Generic lightlike submanifolds of an indefinite Kaehler manifold with an (ℓ, m)-type connection

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Abstract. We study generic lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type connection subject to the condition that the characteristic vector field ζ of $\bar{M}$ belongs to our screen distribution $S(TM)$ of M. We provide several new results on such a generic lightlike submanifold. Also, we investigate generic lightlike submanifolds of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric connection subject such that ζ belongs to $S(TM)$.

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1 Introduction

This author introduced a non-symmetric and non-metric connection on semi-Riemannian manifolds $(\bar{M}, \bar{g})$ in paper [5] as follows:

A linear connection $\bar{\nabla}$ on $(\bar{M}, \bar{g})$ is called an $(\ell,\, m)$*-type connection* if this connection $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\,\ell\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y})\}, \tag{1.1}$$

$$\bar{T}(\bar{X}, \bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}, \tag{1.2}$$

where ℓ and m are smooth functions, J is a tensor field of type $(1,1)$ and θ is a 1-form associated with a smooth unit spacelike vector field ζ, which is called the *characteristic vector field*, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. We set $(\ell, m) \neq (0,0)$ and we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. Then we see that *a linear connection* $\bar{\nabla}$ *on* $\bar{M}$ *is an* $(\ell,\, m)$*-type connection if and only if* $\bar{\nabla}$ *satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\{\ell\bar{X} + mJ\bar{X}\}. \tag{1.3}$$

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A lightlike submanifold M of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, J)$, with an indefinite almost complex structure J, is called *generic* if there exists a screen distribution $S(TM)$, which is a complementary non-degenerate distribution of $Rad(TM) = TM \cap TM^{\perp}$ in TM, such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.4}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$ such that $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds of indefinite almost complex manifolds or indefinite almost contact manifolds was introduced by Jin-Lee [6] and later, studied by several authors [2, 3, 4, 7].

The objective of study in this paper is generic lightlike submanifolds of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, J)$ with an (ℓ, m)-type connection subject to the conditions that (1) the tensor field J, defined by (1.1) and (1.2), is identical with the indefinite almost complex structure tensor J of $\bar{M}$, and (2) the characteristic vector field ζ of $\bar{M}$ belongs to $S(TM)$.

2 (ℓ, m)-type connections

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be an indedinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure;

$$J^2\bar{X} = -\bar{X}, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}} J)\bar{Y} = 0. \tag{2.1}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the (ℓ, m)-type connection $\bar{\nabla}$, the third equation of three equations in (2.1) is reduced to

$$(\bar{\nabla}_{\bar{X}} J)\bar{Y} = \ell\{\theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X}\} + m\{\theta(\bar{Y})\bar{X} + \theta(J\bar{Y})J\bar{X}\}. \tag{2.2}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$ of dimension $(m + n)$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank r $(1 \le r \le \min\{m,\, n\})$. In general, due to [1], we can take two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called the *screen distribution* and the *co-screen distribution* of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \;\in \{1, \ldots, r\}, \qquad a, b, c, \ldots \;\in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_1, \cdots, N_r\}$ be a null basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \cdots, \xi_r\}$ is a null basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

A lightlike submanifold $M = (M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is called an *r-lightlike submanifold* [1] if $1 \leq r < \min\{m, n\}$. For an r-lightlike M, we see that $S(TM) \neq \{0\}$ and $S(TM^{\perp}) \neq \{0\}$. In the sequel, by saying that M is a lightlike submanifold we shall mean that it is an r-lightlike submanifold, with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \cdots, \xi_r, N_1, \cdots, N_r, F_{r+1}, \cdots, F_m, E_{r+1}, \cdots, E_n\},$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{E_{r+1}, \cdots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

In this paper, we consider generic lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ equipped with an (ℓ, m)-type connection and a screen distribution $S(TM)$ which contains the characteristic vector field ζ. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{a=r+1}^{n} h_a^s(X, Y) E_a, \tag{2.3}$$

$$\bar{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_{ia}(X) E_a, \tag{2.4}$$

$$\bar{\nabla}_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \lambda_{ai}(X) N_i + \sum_{b=r+1}^{n} \mu_{ab}(X) E_b; \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY) \xi_i, \tag{2.6}$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j, \tag{2.7}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$, respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai} and μ_{ab} are 1-forms on M.

For a generic lightlike submanifold M, from (1.4), we show that the distributions $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{2.8}$$

Consider r-th local null vector fields U_i and V_i, $(n-r)$-th local non-null unit vector fields W_a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{2.9}$$

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a). \tag{2.10}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{2.11}$$

Applying J to (2.11) and using $(2.1)_1$, (2.9) and (2.11), we have

$$F^2X = -X + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{2.12}$$

By using $(2.1)_2$ and (2.11), we obtain

$$\begin{aligned} g(FX, FY) = g(X,Y) &- \sum_{i=1}^{r}\{u_i(X)v_i(Y) + u_i(Y)v_i(X)\} \\ &- \sum_{a=r+1}^{n} \epsilon_a w_a(X)w_a(Y). \end{aligned} \tag{2.13}$$

We say that F is the *structure tensor field* of M.

Using (1.1), (1.2, (2.3)) and (2.11), we see that

$$\begin{aligned} (\nabla_X g)(Y,Z) = &\sum_{i=1}^{r}\{h_i^\ell(X,Y)\eta_i(Z) + h_i^\ell(X,Z)\eta_i(Y)\}, \\ &- \ell\{\theta(Y)g(X,Z) + \theta(Z)g(X,Y)\} \\ &- m\{\theta(Y)\bar{g}(JX,Z) + \theta(Z)\bar{g}(JX,Y)\}, \end{aligned} \tag{2.14}$$

$$T(X,Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{2.15}$$

$$h_i^\ell(X,Y) - h_i^\ell(Y,X) = m\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\}, \tag{2.16}$$

$$h_a^s(X,Y) - h_a^s(Y,X) = m\{\theta(Y)w_a(X) - \theta(X)w_a(Y)\}, \tag{2.17}$$

where η_i's are 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$.

From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. The above local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X, Y) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y) + m\theta(Y)u_i(X), \tag{2.18}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X, Y) - \sum_{k=1}^{r} \lambda_{ak}(X)\eta_k(Y) + \epsilon_a m\theta(Y)w_a(X), \tag{2.19}$$

$$h_i^*(X, PY) = g(A_{N_i}X, PY) + \theta(PY)\{\ell\eta_i(X) + mv_i(X)\}. \tag{2.20}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$, $g(\xi_i, \xi_j) = 0$, $\bar{g}(\xi_i, E_a) = 0$, $\bar{g}(N_i, N_j) = 0$ and $\bar{g}(N_i, E_a) = 0$ by turns, we obtain $\epsilon_b\mu_{ab} + \epsilon_a\mu_{ba} = 0$ and

$$\begin{aligned} &h^\ell_i(X, \xi_j) + h^\ell_j(X, \xi_i) = 0, \qquad && h^s_a(X, \xi_i) = -\epsilon_a\lambda_{ai}(X), \\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, && \bar{g}(A_{E_a}X, N_i) = \epsilon_a\rho_{ia}(X). \end{aligned} \tag{2.21}$$

Furthermore, using $(2.21)_1$ we see that

$$h^\ell_i(X, \xi_i) = 0, \qquad h^\ell_i(\xi_j, \xi_k) = 0, \qquad A^*_{\xi_i}\xi_i = 0. \tag{2.22}$$

Applying $\bar{\nabla}_X$ to $(2.9)_{1,2,3}$ and (2.11) by turns and using (2.2), (2.3) ~ (2.7), (2.10) ~ (2.12) and (2.9) ~ (2.11), we get

$$\begin{aligned} h^\ell_j(X, U_i) &= u_j(A_{N_i}X) + m\theta(U_i)u_j(X) \\ &= h^*_i(X, V_j) + m\theta(U_i)u_j(X) \\ &\qquad - \theta(V_j)\{\ell\eta_i(X) + mv_i(X)\}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} h^s_a(X, U_i) &= w_a(A_{N_i}X) + m\theta(U_i)w_a(X) \\ &= \epsilon_a h^*_i(X, W_a) + m\theta(U_i)w_a(X) \\ &\qquad - \epsilon_a\theta(W_a)\{\ell\eta_i(X) + mv_i(X)\}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} h^s_a(X, V_i) &= w_a(A^*_{\xi_i}X) + m\theta(V_i)w_a(X) \\ &= \epsilon_a h^\ell_i(X, W_a) + m\{\theta(V_i)w_a(X) - \epsilon_a\theta(W_a)u_i(X)\}, \end{aligned} \tag{2.25}$$

$$h^\ell_j(X, V_i) = h^\ell_i(X, V_j) + m\{\theta(V_i)u_j(X) - \theta(V_j)u_i(X)\}, \tag{2.26}$$

$$\begin{aligned} &\epsilon_b\{h^s_b(X, W_a) - m\theta(W_a)w_b(X)\} \\ &\qquad = \epsilon_a\{h^s_a(X, W_b) - m\theta(W_b)w_a(X)\}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \nabla_X U_i &= F(A_{N_i}X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a \\ &\quad + \theta(U_i)\{\ell X + mFX\}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} \nabla_X V_i &= F(A^*_{\xi_i}X) - \sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h^\ell_j(X, \xi_i)U_j \\ &\quad - \sum_{a=r+1}^n \epsilon_a\lambda_{ai}(X)W_a + \theta(V_i)\{\ell X + mFX\}, \end{aligned} \tag{2.29}$$

$$\begin{aligned} \nabla_X W_a &= F(A_{E_a}X) + \sum_{i=1}^r \lambda_{ai}(X)U_i + \sum_{b=r+1}^n \mu_{ab}(X)W_b, \\ &\quad + \theta(W_a)\{\ell X + mFX\}, \end{aligned} \tag{2.30}$$

$$\begin{aligned} (\nabla_X F)Y &= \sum_{i=1}^r u_i(Y)A_{N_i}X + \sum_{a=r+1}^n w_a(Y)A_{E_a}X \\ &\quad - \sum_{i=1}^r h^\ell_i(X, Y)U_i - \sum_{a=r+1}^n h^s_a(X, Y)W_a \\ &\quad + \ell\{\theta(FY)X - \theta(Y)FX\} + m\{\theta(Y)X + \theta(FY)FX\}. \end{aligned} \tag{2.31}$$

Definition 2.1. We say that a lightlike submanifold M of $\bar{M}$ is called
(1) *irrotational* [9] if $\bar{\nabla}_X\xi_i \in \Gamma(TM)$ for all $i \in \{1, \cdots, r\}$,
(2) *solenoidal* [8] if A_{E_a} and A_{N_i} are $S(TM)$-valued,
(3) *statical* [8] if M is both irrotational and solenoidal.

Remark 2.2. From (2.3) and $(2.21)_2$, the item (1) is equivalent to

$$h^{\ell}_j(X,\xi_i) = 0, \qquad h^s_a(X,\xi_i) = \lambda_{ai}(X) = 0. \tag{2.32}$$

By using $(2.21)_4$, the item (2) is equivalent to

$$\eta_j(A_{N_i}X) = 0, \qquad \rho_{ia}(X) = \eta_i(A_{E_a}X) = 0. \tag{2.33}$$

For an irrotational M, taking $Y = \xi_j$ to (2.17) and using $(2.32)_1$, we get

$$h^{\ell}_i(\xi_j, X) = 0.$$

Taking $X = \xi_j$ to (2.18) and using the last equation, we obtain

$$A^*_{\xi_i}\xi_j = 0. \tag{2.34}$$

3 Some results

Theorem 3.1. *There exist no generic lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ,m)-type connection such that ζ belons to $S(TM)$ and F is parallel with respect to the connection ∇ on M.*

Proof. Taking the scalar product with N_i to (2.31) with $\nabla_X F = 0$, we get

$$\begin{aligned}\sum_{k=1}^{r} u_k(Y)\eta_i(A_{N_k}X) + \sum_{a=r+1}^{n} w_a(Y)\eta_i(A_{E_a}X)& \\ + \{\ell\eta_i(X) + mv_i(X)\}\theta(FY) - \{\ell v_i(X) - m\eta_i(X)\}\theta(Y) &= 0.\end{aligned} \tag{3.1}$$

Replacing Y by ξ_j to (3.1) and using the fact that $F\xi_j = -V_j$, we have

$$\{\ell\eta_i(X) + mv_i(X)\}\theta(V_j) = 0.$$

Taking $X = \xi_j$ and $X = V_j$ to this equation by turns, we obtain

$$\ell\theta(V_i) = 0, \qquad m\theta(V_i) = 0 \qquad \forall\, i. \tag{3.2}$$

Replacing Y by ξ_j to (2.31) with $\nabla_X F = 0$ and using (3.2), we obtain

$$\sum_{k=1}^{r} h^{\ell}_k(X,\xi_j)U_k + \sum_{a=r+1}^{n} h^s_a(X,\xi_j)W_a = 0.$$

Taking the scalar product with V_i and W_b to this by turns, we have

$$h^{\ell}_i(X,\xi_j) = 0, \qquad h^s_a(X,\xi_i) = 0. \tag{3.3}$$

Taking $Y = U_j$ to (3.1) and using the fact that $FU_j = 0$, we obtain

$$\eta_i(A_{N_j}X) = \{\ell v_i(X) - m\eta_i(X)\}\theta(U_j). \tag{3.4}$$

Taking $i = j$ to (3.4) and using $(2.21)_3$, we obtain

$$\{\ell v_i(X) - m\eta_i(X)\}\theta(U_i) = 0.$$

Taking $X = V_j$ and $X = \xi_j$ to this equation by turns, we have

$$\ell\theta(U_i) = 0, \qquad m\theta(U_i) = 0, \qquad \forall\, i. \tag{3.5}$$

From (3.4) and (3.5), we see that

$$\eta_i(A_{N_j}X) = 0. \tag{3.6}$$

Taking $Y = W_a$ to (2.31) with $\nabla_X F = 0$ and using (3.5), we obtain

$$\begin{aligned} A_{E_a}X = \sum_{i=1}^{r} h_i^\ell(X, W_a)U_i + \sum_{b=r+1}^{n} h_b^s(X, W_a)W_b \\ + \theta(W_a)\{\ell FX - mX\}. \end{aligned} \tag{3.7}$$

Taking the scalar product with U_i to (3.7) and using (2.19), we have

$$h_a^s(X, U_i) = -\epsilon_a\theta(W_a)\{\ell\eta_i(X) + mv_i(X)\}.$$

Taking $X = \xi_i$ to this, we have $h_a^s(\xi_i, U_i) = -\epsilon_a\ell\theta(W_a)$. Also, taking $X = U_i$ to $(3.3)_2$, we have $h_a^s(U_i, \xi_i) = 0$. Taking $X = U_i$ and $Y = \xi_i$ to (2.17), we have $h_a^s(U_i, \xi_i) = h_a^s(\xi_i, U_i)$. Therefore, we get $\ell\theta(W_a) = 0$. Taking the scalar product with N_i to (3.7) and using $\ell\theta(W_a) = 0$, we obtain

$$\eta_i(A_{E_a}X) = -\epsilon_a m\theta(W_a)v_i(X). \tag{3.8}$$

Replacing X by ξ_j to (3.1) and using (3.6) and (3.8), we have

$$\ell\theta(FY) + m\theta(Y) = 0. \tag{3.9}$$

Taking $Y = W_a$ to this, we have $m\theta(W_a) = 0$. Thus, from (3.8), we get

$$\eta_i(A_{E_a}X) = 0. \tag{3.10}$$

Using (3.6), (3.9) and (3.10), the equation (3.1) is reduced to

$$m\theta(FY) - \ell\theta(Y) = 0.$$

As $(\ell.m) \neq (0, 0)$, from (3.9) and the last equation, we see that $\theta(X) = 0$, *i.e.*, $g(\zeta, X) = 0$ for all $X \in \Gamma(TM)$. As ζ belongs to $S(TM)$, we see that $\zeta = 0$. Hence $\theta = 0$. It is a contradiction to $\theta \neq 0$. □

Theorem 3.2. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type connection such that ζ belongs to $S(TM)$. If U_is are parallel with respect to the induced connection ∇ of M, then $\tau_{ij} = 0$ for all i and j, and M is solenoidal.*

Proof. Assume that U_is are parallel with respect to the connection ∇. Taking the scalar product with U_j to (2.28) with $\nabla_X U_i = 0$, we have

$$\eta_j(A_{N_i}X) = \theta(U_i)\{\ell v_j(X) - m\eta_j(X)\}.$$

Taking $i = j$ to this equation and using $(2.21)_3$, we obtain

$$\theta(U_j)\{\ell v_j(X) - m\eta_j(X)\} = 0.$$

Taking $X = V_i$ and $X = \xi_i$ to this equation, we have

$$\ell\theta(U_i) = 0, \qquad m\theta(U_i) = 0, \qquad \eta_j(A_{N_i}X) = 0. \tag{3.11}$$

Taking the scalar product with V_j, W_a and N_j to (2.28) by turns, we have

$$\tau_{ij} = 0, \qquad \rho_{ia}(X) = \eta_i(A_{E_a}X) = 0, \qquad h_i^*(X, U_j) = 0. \tag{3.12}$$

From $(3.11)_3$ and $(3.12)_{1,2}$, we see that $\tau_{ij} = 0$ and M is solenoidal. □

Theorem 3.3. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type connection such that ζ belongs to $S(TM)$. If V_is are parallel with respect to ∇, then M is irrotational.*

Proof. Assume that V_is are parallel with respect to ∇. Taking the scalar product with N_j to (2.29) with $\nabla_X V_i = 0$ and using (2.18), we have

$$h_i^\ell(X, U_j) = m\theta(U_j)u_i(X) - \theta(V_i)\{\ell\eta_j(X) + mv_j(X)\}. \tag{3.13}$$

From (2.23) and (3.13), we see that

$$h_i^*(Y, V_j) = 0. \tag{3.14}$$

Taking $X = \xi_i$ to (3.13) and using (2.16) and $(2.22)_2$, we obtain

$$\ell\theta(V_i) = 0. \tag{3.15}$$

Taking the scalar product with V_j and W_a to (2.29) with $\nabla_X V_i = 0$ by turns and using (3.15): $\ell\theta(V_i) = 0$, we have

$$h_i^\ell(X, \xi_j) = 0, \qquad \lambda_{ai}(X) = h_a^s(X, \xi_i) = 0. \tag{3.16}$$

Thus, due to (2.32), we see that M is irrotational. □

4 Indefinite complex space forms

Definition 4.1. An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\begin{aligned} \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{c}{4}\{&\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} \\ &+ \bar{g}(J\bar{Y}, \bar{Z})J\bar{X} - \bar{g}(J\bar{X}, \bar{Z})J\bar{Y} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\}, \end{aligned} \tag{4.1}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Denote by $\bar{R}$ the curvature tensors of the (ℓ, m)-type connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (1.2) and (1.3), we see that

$$\begin{aligned}\bar{R}(\bar{X},\bar{Y})\bar{Z} = {} & \widetilde{R}(\bar{X},\bar{Y})\bar{Z} \\ & + (\bar{\nabla}_{\bar{X}}\theta)(\bar{Z})\{\ell\bar{Y} + mJ\bar{Y}\} - (\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z})\{\ell\bar{X} + mJ\bar{X}\} \\ & + \theta(\bar{Z})\{(\bar{X}\ell)\bar{Y} - (\bar{Y}\ell)\bar{X} + (\bar{X}m)J\bar{Y} - (\bar{Y}m)J\bar{X}\}.\end{aligned} \tag{4.2}$$

Denote by R and R^* the curvature tensor of the induced connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively:

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z \\ & + \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\ & + \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\ + \sum_{i=1}^{r}\{ & (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ & + \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\ & + \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)] \\ & - \ell[\theta(X)h_i^\ell(Y,Z) - \theta(Y)h_i^\ell(X,Z)] \\ & - m[\theta(X)h_i^\ell(FY,Z) - \theta(Y)h_i^\ell(FX,Z)]\}N_i \\ + \sum_{a=r+1}^{n}\{ & (\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\ & + \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_i^\ell(X,Z)] \\ & + \sum_{b=r+1}^{n}[\sigma_{ba}(X)h_b^s(Y,Z) - \sigma_{ba}(Y)h_b^s(X,Z)] \\ & - \ell[\theta(X)h_a^s(Y,Z) - \theta(Y)h_a^s(X,Z)] \\ & - m[\theta(X)h_a^s(FY,Z) - \theta(Y)h_a^s(FX,Z)]\}E_a,\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
(4.4)\qquad R(X,Y)PZ = R^*(X,Y)PZ \\
&+ \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\} \\
&+ \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\
&+ \sum_{k=1}^{r}[\tau_{ik}(Y)h_k^*(X,PZ) - \tau_{ik}(X)h_k^*(Y,PZ)] \\
&- \ell[\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(X,PZ)] \\
&- m[\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)]\}\xi_i.
\end{aligned}
$$

Taking the scalar product with N_i to (4.2) and using (4.1), (4.3) and (4.4), we obtain

$$
\begin{aligned}
(4.5)\qquad &(\nabla_X h_i^*)(Y,\, PZ) - (\nabla_Y h_i^*)(X,\, PZ) \\
&- \sum_{k=1}^{r}\{\tau_{ik}(X)h_k^*(Y,PZ) - \tau_{ik}(Y)h_k^*(X,PZ)\} \\
&- \sum_{k=1}^{r}\{h_k^\ell(Y,PZ)\eta_i(A_{N_k}X) - h_k^\ell(X,PZ)\eta_i(A_{N_k}Y)\} \\
&- \sum_{a=r+1}^{n}\{h_a^s(Y,\, PZ)\eta_i(A_{E_a}X) - h_a^s(X,\, PZ)\eta_i(A_{E_a}Y)\} \\
&- \ell\{\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(X,PZ)\} \\
&- m\{\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)\} \\
&- (\bar\nabla_X\theta)(PZ)\{\ell\eta_i(Y) + mv_i(Y)\} + (\bar\nabla_Y\theta)(PZ)\{\ell\eta_i(X) + mv_i(X)\} \\
&- \theta(PZ)\{(X\ell)\eta_i(Y) - (Y\ell)\eta_i(X) + (Xm)v_i(Y) - (Ym)v_i(X)\} \\
&= \frac{c}{4}\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y) \\
&\quad + v_i(X)\bar g(JY,PZ) - v_i(Y)\bar g(JX,PZ) + 2v_i(PZ)\bar g(X,JY)\}.
\end{aligned}
$$

Theorem 4.1. *Let M be a generic lightlike submanifold of an indefinite complex space form $\bar M(c)$ with an $(\ell,\, m)$-type connection such that ζ belongs to $S(TM)$. If either U_is or V_is are parallel with respect to the connection ∇, then $c = 0$ and $\bar M(c)$ is flat.*

Proof. (1) Assume that U_is are parallel with respect to the connection ∇. Applying $\bar\nabla_X$ to $(3.11)_{1,\,2}$ and using the fact that $\nabla_X U_j = 0$, we get

$$(4.6)\qquad (X\ell)\theta(U_j) + \ell(\bar\nabla_X\theta)(U_j) = 0, \qquad (Xm)\theta(U_j) + m(\bar\nabla_X\theta)(U_j) = 0.$$

Applying ∇_X to $(3.12)_3$ and using the fact that $\nabla_X U_j = 0$, we get

$$(4.7)\qquad (\nabla_X h_i^*)(Y, U_j) = 0.$$

Replacing Z by U_j to (4.5) and using $(3.11)_3$, $(3.12)_{2,\,3}$, (4.6) and (4.7), we obtain

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) - v_i(X)\eta_j(Y) + v_i(Y)\eta_I(X)\} = 0.$$

Taking $Y = V_j$ and $X = \xi_i$ to this equation, we obtain $c = 0$.

(2) Assume that V_is are parallel with respect to the connection ∇ of M. Applying $\bar{\nabla}_X$ to (3.14) and using the fact that $\nabla_X V_j = 0$, we get

$$(\nabla_X h_i^*)(Y, V_j) = 0. \tag{4.8}$$

Taking $X = V_k$ to (3.13), we obtain

$$h_i^\ell(V_k, U_j) = -m\theta(V_i)\delta_{jk}. \tag{4.9}$$

Taking $X = V_k$ and $Y = U_j$ to (2.16) and using (4.9), we get

$$h_i^\ell(U_j, V_k) = m\{\theta(V_k)\delta_{ij} - \theta(V_i)\delta_{jk}\}. \tag{4.10}$$

Taking $X = U_k$ to (2.26) and using (4.10), we have

$$h_j^\ell(U_k, V_i) = 0. \tag{4.11}$$

Taking $i = j$ to (4.10) and using (4.11) and the fact that $r > 1$, we obtain

$$m\theta(V_k) = 0. \tag{4.12}$$

Applying $\bar{\nabla}_X$ to (3.15): $\ell\theta(V_i) = 0$ and (4.12): $m\theta(V_i) = 0$ by turns and using the fact that $\nabla_X V_j = 0$, we get

$$(X\ell)\theta(V_j) + \ell(\bar{\nabla}_X\theta)(V_j) = 0, \quad (Xm)\theta(V_j) + m(\bar{\nabla}_X\theta)(V_j) = 0. \tag{4.13}$$

Replacing X by W_a to (3.13), we obtain

$$h_i^\ell(W_a, U_j) = 0.$$

Taking $X = W_a$ and $Y = U_j$ to (2.16) and using the last equation, we get

$$h_i^\ell(U_j, W_a) = m\theta(W_a)\delta_{ij}.$$

Replacing $X = U_j$ to (2.25) and using the last equation, we have

$$h_a^s(U_j, V_i) = 0. \tag{4.14}$$

Taking $Z = V_j$ to (4.5) and using (3.14), (4.8) and (4.13), we obtain

$$\begin{aligned}
& - \sum_{k=1}^{r} \{h_k^\ell(Y, V_j)\eta_i(A_{N_k}X) - h_k^\ell(X, V_j)\eta_i(A_{N_k}Y)\} \\
& - \sum_{a=r+1}^{n} \{h_a^s(Y, V_j)\eta_i(A_{E_a}X) - h_a^s(X, V_j)\eta_i(A_{E_a}Y)\} \\
& = \frac{c}{4}\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.
\end{aligned}$$

Taking $Y = U_j$ and $X = \xi_i$ to this equation and using (2.16), (2.17), (3.16), (4.11) and (4.14), we obtain $c = 0$. □

Theorem 4.2. *Let M be a solenoidal generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with an (ℓ, m)-type connection such that ζ belongs to $S(TM)$. If W_as are parallel with respect to ∇, then $c = 0$.*

Proof. Assume that W_as are parallel with respect to ∇. Taking the scalar product with W_b to (2.30) with $\nabla_X W_a = 0$, we have

$$\mu_{ab}(X) = -\ell\theta(W_a)w_b(X). \tag{4.15}$$

From (4.15) and the fact that $\epsilon_b\mu_{ab} + \epsilon_a\mu_{ba} = 0$, we have

$$\ell\{\epsilon_b\theta(W_a)w_b(X) + \epsilon_a\theta(W_b)w_a(X)\} = 0.$$

Taking $X = \epsilon_b W_b$ to this equation, we obtain

$$\ell\theta(W_a) = 0, \qquad \mu_{ab} = 0. \tag{4.16}$$

Taking the scalar product with V_i, U_i and N_i to (2.30) with $\nabla_X W_a = 0$ by turns and using (2.19) and $(4.16)_1$: $\ell\theta(W_a) = 0$, we have

$$\begin{aligned} &\lambda_{ai} = 0, \qquad \eta_i(A_{E_a}X) = -m\theta(W_a)\eta_i(X), \\ &h^s_a(X, U_i) = m\{\theta(U_i)w_a(X) - \epsilon_a\theta(W_a)v_i(X)\}. \end{aligned} \tag{4.17}$$

From (2.24) and $(4.17)_3$, we obtain

$$h^*_i(X, W_a) = 0. \tag{4.18}$$

Applying $\bar{\nabla}_X$ to (4.18) and using the fact that $\nabla_X W_a = 0$, we get

$$(\nabla_X h^*_i)(Y, W_a) = 0. \tag{4.19}$$

Now we shall assume that M is solenoidal. Then we obtain (2.33):

$$\eta_i(A_{N_j}X) = 0, \qquad \eta_i(A_{E_a}X) = 0. \tag{4.20}$$

from the second equation of the last equations and $(4.17)_2$, we obtain

$$m\theta(W_a) = 0.$$

Applying $\bar{\nabla}_X$ to $\ell\theta(W_a) = 0$ and $m\theta(W_a) = 0$ and using the fact that $\nabla_X W_a = 0$, we get

$$(X\ell)\theta(W_a) + \ell(\bar{\nabla}_X\theta)(W_a) = 0, \quad (Xm)\theta(W_a) + m(\bar{\nabla}_X\theta)(W_a) = 0. \tag{4.21}$$

Taking $X = W_a$ to (4.5) and using (4.18) $\sim$ (4.21), we have

$$c\{w_a(Y)\eta_i(X) - w_a(X)\eta_i(Y)\} = 0.$$

Taking $Y = W_a$ and $X = \xi_i$ to this equation, we obtain $c = 0$. □

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