler manifold.

Theorem 2.1. *Let (M, L) be a horizontally integrable hyper-generalized recurrent Finsler manifold of dimension $n \geq 3$ with recurrence forms A and B. Then, we have:*
(a) *if $r \neq 0$, then (M, L) is generalized Ricci recurrent (GRF_n).*

(b) *if r is a nonzero constant, then (M, L) is generalized 2-Ricci recurrent $(G(2RF_n))$.*

(c) *if r is non-constant, then the following relation holds*

$$(A + (n-2)B) \circ Ric_o = \frac{r}{2}(A + 2(n-2)B).$$

Proof.

(a) Let (M, L) be a horizontally integrable hyper-generalized recurrent Finsler manifold of dimension $n \geq 3$ with recurrence forms A and B. Then, by Definition 2.2,

$$\begin{aligned}(\nabla_{\beta\bar{U}}\mathbf{R})(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= A(\bar{U})\mathbf{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) + B(\bar{U})\{g(\bar{X}, \bar{Z})\mathrm{Ric}(\bar{Y}, \bar{W}) \\ &\quad +g(\bar{Y}, \bar{W})\mathrm{Ric}(\bar{X}, \bar{Z}) - g(\bar{X}, \bar{W})\mathrm{Ric}(\bar{Y}, \bar{Z}) \\ &\quad -g(\bar{Y}, \bar{Z})\mathrm{Ric}(\bar{X}, \bar{W})\}.\end{aligned} \tag{2.2}$$

Contracting both sides of the above equation with respect to $\bar{Y}$ and $\bar{W}$, noting that Ric is symmetric (Lemma 2.4 of [11]), we get

$$\begin{aligned}(\nabla_{\beta\bar{U}}\mathrm{Ric})(\bar{X}, \bar{Z}) &= (A(\bar{U}) + (n-2)B(\bar{U}))\mathrm{Ric}(\bar{X}, \bar{Z}) + rB(\bar{U})g(\bar{X}, \bar{Z}) \\ &= A_1(\bar{U})\mathrm{Ric}(\bar{X}, \bar{Z}) + B_1(\bar{U})g(\bar{X}, \bar{Z}),\end{aligned} \tag{2.3}$$

where $A_1 := A + (n-2)B$ and $B_1 := rB$. Since r, A and B are nonzero, then A_1 and B_1 are nonzero. Hence, by Definition 2.2 of [11], it follows that (M, L) is generalized Ricci recurrent.

(**b**) Follows from Equation (2.3), taking into account that r is nonzero constant.

(**c**) Since (M, L) is horizontally integrable, then, using Lemma 2.4 of [11], we have [1]

$$\mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}\,\{(\nabla_{\beta\bar{X}}R)(\bar{Y},\bar{Z},\bar{W})\} \;=\; 0. \tag{2.4}$$

Contracting both sides of Equation (2.3) with respect to $\bar{X}$ and $\bar{Z}$, we get

$$(\overset{h}{\nabla}\, r)(\bar{U}) \;=\; r(A(\bar{U}) + (n-2)B(\bar{U})) + rnB(\bar{U}). \tag{2.5}$$

Again, contracting both sides of Equation (2.3) with respect to $\bar{U}$ and $\bar{X}$, taking into account (2.4) and the symmetry of Ric, we obtain

$$\frac{1}{2}(\overset{h}{\nabla}\, r)(\bar{Z}) \;=\; r(A(\mathrm{Ric}_o\bar{Z}) + (n-2)B(\mathrm{Ric}_o\bar{Z})) + rB(\bar{Z}). \tag{2.6}$$

Now, from (2.5) and (2.6), we obtain

$$(A + (n-2)B)\circ \mathrm{Ric}_o = \frac{r}{2}(A + 2(n-2)B).$$

This completes the proof. □

Theorem 2.2. *Let (M, L) be a horizontally integrable hyper-generalized recurrent Finsler manifold of dimension $n \geq 3$ with recurrence forms A and B. If r is a nonzero constant, then we have:*

(**a**) *the associated 1-forms A and B are related by $A + 2(n-1)B = 0$,*

(**b**) *$\frac{r}{n}$ is an eigenvalue of Ric_o and $\bar{\sigma}$, $\bar{\rho}$ are eigenvectors corresponding to $\frac{r}{n}$, where $\bar{\sigma}$ and $\bar{\rho}$ are defined respectively by $g(\bar{\sigma}, \bar{X}) := A(\bar{X})$ and $g(\bar{\rho}, \bar{X}) := B(\bar{X})$.*

Proof.

(**a**) Follows from Equation (2.5), using the assumption that r is a nonzero constant.

(**b**) From Theorem 2.1(**c**) and the fact that $A + 2(n-1)B = 0$, we conclude that

$$A(\mathrm{Ric}_o\,\bar{X}) = \frac{r}{n}A(\bar{X}), \qquad B(\mathrm{Ric}_o\,\bar{X}) = \frac{r}{n}B(\bar{X}). \tag{2.7}$$

From which, using the symmetry of Ric and the nondegeneracy of g, we get

$$\mathrm{Ric}_o\,\bar{\sigma} = \frac{r}{n}\,\bar{\sigma}, \qquad \mathrm{Ric}_o\,\bar{\rho} = \frac{r}{n}\,\bar{\rho}.$$

This proves the result. □

Theorem 2.3. *Let (M, L) be a horizontally integrable hyper-generalized recurrent Finsler manifold of dimension $n \geq 3$ with recurrence forms A and B. If r is a non-vanishing constant, then we have*

[1] $\mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}$ denotes the cyclic sum over $\bar{X}, \bar{Y}, \bar{Z}$.

(a) $\mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}\{(A\otimes \mathbf{R}+B\otimes (g\wedge Ric))(\bar{X},\bar{Y},\bar{Z},\bar{U},\bar{V})\}=0$

(b) $\overset{h}{\nabla} A$ *and* $\overset{h}{\nabla} B$ *are symmetric,*

(c) $R(\bar{X},\bar{Y})\mathbf{R}=0.$

Proof.

(**a**) Follows from Definition 2.2 and Equation (2.4).

(**b**) By Definition 2.2, we have

$$\overset{h}{\nabla}\,\mathrm{Ric}=(A+(n-2)B)\otimes \mathrm{Ric}+rB\otimes g. \tag{2.8}$$

Again, from the same definition, taking (2.8) into account, one can show that

$$\begin{aligned}\overset{h}{\nabla}\overset{h}{\nabla}\,\mathbf{R} \;&=\; (\overset{h}{\nabla} A+A\otimes A)\otimes \mathbf{R}+2rB\otimes B\otimes \mathbf{G}\\ &\quad +(A\otimes B+\overset{h}{\nabla} B+B\otimes A+(n-2)B\otimes B)\otimes (g\wedge \mathrm{Ric}).\end{aligned}$$

From which, using Lemma 2.4(**g**) of [11], we obtain

$$R(\bar{U},\bar{V})\mathbf{R} \;=\; -(\bar{d}A)(\bar{U},\bar{V})\mathbf{R}-(\bar{d}B)(\bar{U},\bar{V})(g\wedge \mathrm{Ric}), \tag{2.9}$$

where $(\bar{d}A)(\bar{U},\bar{V}):=(\overset{h}{\nabla} A)(\bar{U},\bar{V})-(\overset{h}{\nabla} A)(\bar{V},\bar{U})$.

On the other hand, from Theorem 2.1**(a)**, we conclude that

$$\bar{d}B=-\frac{\bar{d}A}{2(n-1)}. \tag{2.10}$$

Hence, (2.9) and (2.10) yield

$$R(\bar{U},\bar{V})\mathbf{R} \;=\; -(\bar{d}A)(\bar{U},\bar{V})\mathcal{H}, \tag{2.11}$$

where $\mathcal{H}:=\{\mathbf{R}-\frac{1}{2(n-1)}(g\wedge \mathrm{Ric})\}$.

From Equation (2.11) and Lemma 2.4(**g**) of [11], we get [2]

$$\mathfrak{S}_{\bar{U},\bar{V};\;\bar{W},\bar{X};\;\bar{Y},\bar{Z}}\{(\bar{d}A)(\bar{U},\bar{V})\mathcal{H}(\bar{W},\bar{X},\bar{Y},\bar{Z})\} \;=\; 0.$$

This, and the fact that $\mathcal{H}(\bar{X},\bar{Y},\bar{Z},\bar{W})=\mathcal{H}(\bar{Z},\bar{W},\bar{X},\bar{Y})$, imply that

$$\bar{d}A=0. \tag{2.12}$$

Hence, by Equation (2.10), $\bar{d}B=0$.

(**c**) Follows from (2.11) and (2.12). □

Theorem 2.4. *Let (M,L) be a horizontally integrable hyper-generalized recurrent Finsler manifold of dimension $n\geq 3$ with recurrence forms A and B. If $r=0$, then the following relations hold:*

[2] $\mathfrak{S}_{\bar{U},\bar{V};\;\bar{W},\bar{X};\;\bar{Y},\bar{Z}}$ denotes the cyclic sum over the three pairs of arguments $\bar{U},\bar{V};\;\bar{W},\bar{X};\;\bar{Y},\bar{Z}$.

(a) $A \circ Ric_o = 0$, $B \circ Ric_o = 0$,

(b) $A(R(\bar{X}, \bar{Y})\bar{\rho}) = 0$,

(c) $\mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}\{A(\bar{X})B(R(\bar{Y}, \bar{Z})\bar{W})\} = 0$.

Proof.

(**a**) Follows from Equation (2.7) since $r = 0$.

(**b**) Using Equation (2.4), we have

$$\begin{aligned} 0 &= \mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}\{A(\bar{X})\mathbf{R}(\bar{Y}, \bar{Z}, \bar{U}, \bar{V})\} + \mathfrak{S}_{\bar{X},\bar{Y},\bar{Z}}\{B(\bar{X})\{g(\bar{Y}, \bar{U})\mathrm{Ric}(\bar{Z}, \bar{V}) \\ &\quad + g(\bar{Z}, \bar{V})\mathrm{Ric}(\bar{Y}, \bar{U}) - g(\bar{Y}, \bar{V})\mathrm{Ric}(\bar{Z}, \bar{U}) - g(\bar{Z}, \bar{U})\mathrm{Ric}(\bar{Y}, \bar{V})\}\}. \end{aligned} \tag{2.13}$$

Contracting both sides of (2.13) with respect to $\bar{Z}$ and $\bar{V}$ and using **(a)** above, we obtain

$$\begin{aligned} &\{A(\bar{X}) + (n-2)B(\bar{X})\}\mathrm{Ric}(\bar{Y}, \bar{U}) + rB(\bar{X})g(\bar{Y}, \bar{U}) \\ &-\{A(\bar{Y}) + (n-2)B(\bar{Y})\}\mathrm{Ric}(\bar{X}, \bar{U}) + rB(\bar{Y})g(\bar{X}, \bar{U}) \\ &+A(R(\bar{X}, \bar{Y})\bar{U}) + B(\bar{Y})\mathrm{Ric}(\bar{X}, \bar{U}) - B(\bar{X})\mathrm{Ric}(\bar{Y}, \bar{U}) = 0. \end{aligned}$$

From which, by setting $\bar{U} = \bar{\rho}$ and noting that $B(\bar{X}) := g(\bar{X}, \bar{\rho})$ and $B \circ \mathrm{Ric}_o = 0$, we conclude that

$$A(R(\bar{X}, \bar{Y})\bar{\rho}) = 0.$$

(**c**) Follows from Equation (2.13) by setting $\bar{U} = \bar{\rho}$ and taking into account the fact that $B \circ \mathrm{Ric}_o = 0$. □

3 Conharmonic recurrence

In this section, we investigate two types of Finsler recurrence, namely the conharmonic and generalized conharmonic recurrences. Some relations between such recurrences and other Finsler recurrences are obtained.

Definition 3.1. Let (M, L) be Finsler manifold of dimension $n \geq 3$ with nonzero h-curvature tensor R. The π-tensor field $\mathbb{C}$ defined by

$$\mathbb{C} := \mathbf{R} - \frac{1}{(n-2)}(g \wedge \mathrm{Ric}) \tag{3.1}$$

will be called the conharmonic curvature tensor, $g \wedge \mathrm{Ric}$ being the Kulkarni-Nomizu product of g and Ric defined by (2.1).

If the conharmonic curvature tensor $\mathbb{C}$ vanishes, then (M, L) is said to be conharmonically flat.

It should be noted that the conharmonic curvature tensor in *Riemannian geometry* has been thoroughly investigated by many authors, see for example [2, 7]. The above definition is the Finsler version of such tensor.

Definition 3.2. Let (M, L) be a Finsler manifold of dimension $n \geq 3$ with nonzero h-curvature tensor R. Then, (M, L) is said to be:

(a) conharmonically recurrent Finsler manifold ($\bar{C}F_n$) if

$$\overset{h}{\nabla}\, \mathbb{C} = A \otimes \mathbb{C},$$

(b) generalized conharmonically recurrent Finsler manifold ($G\bar{C}F_n$) if

$$\overset{h}{\nabla}\, \mathbb{C} = A \otimes \mathbb{C} + B \otimes G,$$

where A and B are nonzero scalar 1-forms on TM, positively homogenous of degree zero in the directional argument, called the recurrence forms.

In particular, if $\overset{h}{\nabla}\, \mathbb{C} = 0$, then (M, L) is called conharmonically symmetric.

Theorem 3.1. *Let (M, L) be a horizontally integrable hyper-generalized recurrent Finsler manifold of dimension $n \geq 3$ with recurrence forms A and B. Then, we have:*

(a) *if $r \neq 0$, then (M, L) is generalized conharmonically recurrent ($G\bar{C}F_n$).*

(b) *if $r = 0$, then (M, L) is conharmonically recurrent ($\bar{C}F_n$).*

Proof. The proof follows from Definition 3.1, taking into account Equations (2.2) and (2.3). □

Theorem 3.2. *Let (M, L) be a Finsler manifold of dimension $n \geq 3$ with nonzero h-curvature tensor R. If (M, L) satisfies $Ric = \frac{r(n-2)}{2n(n-1)}\, g$, then, we have:*

(a) *(M, L) is concircularly recurrent if and only if it is conharmonically recurrent.*

(b) *(M, L) is generalized concircularly recurrent if and only if it is generalized conharmonically recurrent.*

Proof. The proof follows from the fact that the concircular curvature tensor C and the conharmonic curvature tensor $\mathbb{C}$ coincide, under the given assumption. □

Corollary 3.3. *In a Finsler manifold with nonzero h-curvature tensor R satisfying $Ric = \frac{r(n-2)}{2n(n-1)}\, g$, the two notions of being concircularly symmetric and conharmonically symmetric coincide.*

We end our paper with the following result.

Theorem 3.4. *Let (M, L) be a generalized conharmonically recurrent Finsler manifold of dimension $n \geq 3$ with recurrence forms A and B. Then, we have:*

(a) *if $\overset{h}{\nabla}\, Ric = -\frac{(n-2)}{2}\, B \otimes g$, then (M, L) is HGF_n.*

(b) *if (M, L) is Ricci recurrent with recurrence form A, then it is generalized recurrent.*

(c) *if $\overset{h}{\nabla}\, Ric = A \otimes Ric - \frac{(n-2)}{2}\, B \otimes g$, then (M, L) is recurrent.*

Proof.

(**a**) Since (M, L) is generalized conharmonically recurrent with recurrence forms A and B, then, from Definition 3.2(**b**) and Equation (3.1), we have

$$\begin{aligned} \overset{h}{\nabla}\,\mathbb{C} &= \overset{h}{\nabla}\,\mathbf{R} - \frac{1}{(n-2)}(g \wedge \overset{h}{\nabla}\,\mathrm{Ric}) \\ &= A \otimes \{\mathbf{R} - \frac{1}{(n-2)}(g \wedge \mathrm{Ric})\} + B \otimes G. \end{aligned} \tag{3.2}$$

As $\overset{h}{\nabla}\,\mathrm{Ric} = -\frac{(n-2)}{2} B \otimes g$, the above relation reduces to

$$\overset{h}{\nabla}\,\mathbf{R} = A \otimes \mathbf{R} + D \otimes (g \wedge \mathrm{Ric}),$$

where $D := -\frac{1}{(n-2)}B$. Consequently, (M, L) is HGF_n.

(**b**) Since (M, L) is Ricci recurrent with recurrence form A. Then, we have

$$\overset{h}{\nabla}\,\mathrm{Ric} = A \otimes \mathrm{Ric}.$$

From which, together with Equation (3.2), the result follows.

(**c**) the proof is similar to that of item (**a**). $\square$

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