Generalized Wintgen inequality for totally real submanifolds in LCS-manifolds

Aliya Naaz Siddiqui and Kamran Ahmad

Abstract. In this paper, we derive a generalized Wintgen inequality and obtain a relationship between the normalized scalar curvature and the squared norm of mean curvature for totally real submanifolds and C-totally real submanifolds in LCS-manifolds with respect to the Levi-Civita connection and the quarter symmetric metric connection also. Some immediate consequences are also discussed.

M.S.C. 2010: 53C15, 53C25.
Key words: LCS-manifolds; quarter symmetric metric connection; totally real submanifolds.

1 Introduction

The notion of Lorentzian concircular structure manifolds (LCS-manifolds) was initiated by A.A. Shaikh as a generalization of LP-Sasakian manifolds in [11]. These manifolds have many importance in the general theory of relativity and cosmology (for example [12, 13]). Many researchers have studied LCS-manifolds and obtained interesting results (see [1, 4, 5, 10, 14]).

In [2], Friedmann and Schouten defined the notion of semi-symmetric linear connection on smooth manifolds. Later on, Golab [3] put the idea of quarter symmetric linear connection on such smooth manifolds as a generalization of semi-symmetric connection.

For all the geometric quantities, we are interested whether they are intrinsic or extrinsic. Intrinsic: measures that depend on the geometry of the submanifold. Extrinsic: measures that depend on the geometry of the ambient. Several relationships between intrinsic and extrinsic invariants of different submanifolds in Riemannian manifolds are found till now. P. Wintgen [16] gave the relation among Gauss curvature (intrinsic invariant), normal curvature and square mean curvature (extrinsic invariant) of surfaces in $\mathbb{E}^4$. This is known as the Wintgen inequality. In 1999, De Smet et al. [15] conjectured the generalized Wintgen inequality for submanifolds M^m in real space forms $\overline{M}^{n+s}(k)$ with constant sectional curvature k, for all dimensions

Balkan Journal of Geometry and Its Applications, Vol.24, No.2, 2019, pp. 53-62.

$n \geq 2$ and for all co-dimensions $s \geq 1$. This is also known as the DDVV conjecture and it was proved by Ge and Tang (in 2008) and by Lu (in 2011), independently.

2 LCS-manifolds and submanifolds

Definition 2.1. [9, 11] The Lorentzian manifold $\overline{M}$ together with the unit time-like concircular vector field ξ, its associated 1-form η and a $(1,1)$-tensor field φ is said to be a Lorentzian concircular structure manifold (or LCS-manifold).

We consider $\overline{\nabla}$ is the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies the following:

$$\overline{\nabla}_E \alpha = (E\alpha) = d\alpha(E) = \rho\eta(E) \tag{2.1}$$

for any $E \in \Gamma(T\overline{M})$, where ρ being a certain scalar function given by

$$\rho = -(\xi\alpha). \tag{2.2}$$

In an n- dimensional $(\mathrm{LCS})_n$- manifold $\overline{M}$, $n > 2$, the following relations hold [11]:

$$\eta(\xi) = -1, \quad \varphi\xi = 0, \quad \eta o\varphi = 0, \tag{2.3}$$

$$g(\varphi E, \varphi F) = g(E,F) + \eta(E)\eta(F), \quad \varphi^2 E = E + \eta(E)\xi, \tag{2.4}$$

$$\overline{\Re}(E,F)G = \varphi\overline{\Re}(E,F)G + (\alpha^2 - \rho)\Big[g(F,G)\eta(E) - g(E,G)\eta(F)\Big]\xi \tag{2.5}$$

for any $E, F, G \in \Gamma(T\overline{M})$. Also, we have

$$\overline{\Re}(E,F,G,H) = \overline{\Re}(E,F,G,\varphi H) + (\alpha^2 - \rho)\Big[g(F,G)\eta(E) - g(E,G)\eta(F)\Big]\eta(H) \tag{2.6}$$

for any $E, F, G, H \in \Gamma(T\overline{M})$.

Remark 2.2. If we assume that $\alpha = 1$, then Lorentzian concircular structure becomes LP-Sasakian structure [7].

Let M be an m- dimensional submanifold of an n- dimensional manifold $\overline{M}$ with induced metric g. The Gauss equation is given by [18]

$$\overline{\Re}(E,F,G,H) = \Re(E,F,G,H) + g(\zeta(E,G),\zeta(F,H)) - g(\zeta(E,H),\zeta(F,G)) \tag{2.7}$$

for any $E, F, G, H \in \Gamma(TM)$. Here ζ is the second fundamental form of M in $\overline{M}$.

A submanifold M in $\overline{M}$ is called

(i) totally geodesic if $\zeta \equiv 0$.

(ii) totally umbilical if there is a real number ϵ such that $\zeta(E,F) = \epsilon g(E,F)\mathcal{H}$ for any $E, F \in \Gamma(TM)$.

Definition 2.3. [3] A linear connection $\hat{\overline{\nabla}}$ in an n- dimensional smooth manifold $\overline{M}$ is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(E,F) = \hat{\overline{\nabla}}_E F - \hat{\overline{\nabla}}_F E - [E,F] = \eta(F)\varphi E - \eta(E)\varphi F,$$

where η is an 1-form and φ is a tensor of type $(1,1)$.

Remark 2.4. If we assume that $\varphi E = E$, then the quarter symmetric connection reduces to semi-symmetric connection.

Definition 2.5. [3] The quarter symmetric connection $\hat{\overline{\nabla}}$ is said to be a quarter symmetric metric connection if $\hat{\overline{\nabla}}$ satisfies the following condition:

$$(\hat{\overline{\nabla}}_E g)(F,G) = 0$$

for any $E,F,G,H \in \Gamma(T\overline{M})$.

The relation between quarter symmetric metric connection $\hat{\overline{\nabla}}$ and Riemannian connection $\overline{\nabla}$ on a (LCS)$_n$- manifold $\overline{M}$ is given by [4]

$$\hat{\overline{\nabla}}_E F = \overline{\nabla}_E F + \eta(F)\varphi E - g(\varphi E, F)\xi.$$

Let $\hat{\overline{\Re}}$ and $\overline{\Re}$ be the curvature tensors of a (LCS)$_n$- manifold $\overline{M}$ with respect to quarter symmetric metric connection $\hat{\overline{\nabla}}$ and Riemannian connection $\overline{\nabla}$, then [5]

$$\begin{aligned}\hat{\overline{\Re}}(E,F,G,H) &= \overline{\Re}(E,F,G,H) + (2\alpha-1)\Big[g(\varphi E,G)g(\varphi F,H) - g(\varphi F,G)g(\varphi E,H)\Big] \\ &\quad +\alpha\Big[\eta(F)g(E,H) - \eta(E)g(F,H)\Big]\eta(G) \\ &\quad +\alpha\Big[g(F,G)\eta(E) - g(E,G)\eta(F)\Big]\eta(H) \qquad (2.8)\end{aligned}$$

for any $E,F,G,H \in \Gamma(TM)$.

Let M be an m- dimensional submanifold of an n- dimensional (LCS)$_n$- manifold $\overline{M}$ with respect to quarter symmetric metric connection $\hat{\overline{\nabla}}$ and $\hat{\nabla}$ be the induced connection of M associated to the quarter symmetric metric connection. Also let $\hat{\zeta}$ be the second fundamental form of M with respect to $\hat{\nabla}$. Then the relation (2.7) becomes

$$\hat{\overline{\Re}}(E,F,G,H) = \hat{\Re}(E,F,G,H) + g(\hat{\zeta}(E,G),\hat{\zeta}(F,H)) - g(\hat{\zeta}(E,H),\hat{\zeta}(F,G)) \qquad (2.9)$$

for any $E,F,G,H \in \Gamma(TM)$. Here $\hat{\Re}$ is the curvature tensor of M with respect to induced connection associated to the quarter symmetric metric connection.

Definition 2.6. [17, 18] Let M be a submanifold in a contact metric manifold $\overline{M}$. Then M is said to be

(i) anti-invariant in $\overline{M}$ if for any E tangent to M, φE is normal to M, i.e., $\varphi(TM) \subset T^{\perp}M$ at any point $\wp \in M$, where $T^{\perp}M$ is the normal bundle of M. Thus, M is anti-invariant in $\overline{M}$ if M is normal to the structure vector field ξ.

(ii) C-totally real submanifold in $\overline{M}$ if every tangent vector of M belongs to the contact distribution. Thus, M is a C-totally real submanifold if ξ is normal to M. Hence, C-totally real submanifold is anti-invariant because M is normal to ξ.

For a totally real submanifold and a C-totally real submanifold of a $(\text{LCS})_n$-manifold $\overline{M}$, $\hat{\zeta}$ is given by [5]

$$\hat{\zeta}(E, F) = \zeta(E, F) + \eta(F)\varphi E \tag{2.10}$$

and

$$\hat{\zeta}(E, F) = \zeta(E, F), \tag{2.11}$$

respectively, for any $E, F \in \Gamma(TM)$.

Let $\overline{M}$ be an n-dimensional $(\text{LCS})_n$- manifold and M be an m- dimensional submanifold in $\overline{M}$. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ be an orthonormal basis of $T_{\wp}M$ and $\{\mathcal{E}_{m+1}, \ldots, \mathcal{E}_n\}$ be an orthonormal basis of $T_{\wp}^{\perp}M$ at any $\wp \in M$. Then the scalar curvature $\sigma(\wp)$ at $\wp$ is given by

$$\sigma(\wp) = \sum_{1 \le i < j \le m} \mathbb{K}(\mathcal{E}_i \wedge \mathcal{E}_j) \tag{2.12}$$

and the normalized scalar curvature ϱ is given by

$$\varrho = \frac{2\sigma}{m(m-1)}, \tag{2.13}$$

where $\mathbb{K}(\Gamma)$ denotes the sectional curvature of the plane section $\Gamma \subset T_{\wp}M$. The mean curvature vector $\mathcal{H}$ is defined as

$$\mathcal{H} = \frac{1}{m} \sum_{i,j=1}^{m} \zeta(\mathcal{E}_i, \mathcal{E}_i) \tag{2.14}$$

and the squared norm of mean curvature is given by

$$\mathcal{H}^2 = \frac{1}{m^2} \sum_{a=m+1}^{n} \Big(\sum_{i=1}^{m} \zeta_{ii}^a \Big)^2. \tag{2.15}$$

We also put

$$\zeta_{ij}^a = g(\zeta(\mathcal{E}_i, \mathcal{E}_j), \mathcal{E}_a), \; i, j \in \{1, 2, \ldots, m\}, \; a \in \{m+1, m+2, \ldots, n\}.$$

3 Generalized Wintgen inequality

In [8], Mihai has proved the generalized Wintgen inequality for Lagrnagian submanifolds in complex space forms. In the same paper, he also has obtained another Wintgen inequality for totally real submanifolds in the same ambient space forms.

Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ be an orthonormal basis of the tangent space $\overline{M}$ and $\mathcal{U}$ be a unit tangent vector at $\wp \in \overline{M}^n$ such that $\mathcal{E}_1 = \mathcal{E}$ refracting to M^m, $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ is the orthonormal basis to the tangent space $T_\wp M$ with respect to induced quarter symmetric metric connection. Let us denote the the scalar curvature and normalized scalar curvature of M with respect to induced connection associated to the quarter symmetric metric connection by $\hat{\sigma}(\wp)$ at $\wp$ and $\hat{\varrho}$, respectively. We need the scalar normal curvature $\mathcal{K}_N$ [19] and the normalized scalar normal curvature ϱ_N [8] of M^m. Both terms are defined below:

$$
\begin{aligned}
\mathcal{K}_N &= \frac{1}{4}\sum_{a,b=1}^{n-m} trace[\Lambda_a, \Lambda_b]^2 \\
&= \sum_{1\le a<b\le n-m}\ \sum_{1\le i<j\le m}\Big(g([\Lambda_a,\Lambda_b]\mathcal{E}_i,\mathcal{E}_j)\Big)^2 \\
&= \sum_{1\le a<b\le n-m}\ \sum_{1\le i<j\le m}\Big(\sum_{k=1}^{m}(\zeta^a_{jk}\zeta^b_{ik}-\zeta^a_{ik}\zeta^b_{jk})\Big)^2 \qquad (3.1)
\end{aligned}
$$

and

$$
\varrho_N = \frac{2}{m(m-1)}\sqrt{\mathcal{K}_N} \qquad (3.2)
$$

Now, we prove the following:

Theorem 3.1. *Let M be an m- dimensional totally real submanifold in an n- dimensional $(LCS)_n$- manifold $\overline{M}$ with respect to the quarter symmetric metric connection. Then*

$$
\hat{\varrho} + \varrho_N \le \mathcal{H}^2 - \frac{2m-1}{m(m-1)}\alpha - \frac{1}{m-1}\alpha\eta^2(\mathcal{U}). \qquad (3.3)
$$

The equality case holds if and only if, with respect to some suitable orthonormal frame $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$, the shape operator $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1, \ldots, n\}$ of M in $\overline{M}$ take the following forms

$$
\mathcal{S}_{m+1} = \begin{pmatrix} c_1 & d & 0 & \ldots & 0 \\ d & c_1 & 0 & \ldots & 0 \\ 0 & 0 & c_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & c_1 \end{pmatrix},
$$

$$\mathcal{S}_{m+2} = \begin{pmatrix} c_2+d & 0 & 0 & \dots & 0 \\ 0 & c_2-d & 0 & \dots & 0 \\ 0 & 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_2 \end{pmatrix},$$

$$\mathcal{S}_{m+2} = \begin{pmatrix} c_3 & 0 & 0 & \dots & 0 \\ 0 & c_3 & 0 & \dots & 0 \\ 0 & 0 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_3 \end{pmatrix}, \ \mathcal{S}_{m+4} = \dots = \mathcal{S}_n = 0,$$

where c_1, c_2, c_3 and d are real functions on M.

Proof. The Guass equation gives

$$\sum_{a=1}^{n-m} \sum_{1\le i<j\le m} [\zeta_{ii}^a \zeta_{jj}^a - (\zeta_{ij}^a)^2] = \hat{\sigma} + \frac{2m-1}{2}\alpha + \frac{m}{2}\alpha\eta^2(\mathcal{U}). \tag{3.4}$$

Due to Gauss and Ricci equations, we get (see [8])

$$\begin{aligned} &\textstyle\sum_{a=1}^{n-m} \sum_{1\le i<j\le} (\zeta_{ii}^a - \zeta_{jj}^a)^2 + 2m \sum_{a=1}^{n-m} \sum_{1\le i<j\le m} (\zeta_{ij}^a)^2 \\ &\ge 2m \Big[\textstyle\sum_{1\le a<b\le n-m} \sum_{1\le i<j\le m} \Big(\sum_{k=1}^{m} (\zeta_{jk}^a \zeta_{ik}^b - \zeta_{ik}^a \zeta_{jk}^b) \Big)^2 \Big]^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

On the other hand, we see that

$$\begin{aligned} m^2\mathcal{H}^2 &= \sum_{a=1}^{n-m} \left(\sum_{i=1}^{n} \zeta_{ii}^a \right)^2 \\ &= \frac{1}{m-1} \sum_{a=1}^{n-m} \sum_{1\le i<j\le m} (\zeta_{ii}^a - \zeta_{jj}^a)^2 + 2m(\zeta_{ii}^a \zeta_{jj}^a). \end{aligned} \tag{3.6}$$

Combining (3.1), (3.2), (3.5) and (3.6), we arrive at

$$m^2\mathcal{H}^2 - m^2\varrho_N \ge \frac{2m}{m-1} \sum_{a=1}^{n-m} \sum_{1\le i<j\le m} [\zeta_{ii}^a \zeta_{jj}^a - (\zeta_{ij}^a)^2]. \tag{3.7}$$

Putting (3.7) into (3.4), we find that

$$m^2\mathcal{H}^2 - m^2\varrho_N \ge \frac{2m}{m-1} \Big[\hat{\sigma} + \frac{2m-1}{2}\alpha + \frac{m}{2}\alpha\eta^2(\mathcal{U}) \Big]$$

or

$$\mathcal{H}^2 - \varrho_N \geq \frac{2}{m(m-1)}\left[\hat{\sigma} + \frac{2m-1}{2}\alpha + \frac{m}{2}\alpha\eta^2(\mathcal{U})\right]. \tag{3.8}$$

Since

$$\hat{\varrho} = \frac{2\hat{\sigma}}{m(m-1)},$$

then inequality (3.8) becomes

$$\mathcal{H}^2 - \varrho_N \geq \hat{\varrho} + \frac{2m-1}{m(m-1)}\alpha + \frac{1}{m-1}\alpha\eta^2(\mathcal{U}).$$

Thus, we get our desired inequality (3.3).

The equality case of (3.3) at a point $\wp \in M$ holds identically if and only if we have equality in (3.7). According to [6], the shape operators take the above forms with respect to suitable frames. □

Next, we have a DDVV-type inequality for totally real submanifolds in LCS-manifolds with respect to the Levi-Civita connection.

Theorem 3.2. *Let M be an m- dimensional totally real submanifold in an n- dimensional (LCS)$_n$- manifold $\overline{M}$. Then*

$$\varrho + \varrho_N \leq \mathcal{H}^2 - \frac{\alpha^2 - \rho}{m}.$$

We have some applications:

Corollary 3.3. *Let M be an m- dimensional totally real submanifold in an n- dimensional LP-Sasakian manifold $\overline{M}$ with respect to the quarter symmetric metric connection. Then*

$$\hat{\varrho} + \varrho_N \leq \mathcal{H}^2 - \frac{2m-1}{m(m-1)} - \frac{1}{m-1}\eta^2(\mathcal{U}).$$

Corollary 3.4. *Let M be an m- dimensional totally real submanifold in an n- dimensional LP-Sasakian manifold $\overline{M}$. Then*

$$\varrho + \varrho_N \leq \mathcal{H}^2 - \frac{1-\rho}{m}.$$

Since the scalar curvature and hence the normalized scalar curvature of C-totally real submanifold of a (LCS)$_n$- manifold with respect to induced Levi-Civita connection and induced quarter symmetric metric connection are identical (see [5]). Thus, we have the following:

Theorem 3.5. *Let M be an m- dimensional C-totally real submanifold in an n-dimensional (LCS)$_n$- manifold $\overline{M}$ with respect to the quarter symmetric metric connection (resp. the Levi-Civita connection). Then we have $\varrho + \varrho_N \leq \mathcal{H}^2$.*

4 Some other inequalities

In this section, we establish a sharp inequality between the normalized scalar curvature and the squared norm of mean curvature for totally real submanifolds in $(\mathrm{LCS})_n$-manifold with respect to the quarter symmetric metric connection and the Levi-Civita connection.

Theorem 4.1. *Let M be an m- dimensional totally real submanifold in an n- dimensional $(LCS)_n$- manifold $\overline{M}$ with respect to the quarter symmetric metric connection. Then*

$$\mathcal{H}^2 \geq \hat{\varrho} + \frac{(2m-1)}{m(m-1)}\alpha + \frac{1}{m-1}\alpha\eta^2(\mathcal{U}). \tag{4.1}$$

The equality holds at a point $\wp \in M$ if and only if $\wp$ is a totally umbilical point.

Proof. From Gauss equation, we have

$$m^2\mathcal{H}^2 = 2\hat{\sigma} + (2m-1)\alpha + m\alpha\eta^2(\mathcal{U}) + \sum_{a=1}^{n-m}\sum_{1\leq i<j\leq m}[(\zeta_{ij}^a)^2]. \tag{4.2}$$

We choose the normal vector $\mathcal{E}_{m+1}$ in the direction of the mean curvature vector and the shape operator is given by

$$\mathcal{S}_{m+1} = diag\Big(c_1, c_2, \ldots, c_m\Big)$$

and for $i, j \in \{1, 2, \ldots, m\}, a \in \{m+2, \ldots, n\}$

$$\mathcal{S}_a = (\zeta_{ij}^a), \qquad \sum_{i=1}^{m}\zeta_{ii}^a = 0.$$

Now equation (4.2) takes the following form:

$$m^2\mathcal{H}^2 = \hat{\sigma} + \frac{2m-1}{2}\alpha + \frac{m}{2}\alpha\eta^2(\mathcal{U}) + \sum_{a=m+2}^{n}\sum_{1\leq i<j\leq m}[(\zeta_{ij}^a)^2] + \sum_{i=1}^{m}c_i^2. \tag{4.3}$$

Since

$$\sum_{1\leq i<j\leq m}(c_i - c_j)^2 = (m-1)\sum_{i=1}^{m}c_i^2 - 2\sum_{1\leq i<j\leq m}c_ic_j,$$

then we have

$$\begin{aligned} m^2\mathcal{H}^2 &= \left(\sum_{i=1}^{m}c_i\right)^2 = \sum_{i=1}^{m}c_i^2 + 2\sum_{1\leq i<j\leq m}c_ic_j \\ &= \sum_{i=1}^{m}c_i^2 + (m-1)\sum_{i=1}^{m}c_i^2 - \sum_{1\leq i<j\leq m}(c_i-c_j)^2 \leq m\sum_{i=1}^{m}c_i^2. \end{aligned} \tag{4.4}$$

Thus, equations (4.3) and (4.4) give

$$m(m-1)\mathcal{H}^2 \geq 2\hat{\sigma} + (2m-1)\alpha + m\alpha\eta^2(\mathcal{U}). \tag{4.5}$$

From this we can get our desired inequality.

The equality case of (4.1) at a point $\wp \in M$ holds identically if we have equality in (4.5). Accordingly, $\mathcal{S}_a = 0$ for $a \in \{m+2, \dots, n\}$ and $c_1 = c_2 = \cdots = c_m$. Thus, $\wp$ is a totally umbilical point. Converse part is trivial. □

Similarly, one can prove the following results.

Theorem 4.2. *Let M be an m- dimensional totally real submanifold in an n- dimensional $(LCS)_n$- manifold $\overline{M}$ with respect to the Levi Civita connection. Then*

$$\mathcal{H}^2 \geq \varrho + \frac{\alpha^2 - \rho}{m}. \tag{4.6}$$

The equality holds at a point $\wp \in M$ if and only if $\wp$ is a totally umbilical point.

Theorem 4.3. *Let M be an m- dimensional C-totally real submanifold in an n-dimensional $(LCS)_n$- manifold $\overline{M}$ with respect to the quarter symmetric metric connection (resp. the Levi-Civita connection). Then $\varrho \leq \mathcal{H}^2$.*

References

[1] M. Ateceken, S. K. Hui, *Slant and pseudo-slant submanifolds of LCS-manifolds*, Czechoslovak Math. J., 63 (2013), 177-190.

[2] A. Friedmann, J. A. Schouten, *Uber die geometric derhalbsymmetrischen Ubertragung*, Math. Zeitscr., 21 (1924), 211-223.

[3] S. Golab, *On semi-symmetric and quarter symmetric linear connections*, Tensor N.S., 29 (1975), 249-254.

[4] S. K. Hui, L. I. Piscoran, T. Pal, *Invariant submanifolds of $(LCS)_n$- manifolds with respect to quarter symmetric metric connection*, arXiv:1706.09159 [math.DG] (2017).

[5] S. K. Hui, T. Pal, *Totally real submanifolds of $(LCS)_n$-manifolds*, arXiv:1710.04873v1 [math.DG] (2017).

[6] Z. Lu, *Normal scalar curvature conjecture and its application*, J. Funct. Anal., 261 (2011), 1284-1308.

[7] K. Matsumoto, *On Lorentzian almost paracontact manifolds*, Bull. of Yamagata Univ. Nat. Sci., 12 (1989), 151-156.

[8] I. Mihai, *On the generalized Wintgen inequality for Lagrangian submanifolds in complex space form*, Nonlinear Anal., 95 (2014), 714-720.

[9] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

[10] M. H. Shahid, A. N. Siddiqui, *Optimizations on totally real submanifolds of LCS-manifolds using Casorati curvatures*, Comm. Korean Math. Soc., 34, 2 (2019), 603-614.

[11] A. A. Shaikh, *On Lorentzian almost paracontact manifolds with a structure of the concircular type*, Kyungpook Math. J., 43 (2003), 305-314.

[12] A. A. Shaikh, K. K. Baishya, *On concircular structure spacetimes*, J. Math. Stat., 1 (2005), 129-132.

[13] A. A. Shaikh, K. K. Baishya, *On concircular structure spacetimes II*, American J. Appl. Sci., 3 (2006), 1790-1794.

[14] A. A. Shaikh, Y. Matsuyama, S. K. Hui, *On invariant submanifold of $(LCS)_n$-manifolds*, J. Egyptian Math. Soc., 24 (2016), 263-269.

[15] D. Smet, P. J. Dillen, F. Verstraelen, L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. (Brno) 35 (1999), 115-128.

[16] P. Wintgen, *Sur l'inegalite de Chen-Wilmore*, C. R. Acad. Sci. Paris, Ser. A-B 288 (1979), A993-A995.

[17] S. Yamaguchi, M. Kon, T. Ikawa, *C-totally real submanifolds*, J. Diff. Geom., 11 (1976), 53-64.

[18] K. Yano, M. Kon, *Structures on manifolds*, World Scientific publishing, 1984.

[19] K. Yano, M. Kon, *Anti-invariant Submanifolds*, M. Dekker, New York, 1976.

Authors' addresses:

Aliya Naaz Siddiqui
Department of Mathematics,
Faculty of Natural Sciences,
Jamia Millia Islamia,
New Delhi-110025, India.
E-mail: aliyanaazsiddiqui9@gmail.com

Kamran Ahmad
M.M. Engineering College,
Maharishi Markandeshwar *University*,
Mullana, Ambala-133207, India.
E-mail: qskamrankhan@gmail.com