Generalized Wintgen inequality for some submanifolds in Golden Riemannian space forms

M. A. Choudhary, O. Bahadir and H. Alsulami

Abstract. In the present paper, we obtain the generalized Wintgen inequalities for some submanifolds in Golden Riemannian space forms. We have also discussed the equality cases.

1 Introduction

The theory of Golden ratio has been very interesting topic for researchers of diverse interests for more than 2000 years. In fact it is probably not wrong to say that this number has inspired thinkers of all times like no other number in the history of number theory [13].

In the year 2009, some properties of the induced structure on an invariant submanifold in a Golden Riemannian manifold were investigated by C. Hretcanu and M. Crasmareanu [10]. They [4] also studied geometry of Golden structure on a manifold by using a corresponding almost product structure. A. Gezer, N. Cengiz and A. Salimov [5], worked on Golden semi-Riemannian manifold and defined the horizontal lift of Golden structure in tangent bundle. N. O. Poyraz and Erol Yasar [16] studied lightlike hypersurfaces of Golden semi-Riemannian manifold and obtained several results for screen semi-invariant lightlike hypersurfaces of a Golden semi-Riemannian manifold.

On the other hand, Wintgen inequality is a sharp geometric inequality for surfaces in four dimensional Euclidean space involving Gauss curvature (intrinsic invariants), normal curvature and square mean curvature (extrinsic invariants). In 1979, P. Wintgen [17] established an inequality to prove that the Gauss curvature $\mathcal{K}$, the normal curvature $\mathcal{K}^{\perp}$ and the squared mean curvature $||\mathcal{H}||^2$ for any surface M^2 in E^4 satisfy the following inequality:

$$||\mathcal{H}||^2 \geq \mathcal{K} + |\mathcal{K}^{\perp}|,$$

and the equality holds if and only if the ellipse of curvature of M^2 in E^4 is a circle. Moreover, I. V. Guadalupe et.al. [8] extended it for arbitrary codimension m in real space forms $\overline{M}^{m+2}(c)$ as follows

$$||\mathcal{H}||^2 + c \geq \mathcal{K} + |\mathcal{K}^\perp|,$$

equality case was also discussed by them.

As a generalization to Wintgen inequality, De Smet, Dillen, Verstraelen and Vrancken conjectured inequality for submanifolds in real space form known as generalized Wintgen inequality or DDVV conjecture and it was also independently proved by Ge and Tang [7]. In the recent years, DDVV inequality has been obtained by distinct researchers for different classes of submanifolds in different ambient manifolds (see [15]).

In the present paper, we obtain generalized Wintgen type inequalities for slant, invariant, C-totally real and Lagrangian submanifolds in Golden Riemannian space forms and also discuss the equality cases.

2 Submanifolds in Golden Riemannian manifolds

[4, 6] Let $(\overline{M}, \overline{g})$ be Riemannian manifold and let $\mathcal{G}$ be a $(1,1)$-tensor field on $\overline{M}$ satisfying the following equation

$$\mathcal{D}(X) = X^n + a_n X^{n-1} + ... + a_2 X + a_1 I = 0,$$

where I is the identity transformation and (for $X = \mathcal{G}$) $\mathcal{G}^{n-1}(p), \mathcal{G}^{n-2}(p), ..., \mathcal{G}(p), I$ are linearly independent at every point $p \in \overline{M}$. Then, the polynomial $\mathcal{D}(X)$ is called the structure polynomial. Moreover, if we select the structure polynomial as

- $\mathcal{D}(X) = X^2 + I$, we have an almost complex structure,

- $\mathcal{D}(X) = X^2 - I$, we have an almost product structure,

- $\mathcal{D}(X) = X^2$, we have an almost tangent structure.

[6, 11] A $(1,1)$-tensor field φ is called a Golden structure on $\overline{M}$ if it satisfies

$$\varphi^2 = \varphi + I$$

and $(\overline{M}, \overline{g}, \varphi)$ is called a Golden Riemannian manifold if the Riemannian metric $\overline{g}$ is φ compatible [4, 1]. For a φ-compatible metric, we also observe that

$$\overline{g}(\varphi X, Y) = \overline{g}(X, \varphi Y), \qquad \forall X, Y \in \Gamma(T\overline{M}),$$

where $\Gamma(T\overline{M})$ is the set of all vector fields on $\overline{M}$. Using φX in place of X, we have

$$\overline{g}(\varphi X, \varphi Y) = \overline{g}(\varphi^2 X, Y) = \overline{g}(\varphi X, Y) + \overline{g}(X, Y).$$

[1] Let $\overline{M}$ be differentiable manifold with a tensor field $\mathcal{G}$ of type $(1,1)$ on $\overline{M}$ defining an almost product structure on $\overline{M}$ and admitting a Riemannian metric $\overline{g}$ such that

$$\overline{g}(\mathcal{G}X, Y) = \overline{g}(X, \mathcal{G}Y), \forall X, Y \in \Gamma(T\overline{M}),$$

then $(\overline{M}, \overline{g})$ is called almost product Riemannian manifold. We also notice that Golden structures appear in pairs. Furthermore, we relate Golden structure and product structure as follows [4, 1]:

- An almost product structure $\mathcal{G}$ induces a Golden structure

$$\varphi = \frac{1}{2}(I + \sqrt{5}\mathcal{G})$$

- A Golden structure φ induces an almost product structure

$$\mathcal{G} = \frac{1}{\sqrt{5}}(2\varphi - I).$$

Next, we give example of a Golden Riemannian manifold.

Example 1 [12] Consider the Euclidean 6-space R^6 with standard coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$ and let $\varphi : R^6 \to R^6$ represents $(1, 1)$-tensor field defined by

$$\varphi(x_1, x_2, x_3, x_4, x_5, x_6) = (\psi x_1, \psi x_2, \psi x_3, (1 - \psi)x_4, (1 - \psi)x_5, (1 - \psi)x_6)$$

for any vector field $(x_1, x_2, x_3, x_4, x_5, x_6) \in R^6$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $1 - \psi = \frac{1-\sqrt{5}}{2}$ are the roots of the equation $x^2 = x + 1$. Then we obtain

$$
\begin{aligned}
\varphi^2(x_1, x_2, x_3, x_4, x_5, x_6) &= (\psi^2 x_1, \psi^2 x_2, \psi^2 x_3, (1 - \psi)^2 x_4, (1 - \psi)^2 x_5, (1 - \psi)^2 x_6) \\
&= (\psi x_1, \psi x_2, \psi x_3, (1 - \psi)x_4, (1 - \psi)x_5, (1 - \psi)x_6) \\
&\quad + (x_1, x_2, x_3, x_4, x_5, x_6).
\end{aligned}
$$

Thus, we have $\varphi^2 - \varphi - I = 0$. Moreover, we get

$$< \varphi(x_1, ..., x_6), (y_1, ..., y_6) > = < (x_1, ..., x_6), \varphi(y_1, ..., y_6) >,$$

for each vector fields $(x_1, x_2, x_3, x_4, x_5, x_6), (y_1, y_2, y_3, y_4, y_5, y_6) \in R^6$, where $<,>$ is the standard metric on R^6. Hence, $(R^6, <,>, \varphi)$ is a Golden Riemannian manifold.

Now, let us suppose that M_p and M_q be two real-space forms with constant sectional curvatures c_p and c_q, respectively. Then, the Riemannian curvature tensor R of a locally Golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), g, \varphi)$ is given by [16]

$$
\begin{aligned}
R(X, Y)Z &= \Big(-\frac{(1 - \psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\{g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \\
&\quad - g(\varphi X, Z)\varphi Y\} + \Big(-\frac{(1 - \psi)c_p + \psi c_q}{4}\Big)\{g(\varphi Y, Z)X \\
(2.1) &\quad - g(\varphi X, Z)Y + g(Y, Z)\varphi X - g(X, Z)\varphi Y\}.
\end{aligned}
$$

Let M be submanifold of Golden Riemannian manifold $\overline{M}$ with induced metric g. If ∇ and $\nabla^\perp$ be induced connections on the tangent bundle TM and $TM^\perp$ of M, respectively, then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(TM)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \qquad \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^\perp)$$

where h is the second fundamental form of M and A_N is the shape operator of M with respect to N. The shape operator A_N is related to h by

$$g(A_N X, Y) = g(h(X,Y), N) \qquad \forall X, Y \in \Gamma(TM), \forall N \in \Gamma(TM^\perp).$$

Let us denote the curvature tensors of $\overline{M}$ and M by $\overline{R}$ and R respectively. Then, recall the equation of Gauss as follows [18]

$$\begin{aligned}
\overline{R}(X,Y,Z,W) \quad = \quad & R(X,Y,Z,W) - g(h(X,W), h(Y,Z)) \\
& + g(h(X,Z), h(Y,W)), \quad \forall X, Y, Z, W \in TM,
\end{aligned} \tag{2.2}$$

and the equation of Ricci by [18]

$$g(\overline{R}(X,Y), \xi, \eta) \quad = \quad g(R^\perp(X,Y), \xi, \eta) + g([A_\xi, A_\eta] X, Y), \tag{2.3}$$

for all $\xi, \eta \in \Gamma(TM^\perp)$, where $R^\perp$ is the Riemannian curvature tensor on $TM^\perp$ and $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$.

Let us consider a local orthonormal tangent frame $\{E_1, \ldots, E_n\}$ of the tangent bundle TM of M and a local orthonormal normal frame $\{E_{n+1}, \ldots, E_{2m}\}$ of the normal bundle $T^\perp M$ of M in $\overline{M}$. The mean curvature vector denoted by $\mathcal{H}$ of M is given by

$$\mathcal{H} = \sum_{i=1}^n \frac{1}{n} h(E_i, E_i), \tag{2.4}$$

and the squared norm of the second fundamental form h is defined as

$$||h||^2 = \sum_{i,j=1}^n g\big(h(E_i, E_j), h(E_i, E_j)\big). \tag{2.5}$$

For any $X \in \Gamma(TM)$, we can write $\varphi X = PX + QX$, where P and Q are tangential and normal components of φX. We recall the scalar curvature τ at $p \in M$ by

$$\tau = \sum_{1 \le i < j \le n} R(E_i, E_j, E_j, E_i) \tag{2.6}$$

and the normalized scalar curvature ρ of M as

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \mathcal{K}(e_i \wedge e_j), \tag{2.7}$$

where $\mathcal{K}$ is the sectional curvature function on M. The scalar normal curvature $\mathcal{K}_N$ in terms of the components of the second fundamental form is expressed by [15]

$$\mathcal{K}_N = \sum_{1 \le \alpha < \beta \le 2m-n} \sum_{1 \le i < j \le n} \left(\sum_{k=1}^n h_{jk}^r h_{ik}^s - h_{jk}^r h_{ik}^s \right)^2, \tag{2.8}$$

for all $i, j \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{1, \ldots, 2m - n\}$.

Recall the scalar normal curvature by [15]

$$(2.9) \qquad \rho_N = \frac{2}{n(n-1)} \sqrt{\mathcal{K}_N}.$$

We note that for a C-totally real submanifold M of a Golden Riemannian manifold $\overline{M}$, φ maps each tangent space of M into the normal space, i.e., $\varphi(TM) \subset T^\perp M$. We prove the following result for C-totally real submanifolds.

Lemma 2.1. *Let M be an n-dimensional C-totally real submanifold of a locally Golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), g, \varphi)$. Then*

$$(2.10) \qquad \rho_N \;\leq\; 2\Big(\frac{||\mathcal{H}||^2}{2} - \rho\Big) - 2\Big(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big).$$

Furthermore, (2.10) holds for equality if and only if with respect to some orthonormal tangent frame $\{E_1, \ldots, E_n\}$ and orthonormal normal frame $\{E_{n+1}, \ldots, E_{2m}\}$, the shape operator A, takes the following form:

$$(2.11) \qquad A_{n+1} = \begin{pmatrix} f_1 & g & 0 & \ldots & 0 & 0 \\ g & f_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & f_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & f_1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & f_1 \end{pmatrix},$$

$$(2.12) \qquad A_{n+2} = \begin{pmatrix} f_2 + g & 0 & 0 & \ldots & 0 & 0 \\ 0 & f_2 - g & 0 & \ldots & 0 & 0 \\ 0 & 0 & f_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & f_2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & f_2 \end{pmatrix},$$

$$(2.13) \quad A_{n+3} = \begin{pmatrix} f_3 & 0 & 0 & \ldots & 0 & 0 \\ 0 & f_3 & 0 & \ldots & 0 & 0 \\ 0 & 0 & f_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & f_3 & 0 \\ 0 & 0 & 0 & \ldots & 0 & f_3 \end{pmatrix}, \qquad A_{n+4} = \cdots = A_{2m} = 0,$$

where f_1, f_2, f_3 and g are real functions on M.

Proof. Let M be C-totally real submanifold of a locally Golden product space form $\overline{M}$ and $\{E_1, \ldots, E_n\}$ and $\{E_{n+1}, \ldots, E_{2m}\}$ be orthonormal tangent frame and orthonormal normal frame on M respectively. Then, using (2.1), we have

$$
\begin{aligned}
\overline{R}(E_i, E_j, E_j, E_i) &= \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ g(E_j, E_j)g(E_i, E_i) \\
&\quad - g(E_i, E_j)g(E_j, E_i) + g(\varphi E_j, E_j)g(\varphi E_i, E_i) \\
&\quad - g(\varphi E_i, E_j)g(\varphi E_j, E_i) \Big\} \\
&\quad + \Big(-\frac{(1-\psi)c_p + \psi c_q}{4} \Big) \Big\{ g(\varphi E_j, E_j)g(E_i, E_i) \\
&\quad - g(\varphi E_i, E_j)g(E_j, E_i) + g(E_j, E_j)g(\varphi E_i, E_i) \\
&\quad - g(E_i, E_j)g(\varphi E_j, E_i) \Big\}
\end{aligned}
$$
(2.14)

which in the light of (2.2) implies

$$
(2.15) \quad \tau = n(1-n)\Big(\frac{(1-\psi)c_p - \psi c_q}{4\sqrt{5}} \Big) + \sum_{\alpha=n+1}^{2m} \sum_{1 \le i < j \le n} \Big[h_{ii}^{\alpha} h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2 \Big],
$$

where we have used (2.6). One can also observe that

$$
\begin{aligned}
n^2 ||\mathcal{H}||^2 &= \sum_{\alpha=n+1}^{2m-n} \Big(\sum_{i=1}^{n} h_{ii}^{\alpha} \Big)^2 = \frac{1}{n-1} \sum_{\alpha=n+1}^{2m-n} \sum_{1 \le i < j \le n} (h_{ii}^{\alpha} - h_{jj}^{\alpha})^2 \\
&\quad + \frac{2n}{n-1} \sum_{\alpha=n+1}^{2m-n} \sum_{1 \le i < j \le n} h_{ii}^{\alpha} h_{jj}^{\alpha}.
\end{aligned}
$$
(2.16)

On the other hand, from [14] we have

$$
\begin{aligned}
&\sum_{\alpha=n+1}^{2m-n} \sum_{1 \le i < j \le n} (h_{ii}^{\alpha} - h_{jj}^{\alpha})^2 + 2n \sum_{\alpha=n+1}^{2m-n} \sum_{1 \le i < j \le n} (h_{ij}^{\alpha})^2 \ge \\
&\quad 2n \Big[\sum_{n+1 \le \alpha < \beta \le 2m-n} \sum_{1 \le i < j \le n} \Big(\sum_{k=1}^{n} (h_{jk}^{\alpha} h_{ik}^{\beta} - h_{ik}^{\alpha} h_{jk}^{\beta}) \Big)^2 \Big]^{\frac{1}{2}}
\end{aligned}
$$
(2.17)

Taking into account (2.16) and (2.17) and in view of (2.8), we get

$$
(2.18) \quad n^2 ||\mathcal{H}||^2 - n^2 \rho_N \ge \frac{2n}{n-1} \sum_{\alpha=n+1}^{2m-n} \sum_{1 \le i < j \le n} [h_{ii}^{\alpha} h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2].
$$

Finally, combining (2.15) and (2.18), we find our required result.

Furthermore, equality sign holds in (2.10) if and only if with respect to suitable orthonormal tangent and orthonormal normal frames, the shape operator takes the forms of (2.11), (2.12) and (2.13).   □

Note : A submanifold M for which equality holds in (2.10) is called Wintgen ideal submanifold [9].

3 Generalized Wintgen inequality for slant submanifolds

Let (M, g) be a submanifold of a Golden Riemannian manifold $(\overline{M}, \overline{g}, \varphi)$. For each nonzero vector X tangent to M at any point p if the slant angle between TM and φX is independent of the choice of $p \in M$ and $X \in T_pM$, then M is called a slant submanifold. Observe that submanifold M becomes φ-invariant and φ-anti-invariant if the slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called proper slant (or θ-slant proper) submanifold.

Following the way of [2, 3], we recall the following characterization for slant submanifolds in Golden Riemannian manifolds.

Lemma 3.1. *[1] Let (M, g) be a submanifold of a Golden Riemannian manifold $(\overline{M}, \overline{g}, \varphi)$. Then,*

1. *M is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $P^2 = \lambda(\varphi + I)$. Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.*

2. *$g(PX, PY) = \cos^2\theta(g(X, Y) + g(X, PY))$, for any $X, Y \in \Gamma(TM)$.*

Now, we prove the generalized Wintgen inequality for slant submanifolds of a locally Golden product space form.

Theorem 3.2. *Let M be an n-dimensional θ-slant proper submanifold of a locally Golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), g, \varphi)$. Then*

$$
\begin{aligned}
\rho_N \quad \leq \quad & ||\mathcal{H}||^2 - 2\rho - 2\Big(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{1 + \frac{1}{n(n-1)}tr^2\varphi\Big\} \\
& +2\Big(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\cos^2\theta\Big\{\frac{1}{n-1} + \frac{1}{n(n-1)}trP\Big\} \\
& -\Big(\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\frac{4}{n}tr\varphi.
\end{aligned}
$$

(3.1)

Proof. Assume $\{E_1, \ldots, E_n\}$ and $\{E_{n+1}, \ldots, E_{2m}\}$ to be local orthonormal frame and local orthonormal normal frame on M, respectively. Then, using Gauss equation, we have

$$
\begin{aligned}
R(E_i, E_j, E_j, E_i) \quad = \quad & \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{g(E_j, E_j)g(E_i, E_i) \\
& - \quad g(E_i, E_j)g(E_j, E_i) + g(\varphi E_j, E_j)g(\varphi E_i, E_i) \\
& - \quad g(\varphi E_i, E_j)g(\varphi E_j, E_i)\Big\} \\
& + \quad \Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\Big\{g(\varphi E_j, E_j)g(E_i, E_i) \\
& - \quad g(\varphi E_i, E_j)g(E_j, E_i) + g(E_j, E_j)g(\varphi E_i, E_i) \\
& - \quad g(E_i, E_j)g(\varphi E_j, E_i)\Big\} + g(h(E_i, E_i), h(E_j, E_j)) \\
& - \quad g(h(E_i, E_j), h(E_i, E_j))
\end{aligned}
$$

(3.2)

which in the light of Lemma 3.1 implies

$$
\begin{aligned}
\sum_{1 \le i < j \le n} R(E_i, E_j, E_j, E_i) &= \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{n(1-n) - tr^2\varphi\right\} \\
&+ \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)cos^2\theta(n + trP) \\
&+ \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(1-n)tr\varphi \\
&+ \sum_{\alpha=n+1}^{2m}\sum_{1 \le i < j \le n}\left[h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2\right].
\end{aligned}
\tag{3.3}
$$

But, we also have

$$
2\tau = \sum_{1 \le i < j \le n} R(E_i, E_j, E_j, E_i).
\tag{3.4}
$$

Combining (3.3) and (3.4), we get

$$
\begin{aligned}
2\tau &= \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{n(1-n) - tr^2\varphi\right\} \\
&+ \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)cos^2\theta(n + trP) \\
&+ \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)2(1-n)tr\varphi \\
&+ \sum_{\alpha=n+1}^{2m}\sum_{1 \le i < j \le n}\left[h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2\right].
\end{aligned}
\tag{3.5}
$$

On the other hand, from (2.18) we have

$$
n^2||\mathcal{H}||^2 - n^2\rho_N \ge \frac{2n}{n-1}\sum_{\alpha=n+1}^{2m-n}\sum_{1 \le i < j \le n}[h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2].
\tag{3.6}
$$

Thus, thanks to (2.9), (3.5) and (3.6), we find

$$
\begin{aligned}
\rho_N - ||\mathcal{H}||^2 &\le -2\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right) + 2\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\frac{1}{n(1-n)}tr^2\varphi \\
&+ 2\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)cos^2\theta\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}trP\right\} \\
&- \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{4}{n}tr\varphi - 2\rho
\end{aligned}
$$

whereby proving the required result. ☐

With the help of Theorem 3.2, we establish the generalized Wintgen inequality for invariant submanifold of Golden Riemannian space forms.

Theorem 3.3. *Let M be an n-dimensional invariant submanifold of a locally Golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), g, \varphi)$. Then*

$$\rho_N \leq ||\mathcal{H}||^2 - 2\rho - 2\Big(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{1 + \frac{1}{n(n-1)}tr^2\varphi\Big\}$$
$$+2\Big(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{\frac{1}{n-1} + \frac{1}{n(n-1)}trP\Big\}$$
$$(3.7) \qquad -\Big(\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\frac{4}{n}tr\varphi.$$

Next, we derive the generalized Wintgen inequality for a Lagrangian submanifold of a locally Golden product space form.

Theorem 3.4. *Let M be a Lagrangian submanifold of a locally Golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), g, \varphi)$. Then*

$$(3.8) \quad (\rho^\perp)^2 \geq \frac{2}{n(n-1)}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)^2 + \rho_N^2$$
$$-\frac{4}{n(n-1)}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big) - \rho\Big\}.$$

Proof. Let M be a Lagrangian submanifold of a locally Golden product space form $(\overline{M} = M_p(c_p) \times M_q(c_q), g, \varphi)$ and $\{E_1, \ldots, E_n\}$ be local orthonormal frame on M; then $\{\xi_1 = \phi E_1, \ldots, \xi_n = \phi E_n\}$ is the orthonormal frame in the normal bundle $\Gamma(TM^\perp)$. In the light of Gauss equation, we have

$$2\tau = n(n-1)\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)$$
$$(3.9) \qquad +n^2||\mathcal{H}||^2 - g(h(E_i, E_j), h(E_i, E_j)),$$

where we have taken account of (2.4) and (2.6). Also, using (2.9), in the above equation (3.9), we obtain

$$\rho = \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big) + \frac{n}{n-1}||\mathcal{H}||^2$$
$$(3.10) \qquad -\frac{1}{n(n-1)}g(h(E_i, E_j), h(E_i, E_j)).$$

In view of Cauchy-Schwarz inequality above equation yields

$$(3.11) \qquad ||h||^2 \leq n(n-1)(w_1 - \rho) + n^2||\mathcal{H}||^2,$$

where $w_1 = \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)$.

On the other hand, from (2.3) we have

$$R^\perp(E_i, E_j, \xi_r, \xi_s) = \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\{-(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is})\}$$
$$(3.12) \qquad +g([A_{\xi_r}, A_{\xi_s}]E_i, E_j),$$

for all $i, j \in \{1, \ldots, n\}$ and $r, s \in \{1, \ldots, n\}$. Hence, we obtain

$$
\begin{aligned}
(\tau^{\perp})^2 &= (R^{\perp}(E_i, E_j, \xi_r, \xi_s))^2 \\
&= \frac{n(n-1)}{2}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big)^2 + \mathcal{K}_N \\
&\quad -\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) g(h(E_i, E_j), h(E_i, E_j)) \\
&\quad +\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) g(h(E_i, E_i), h(E_j, E_j))
\end{aligned}
\tag{3.13}
$$

or, we can write above equation as

$$
\begin{aligned}
(\rho^{\perp})^2 \;\geq\; & \frac{2}{n(n-1)}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big)^2 + \rho_N^2 \\
& -\frac{4}{n^2(n-1)^2}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big)||h||^2 \\
& +\frac{4}{(n-1)^2}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big)||\mathcal{H}||^2
\end{aligned}
\tag{3.14}
$$

where we have used (2.9).

Finally, taking account of (3.11) and (3.14), we find

$$
\begin{aligned}
(\rho^{\perp})^2 \;\geq\; & \frac{2}{n(n-1)}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big)^2 + \rho_N^2 \\
& -\frac{4}{n(n-1)}\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big)\Big\{ \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) - \rho \Big\}
\end{aligned}
\tag{3.15}
$$

whereby proving the required result. $\square$

Remark 3.1. (i) The proof of Theorem 3.3 is similar to Theorem 3.2. In fact, using Theorem 3.2 we can obtain inequality (3.7) by putting $\theta = 0$.

(ii) Equality cases hold in the inequalities (3.1) and (3.7) if and only if the shape operator takes the forms as stated in Lemma 2.1.

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Author's address:

Majid Ali Choudhary
Department of Mathematics, School of Sciences,
Maulana Azad National Urdu University,
Hyderabad, India.
E-mail: majid_alichoudhary@yahoo.co.in

Oguzhan Bahadir
Department of Mathematics,
Faculty of Arts and Sciences,
K.S.U. Kahramanmaras, Turkey.
E-mail: oguzbaha@gmail.com

Hamed Alsulami
Department of Mathematics,
Faculty of Science,
King Abdulaziz University, 21589 Jeddah, Saudi Arabia.
E-mail: hhaalsalmi@kau.edu.sa