Lightlike submanifolds of semi-Riemannian statistical manifolds

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Abstract. In this paper, we define lightlike submanifolds of statistical manifolds. We prove that induced connections from statistical connections on a lightlike submanifold are not statistical, in spite of the Riemannian case. Necessary and sufficient conditions that the induced connections to be statistical are obtained. Moreover, we investigate curvature tensor for tangential and transversal vector fields when the submanifold is totally umbilical. Finally, non-trivial examples of lightlike submanifolds of statistical manifolds are given.

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1 Introduction

Lightlike submanifolds of semi-Riemannian manifolds were introduced by Duggal and Bejancu [3]. A submanifold (M, g) of semi-Riemannian manifold $\bar{M}$ is called a lightlike submanifold if g is degenerate. It means that in lightlike submanifolds the normal vector bundle intersects with the tangent bundle, so the investigation of these submanifolds are different from non-degenerate case. In [3], they defined a non-degenerate screen distribution of tangent bundle that has not intersection with the transversal vector bundle and studied the classical submanifolds theory, induced connections and integrability of these distributions. Lightlike hypersurfaces have many applications in general relativity particularly in black hole theory and electromagnetism ([3].ch.8). So, many authors have studied the lightlike submanifolds from different view points and for various structures [4, 6, 8].

On the other hand, the semi-Riemannian manifold $\bar{M}$ with an affine and torsion-free conjugate connections (∇, ∇^*) is a statistical manifold if ∇g and $\nabla^* g$ are symmetric [1]. Conjugate and statistical structures are interesting for various fields [2, 7, 9]. In motivated of applications of these two types of structures, here we define lightlike submanifolds of statistical manifolds.

The paper is organized as follows. In Section 2 we provide a review of statistical manifolds and lightlike submanifolds. In Section 3 by using the approach of [4]

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and [11], the corresponding Gauss and Weingarten fundamental formulas for lightlike statistical submanifolds are obtained. In Riemannian case, a submanifold of a statistical manifold is also statistical with the induced connection. But in this paper we prove that a lightlike submanifold of a statistical manifold is not statistical in general and we obtain necessary and sufficient conditions that the submanifold to be statistical. Moreover, we show the induced connections of a lightlike submanifold on the screen distribution are statistical and lightlike second fundamental forms on the null distribution are not equal to zero, in spite of Levi-Civita case. In Section 4 we obtain some equations for curvature tensor of these submanifolds like the Gauss and Codazzi equations. Specially these equations for curvature tensor of totally umbilical submanifolds are investigated.

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. In all of the paper we assume $(\bar{M}, \bar{g})$ be an $(m+n)$-dimensional manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and $\hat{\nabla}$ be the Levi-Civita connection on $\bar{M}$.

A pair $(\bar{\nabla}, \bar{g})$ is called a statistical structure on $\bar{M}$ if $\bar{\nabla}$ is an affine and torsion-free connection and for all $X, Y, Z \in \Gamma(T\bar{M})$

$$(\bar{\nabla}_X \bar{g})(Y, Z) = (\bar{\nabla}_Y \bar{g})(X, Z). \tag{2.1}$$

Also $(\bar{M}, \bar{g}, \bar{\nabla})$ is said to be a statistical manifold.

Moreover, an affine connection $\bar{\nabla}^*$ is called a dual connection of $\bar{\nabla}$ with respect to $\bar{g}$ if [5]

$$X\bar{g}(Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}^*_X Z). \tag{2.2}$$

It is well-known $(\bar{\nabla}^*)^* = \bar{\nabla}$ and $\bar{\nabla}^*$ satisfies in (2.1).

$(1, 2)$-tensor field $\bar{K}$ is defined

$$\bar{K}_X Y = \bar{\nabla}_X Y - \hat{\nabla}_X Y = \frac{1}{2}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y). \tag{2.3}$$

It can be verify that $\bar{K}$ is symmetric so,

$$g(\bar{K}_X Y, Z) = g(\bar{K}_X Z, Y), \qquad \bar{K}_X Y = \bar{K}_Y X. \tag{2.4}$$

The statistical curvature tensor is defined

$$\bar{S}(X, Y)Z = \frac{1}{2}(\bar{R}(X, Y)Z + \bar{R}^*(X, Y)Z), \tag{2.5}$$

where $\bar{R}, \bar{R}^*$ are curvature tensors of $\bar{\nabla}, \bar{\nabla}^*$, respectively.

For a statistical manifold $(\bar{M}, \bar{g})$ the following relation holds [10]

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) &= \bar{g}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y, Z) + X\bar{g}(Y, Z) + Y\bar{g}(Z, X) - Z\bar{g}(X, Y) \\ &+ \bar{g}([X, Y], Z) + \bar{g}([Y, Z], X) - \bar{g}([Z, X], Y). \end{aligned} \tag{2.6}$$

Definition 2.1. [12] A vector field X on $\bar{M}$ is said to be Killing vector field if $\mathcal{L}_X \bar{g} = 0$, where $\mathcal{L}$ is the Lie derivative. A distribution D on $\bar{M}$ is called Killing distribution if each vector field on D be a Killing vector field.

D is called parallel with respect to $\bar{\nabla}$ if for all $X \in \Gamma(T\bar{M})$ and $Y \in \Gamma(D)$, $\bar{\nabla}_X Y \in \Gamma(D)$.

Let (M, g) be an immersed m-dimensional submanifold in a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla})$ and g be a induced metric of $\bar{g}$ on M. The submanifold M is called lightlike submanifold if there exists a non-zero $X \in \Gamma(TM)$ such that $g(X, Y) = 0, \forall Y \in \Gamma(TM)$. In this case, there exists a distribution $Rad(TM) = TM \cap TM^{\perp}$ of rank r, $(1 \leq r \leq m)$ which is known as radical (null) distribution, where

$$TM^{\perp} = \bigcup_{p \in M} \{X \in T_p\bar{M} : \bar{g}(X, Y) = 0, \forall Y \in T_pM\}.$$

The screen distribution $S(TM)$ and screen transversal vector bundle $S(TM^{\perp})$ are semi-Riemannian complementary distribution of $Rad(TM)$ in TM and $TM^{\perp}$, respectively.

Theorem 2.1. *[4] Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of $(\bar{M}, \bar{g})$ such that $r > 1$. Let $\mathcal{U}$ be a coordinate neighborhood of M and for $i \in \{1, \cdots, r\}$, $\{\xi_i\}$ be a basis for $\Gamma(Rad(TM)) \mid_{\mathcal{U}}$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp} \mid_{\mathcal{U}}$ where $\{N_i\}$ is a basis of $ltr(TM)$ and*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \tag{2.7}$$

$$\bar{g}(N_i, N_j) = 0, \qquad \forall i, j \in \{1, \cdots, r\}. \tag{2.8}$$

Let $tr(TM)$ be the complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M} \mid_M = S(TM) \perp [RadTM \oplus ltr(TM)] \perp S(TM^{\perp}). \tag{2.9}$$

For the statistical manifold $\bar{M}$ and lightlike submanifold M the Gauss formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\bar{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \qquad \forall X, Y \in \Gamma(TM)$$

where $\{\nabla_X Y, \nabla^*_X Y\}$ and $\{h(X, Y), h^*(X, Y)\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively.

Consider the projection morphism P from TM to $S(TM)$, then Gauss formulas become [4]

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.10}$$

$$\nabla_X PY = \nabla'_X PY + h'(X, PY), \tag{2.11}$$

where h^l, h^s and h' are $\Gamma(ltr(TM))$-valued, $\Gamma(S(TM^{\perp}))$-valued and $\Gamma(Rad(TM))$-valued which are called lightlike second fundamental form, screen second fundamental form and radical second fundamental form, respectively. Also ∇' is the tangential projection of ∇ on $\Gamma(S(TM))$. In above formulas by changing $\bar{\nabla}$ to $\bar{\nabla}^*$ we get the conjugate equations.

Example 2.2. Let $\bar{M} = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in \mathbb{R}, i = 1, \cdots, 5\}$ be a 5-dimensional semi-Riemannian manifold with metric $\bar{g} = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2$.
By taking $\dfrac{\partial}{\partial x^i} = e_i$, $i = 1, \cdots, 5$, we define statistical connections $\bar{\nabla}$ and $\bar{\nabla}^*$ on $\bar{M}$ as below

$$\begin{array}{llll}
\bar{\nabla}_{e_1} e_1 = e_2, & \bar{\nabla}_{e_2} e_2 = -e_2, & \bar{\nabla}_{e_2} e_1 = -e_5 + e_1, & \bar{\nabla}_{e_1} e_2 = e_5 + e_1, \\
\bar{\nabla}_{e_3} e_3 = e_4, & \bar{\nabla}_{e_4} e_4 = -e_4, & \bar{\nabla}_{e_4} e_3 = e_5 + e_3, & \bar{\nabla}_{e_3} e_4 = -e_5 + e_3 \\
\bar{\nabla}^*_{e_1} e_1 = -e_2, & \bar{\nabla}^*_{e_2} e_2 = e_2, & \bar{\nabla}^*_{e_2} e_1 = -e_5 - e_1, & \bar{\nabla}^*_{e_1} e_2 = e_5 - e_1, \\
\bar{\nabla}^*_{e_3} e_3 = -e_4, & \bar{\nabla}^*_{e_4} e_4 = e_4, & \bar{\nabla}^*_{e_4} e_3 = e_5 - e_3, & \bar{\nabla}^*_{e_3} e_4 = -e_5 - e_3
\end{array}$$

$$\bar{\nabla}_{e_1} e_5 = \bar{\nabla}_{e_5} e_1 = \bar{\nabla}^*_{e_1} e_5 = \bar{\nabla}^*_{e_5} e_1 = e_2,$$
$$\bar{\nabla}_{e_2} e_5 = \bar{\nabla}_{e_5} e_2 = \bar{\nabla}^*_{e_2} e_5 = \bar{\nabla}^*_{e_5} e_2 = -e_1,$$
$$\bar{\nabla}_{e_3} e_5 = \bar{\nabla}_{e_5} e_3 = \bar{\nabla}^*_{e_3} e_5 = \bar{\nabla}^*_{e_5} e_3 = e_4,$$
$$\bar{\nabla}_{e_4} e_5 = \bar{\nabla}_{e_5} e_4 = \bar{\nabla}^*_{e_4} e_5 = \bar{\nabla}^*_{e_5} e_4 = -e_3.$$

and other components be zero. Then $\bar{M}$ is semi-Riemannian statistical manifold.

3 Lightlike submanifolds of statistical manifolds

Definition 3.1. [4] A lightlike submanifold (M, g) of statistical manifold $\bar{M}$ is said to be totally umbilical in $\bar{M}$ if there exists a smooth vector field $H^l, H^{l*} \in \Gamma(tr(TM))$ and $H^s, H^{s*} \in \Gamma(S(TM^\perp))$ on M such that

$$h^l(X,Y) = H^l \bar{g}(X,Y), \quad h^s(X,Y) = H^s \bar{g}(X,Y), \quad \forall X, Y \in \Gamma(TM)$$

$$h^{l*}(X,Y) = H^{l*} \bar{g}(X,Y), \qquad h^{s*}(X,Y) = H^{s*} \bar{g}(X,Y).$$

M is called totally geodesic if h^l, h^{l*} and h^s, h^{s*} vanish identically on M.

Proposition 3.1. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. The induced connection ∇, ∇^* are affine and torsion-free connection on M. Moreover, h^l, h^{l*}, h^s and h^{s*} are symmetric and $C^\infty(M)$-bilinear forms.*

Proof. For any $f, g \in C^\infty(M)$ and $X, Y \in \Gamma(T(M))$

$$\begin{aligned}
\bar{\nabla}_{fX}\, gY &= f(Xg)Y + fg\bar{\nabla}_X Y \\
&= f(Xg)Y + fg\nabla_X Y + fg\, h^l(X,Y) + fg\, h^s(X,Y),
\end{aligned}$$

on the other hand, from Gauss formula we have

$$\bar{\nabla}_{fX}\, gY = \nabla_{fX}\, gY + h^l(fX, gY) + h^s(fX, gY).$$

Considering tangential and transversal components of above equations we get

$$\nabla_{fX}\, gY = f(Xg)Y + fg\nabla_X Y,$$

$$h^l(fX, gY) = fg\, h^l(X,Y), \qquad h^s(fX, gY) = fg\, h^s(X,Y),$$

since $ltr(TM)$ and $S(TM^\perp)$ are orthogonal to each other.

Moreover, since $\bar{\nabla}$ is torsion-free on $\bar{M}$

$$\begin{aligned}0 &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \\ &= \nabla_X Y + h^l(X,Y) + h^s(X,Y) - \nabla_Y X - h^l(Y,X) - h^s(Y,X) - [X,Y],\end{aligned}$$

by equating tangential and transversal parts we obtain

$$[X,Y] = \nabla_X Y - \nabla_Y X, \quad h^l(X,Y) = h^l(Y,X), \quad h^s(X,Y) = h^s(Y,X),$$

which proves the assertions. □

For all $Z \in \Gamma(tr(M))$ and $X \in \Gamma(T(M))$ the Weingarten formulas are as follows [11]

$$\bar{\nabla}_X Z = -A^*_Z X + \nabla^{tr}_X Z, \tag{3.1}$$

$$\bar{\nabla}^*_X Z = -A_Z X + \nabla^{tr*}_X Z,$$

where $A^*_Z X$, $A_Z X$ are shape operators on $\Gamma(T(M))$ and $\nabla^{tr}_X ZX$, $\nabla^{tr*}_X ZX$ are linear connections on $\Gamma(tr(M))$.

Decomposition (2.9) and (3.1) give the Weingarten formulas for the lightlike submanifold M

$$\bar{\nabla}_X N = -A^*_N X + \nabla^l_X N + D^s(X,N), \qquad \forall N \in \Gamma(ltr(TM)) \tag{3.2}$$

$$\bar{\nabla}_X W = -A^*_W X + \nabla^s_X W + D^l(X,W), \qquad \forall W \in \Gamma(S(TM^\perp)) \tag{3.3}$$

for linear connections ∇^l on $\Gamma(ltr(TM))$ and ∇^s on $S(TM^\perp)$. D^l, D^{l*} and D^s, D^{s*} are $C^\infty(M)$-bilinear mappings on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively. By changing $\bar{\nabla}$ to $\bar{\nabla}^*$ we get the conjugate Weingarten formulas.

On the other hand, if we take the vector fields $\xi \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$ we have the following relations like Weingarten formulas.

$$\nabla_X \xi = -A'^*_\xi X + D_X \xi, \quad \nabla^*_X \xi = -A'_\xi X + D^*_X \xi, \tag{3.4}$$

where $A'_\xi X, A'^*_\xi X$ and $D_X \xi, D^*_X \xi$ are shape operators on $\Gamma(S(TM))$ and linear connections on $\Gamma(Rad(TM))$, respectively.

Proposition 3.2. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. Then for all $N \in \Gamma(ltr(TM))$*

$$\bar{g}(h'(X,PY),N) = \bar{g}(PY, A_N X), \ \bar{g}(h'^*(X,PY),N) = \bar{g}(PY, A^*_N X). \tag{3.5}$$

Proof. For all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$ from (2.2), (2.10) and (2.11)

$$\begin{aligned}0 = X\bar{g}(PY,N) &= \bar{g}(\bar{\nabla}^*_X PY, N) + \bar{g}(PY, \bar{\nabla}_X N) \\ &= \bar{g}(\nabla^*_X PY, N) - \bar{g}(PY, A^*_N X) \\ &= \bar{g}(h'^*(X,PY),N) - \bar{g}(PY, A^*_N X).\end{aligned}$$

Now, with similar computation for $\bar{\nabla}$ we get the result. □

Proposition 3.3. *Let M be a lightlike submanifold of statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g})$. Then for all $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$*

$$\bar{g}(h^{s*}(X,Y), W) + \bar{g}(D^l(X,W), Y) = g(A^*_W X, Y). \tag{3.6}$$

Proof. Since $S(TM^{\perp})$ is orthogonal to TM and $ltr(TM)$ so for all $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$ we get

$$0 = X\bar{g}(W,Y) = \bar{g}(\bar{\nabla}_X W, Y) + \bar{g}(W, \bar{\nabla}^*_X Y),$$

now,by using Gauss formula and (3.3) we have

$$0 = \bar{g}(Y, -A^*_W X + D^l(X,W)) + \bar{g}(W, h^{s*}(X,Y)).$$

□

Proposition 3.4. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. Then for all $\xi \in \Gamma(Rad(TM))$ and $X, Y \in \Gamma(TM)$*

$$\bar{g}(h^l(X, PY), \xi) = g(A'_\xi X, PY), \ \bar{g}(h^{l*}(X, PY), \xi) = g(A'^*_\xi X, PY) \tag{3.7}$$

Proof. From (2.11), (3.4) and Gauss formula we obtain

$$\begin{aligned} 0 = X\bar{g}(PY, \xi) &= \bar{g}(\bar{\nabla}^*_X PY, \xi) + \bar{g}(PY, \bar{\nabla}_X \xi) = \bar{g}(h^{l*}(X, PY), \xi) \\ &+ \bar{g}(PY, \nabla_X \xi) = \bar{g}(h^{l*}(X, PY), \xi) + g(-A'^*_\xi X, PY), \end{aligned}$$

this completes the proof. □

By a simple computation such as previous propositions from (2.11), (3.4), Gauss and Weingarten formulas we get the following relations

$$\bar{g}(D^s(X,N), W) = \bar{g}(A_W X, N), \tag{3.8}$$

$$g(\nabla^*_X \xi, Y) + \bar{g}(h^l(X,Y), \xi) + \bar{g}(h^{l*}(X, \xi), Y) = 0, \tag{3.9}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Remark 3.2. The induced connections on non-degenerate submanifolds of statistical semi-Riemannian manifolds are statistical. In the next theorem we show that on lightlike submanifolds of statistical manifolds this does not satisfy in general (cf. Theorem 3.5). In the Theorem 3.6 we obtain the necessary and sufficient condition that induced connection and its dual be statistical.

Theorem 3.5. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. Then the induced connections ∇ and ∇^* on M are not necessarily statistical.*

Proof. For all $X, Y, Z \in \Gamma(TM)$ from (2.2) and Gauss formula

$$\begin{aligned} Xg(Y,Z) = X\bar{g}(Y,Z) &= \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}^*_X Z) \\ &= \bar{g}(\nabla_X Y + h^l(X,Y) + h^s(X,Y), Z) \\ &+ \bar{g}(Y, \nabla^*_X Z + h^{l*}(X,Z) + h^{s*}(X,Z)) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) + \bar{g}(h^l(X,Y), Z) + \bar{g}(Y, h^{l*}(X,Z)). \end{aligned} \tag{3.10}$$

So in general (3.10) is not satisfied. □

Theorem 3.6. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. The induced connections ∇ and ∇^* on M are statistical if and only if*

$$\bar{g}((h^l(X,Y),Z) + \bar{g}(Y, h^{l*}(X,Z)) = 0, \quad \forall X,Y,Z \in \Gamma(TM). \tag{3.11}$$

Corollary 3.7. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. The Equation (3.11) implies one of the following conditions holds*

a) h^l *and* h^{l*} *vanish identically,*

a) $h^l(X,Y) = -h^{l*}(X,Z), \quad \forall X,Y,Z \in \Gamma(TM)$,

c) $Y, Z \in \Gamma(S(TM))$.

Corollary 3.8. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. Then for any vector fields on distribution $S(TM)$, M satisfies Equation (2.2).*

Proof. If $X, Y, Z \in \Gamma(S(TM))$, (3.10) implies that the relation (3.11) holds, so from Theorem 3.6, Eq. (2.2) is satisfied. □

Proposition 3.9. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. Then for all $\xi, \xi' \in \Gamma(Rad(TM))$*

$$A'_{\xi}\xi' + A'^{*}_{\xi'}\xi = 0. \tag{3.12}$$

Proof. Changing Y by $\xi' \in \Gamma(Rad(TM))$ in (3.9) we obtain

$$\bar{g}(h^l(X,\xi'),\xi) + \bar{g}(h^{l*}(X,\xi),\xi') = 0. \tag{3.13}$$

Substituting PX by X in (3.13) we have

$$\bar{g}(h^l(PX,\xi'),\xi) + \bar{g}(h^{l*}(PX,\xi),\xi') = 0, \tag{3.14}$$

using (3.7) we get

$$0 = \bar{g}(h^l(PX,\xi'),\xi) + \bar{g}(h^{l*}(PX,\xi),\xi') = g(A'_{\xi}\xi', PX) + g(A'^{*}_{\xi'}\xi, PX).$$

the assertion follows since $S(TM)$ is non-degenerate. □

In spite of Levi-Civita case that the lightlike second fundamental form on the null distribution is always equal to zero, in the next proposition we show that in general for lightlike submanifolds of statistical manifolds h^l and h^{l*} do not vanish on $Rad(TM)$.

Proposition 3.10. *Let M be a lightlike submanifold of the statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g})$ and h^l and h^{l*} are not identically equal to zero. Then one of the following statements holds*

a) h^l *and* h^{l*} *vanish on* $Rad(TM)$,

b) $h^l(\xi', \xi) = -h^{l*}(\xi, \xi')$

for all $\xi, \xi' \in \Gamma(Rad(TM))$.

Proof. For all $\xi, \xi' \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$ from (2.6) we get

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_\xi \xi', X) &= \bar{g}(\bar{\nabla}_\xi \xi' - \bar{\nabla}^*_\xi \xi', X) + \xi \bar{g}(\xi', X) + \xi' \bar{g}(X, \xi) - X\bar{g}(\xi, \xi') \\ &+ \bar{g}([\xi, \xi'], X) + \bar{g}([\xi', X], \xi) - \bar{g}([X, \xi], \xi') \\ &= \bar{g}(\bar{\nabla}_\xi \xi' - \bar{\nabla}^*_\xi \xi', X) + \bar{g}(\bar{\nabla}_\xi \xi' - \bar{\nabla}_{\xi'} \xi, X), \end{aligned} \tag{3.15}$$

so we get

$$\bar{g}(\bar{\nabla}^*_\xi \xi', X) + \bar{g}(\bar{\nabla}_{\xi'} \xi, X) = 0.$$

From (2.10) we have

$$\bar{g}(\nabla^*{}_\xi \xi' + h^{l*}(\xi, \xi') + h^{s*}(\xi, \xi'), X) + \bar{g}(\nabla_{\xi'} \xi + h^l(\xi', \xi) + h^s(\xi', \xi), X) = 0.$$

By putting $X = \xi''$ in last equation we obtain

$$\bar{g}(h^l(\xi', \xi), \xi'') = -\bar{g}(h^{l*}(\xi, \xi'), \xi''), \tag{3.16}$$

so (3.16) implies that one of the statements (a) and (b) satisfies. □

In the last of this paper we construct examples that shows the items (a) and (b) hold and the Equation (3.16) satisfies.

Theorem 3.11. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a statistical lightlike submanifold of $\bar{M}$. For all $\xi \in \Gamma(Rad(TM))$, A'_ξ and A'^*_ξ vanish on $\Gamma(TM)$ if and only if one of the following relations hold*

a) $Rad(TM)$ is a parallel distribution with respect to ∇ and ∇^*.

b) $Rad(TM)$ is a Killing distribution.

Proof. a) From (3.4), A'_ξ and A'^*_ξ vanish if and only if $Rad(TM)$ is a parallel distribution with respect to ∇ and ∇^*.

b) For all $X, Y \in \Gamma(TM)$ Equation (2.2) implies

$$\begin{aligned} (\mathcal{L}_\xi \bar{g})(X, Y) &= \xi \bar{g}(X, Y) - \bar{g}([\xi, X], Y) - \bar{g}(X, [\xi, Y]) \\ &= \xi \bar{g}(X, Y) - \bar{g}(\bar{\nabla}_\xi X, Y) + \bar{g}(\bar{\nabla}_X \xi, Y) - \bar{g}(X, \bar{\nabla}_\xi Y) + \bar{g}(X, \bar{\nabla}_Y \xi) \\ &= \xi \bar{g}(X, Y) - \xi \bar{g}(X, Y) + \bar{g}(X, \bar{\nabla}^*_\xi Y) + \bar{g}(\bar{\nabla}_X \xi, Y) - \bar{g}(X, \bar{\nabla}_\xi Y) \\ &+ \bar{g}(X, \bar{\nabla}_Y \xi) = -2\bar{g}(X, \bar{K}_\xi Y) + \bar{g}(\bar{\nabla}_X \xi, Y) + \bar{g}(X, \bar{\nabla}_Y \xi), \end{aligned}$$

so from (2.4) and (2.10) we have

$$\begin{aligned} (\mathcal{L}_\xi \bar{g})(X, Y) &= -2\bar{g}(X, \bar{K}_Y \xi) + \bar{g}(\nabla_X \xi + h^l(X, \xi), Y) + \bar{g}(X, \bar{\nabla}_Y \xi) \\ &= g(X, \nabla^*_Y \xi) + g(\nabla_X \xi, Y) + \bar{g}(X, h^{l*}(Y, \xi)) + \bar{g}(Y, h^l(X, \xi)), \end{aligned} \tag{3.17}$$

by using (3.4) it turns into

$$\begin{aligned} (\mathcal{L}_\xi \bar{g})(X, Y) &= g(X, -A'^*_\xi Y) + g(Y, -A'_\xi X) + \bar{g}(X, h^{l*}(Y, \xi)) \\ &+ \bar{g}(Y, h^l(X, \xi)). \end{aligned} \tag{3.18}$$

If A'_ξ and A'^*_ξ vanish, since the submanifold is statistical from Theorem (3.6)

$$\bar{g}(X, h^{l*}(Y,\xi)) + \bar{g}(Y, h^l(X,\xi)) = 0, \tag{3.19}$$

so (3.18) implies $(\mathcal{L}_\xi \bar{g}) = 0$ and $Rad(TM)$ is a Killing distribution. Conversely, replacing X by $\xi' \in \Gamma(Rad(TM))$ and Y by PY in (3.18) and using (3.19) we obtain

$$g(PY, A'_\xi \xi') = 0,$$

so $A'_\xi \xi' = 0$. On the other hand replacing X, Y by PX, PY in (3.17) and using (3.4) we get

$$\begin{aligned} 0 &= g(PX, \nabla^*_{PY}\xi) + g(\nabla_{PX}\xi, PY) = PYg(PX,\xi) - g(\nabla_{PY}PX, \xi) \\ &\quad + g(\nabla_{PX}\xi, PY) = g(-A'^*_\xi PX, PY) \end{aligned} \tag{3.20}$$

so $A'^*_\xi PX = 0$. Thus A'^*_ξ vanishes for any vector field in $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$. By similar computation we get $A'_\xi = 0$. □

Remark 3.3. One can show ∇' and ∇'^* are linear connections on $S(TM)$ and h' and h'^* are $C^\infty(M)$-bilinear forms. In general ∇' and ∇'^* are not statistical connections and h' and h'^* are not symmetric second fundamental forms. In the next theorems we prove the necessary condition that h' and h'^* be symmetric and ∇' and ∇'^* be statistical.

Proposition 3.12. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. $S(TM)$ is integrable distribution if and only if h' and h'^* are symmetric on $S(TM)$.*

Proof. For all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$ from (2.10) and (2.11) we have

$$\begin{aligned} \bar{g}([PX, PY], N) &= \bar{g}(\bar{\nabla}_{PX}PY - \bar{\nabla}_{PY}PX, N) = g(\nabla_{PX}PY - \nabla_{PY}PX, N) \\ &= g(h'(PX, PY) - h'(PY, PX), N). \end{aligned}$$

The above equation implies the equivalence of assertions. □

Proposition 3.13. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. If $S(TM)$ is an integrable distribution then the induced connections ∇' and ∇'^* are affine and torsion-free connections on $S(TM)$.*

Proof. For all $X, Y, \in \Gamma(TM)$ since ∇ is torsion-free, (2.11) implies

$$\begin{aligned} 0 &= \nabla_{PX}PY - \nabla_{PY}PX - [PX, PY] \\ &= \nabla'_{PX}PY - \nabla'_{PY}PX - [PX, PY] + h'(PX, PY) - h'(PY, PX), \end{aligned}$$

from Proposition 3.12 equating screen and radical parts gives

$$[PX, PY] = \nabla'_{PX}PY - \nabla'_{PY}PX, \qquad h'(PX, PY) = h'(PY, PX).$$

□

Theorem 3.14. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. If $S(TM)$ is an integrable distribution then the induced connections ∇' and ∇'^* are statistical connections on $S(TM)$.*

Proof. For all $X, Y, Z \in \Gamma(TM)$, (2.2) and (2.11) and Gauss formula for lightlike submanifolds imply

$$\begin{aligned} PXg(PY,PZ) &= PX\bar{g}(PY,PZ) = \bar{g}(\bar{\nabla}_{PX}PY,PZ) + \bar{g}(PY,\bar{\nabla}^*_{PX}PZ) \\ &= g(\nabla_{PX}PY,PZ) + g(PY,\nabla^*_{PX}PZ) \\ &= g(\nabla'_{PX}PY + h'(PX,PY),PZ) + g(PY,\nabla'^*_{PX}PZ + h'^*(PX,PZ)) \\ &= g(\nabla'_{PX}PY,PZ) + g(PY,\nabla'^*_{PX}PZ). \end{aligned}$$

Moreover, from Proposition 3.13, ∇' and ∇'^* are affine and torsion-free so, M is statistical on $\Gamma(TM)$. □

4 Curvature tensors

In this section according to the Gauss and Codazzi equations for statistical manifolds in [11] we obtain these equations for lightlike case.

Lemma 4.1. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. If $\bar{S}$ and S be the curvature tensors of $\bar{M}$ and M, respectively, then for all $X, Y, Z, W \in \Gamma(TM)$ we get*

$$\begin{aligned} 2\bar{g}(\bar{S}(X,Y)Z,W) &= 2g(S(X,Y)Z,W) + g(A_{h^l(X,Z)}Y,W) - g(A_{h^l(Y,Z)}X,W) \\ &+ g(A_{h^s(X,Z)}Y,W) - g(A_{h^s(Y,Z)}X,W) + g(A^*_{h^{l*}(X,Z)}Y,W) \\ &- g(A^*_{h^{l*}(Y,Z)}X,W) + g(A^*_{h^{s*}(X,Z)}Y,W) - g(A^*_{h^{s*}(Y,Z)}X,W) \\ &+ \bar{g}((\nabla_X h^l)(Y,Z),W) - \bar{g}((\nabla_Y h^l)(X,Z),W) \\ &+ \bar{g}((\nabla^*_X h^{l*})(Y,Z),W) - \bar{g}((\nabla^*_Y h^{l*})(X,Z),W) \\ &+ \bar{g}(D^l(X,h^s(Y,Z)),W) - \bar{g}(D^l(Y,h^s(X,Z)),W) \\ &+ \bar{g}(D^{l*}(X,h^{s*}(Y,Z)),W) - \bar{g}(D^{l*}(Y,h^{s*}(X,Z)),W). \end{aligned} \tag{4.1}$$

Proof. Let $\bar{R}$ and R be the curvature tensors of $\bar{\nabla}$ and ∇, respectively. Then for all $X, Y, Z \in \Gamma(TM)$, we can obtain

$$\begin{aligned} \bar{R}(X,Y)Z &= \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \\ &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ &+ (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) \\ &+ D^s(X,h^l(Y,Z)) - D^s(Y,h^l(X,Z)) \\ &+ D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z)). \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} (\nabla_X h^l)(Y,Z) &= \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y,Z) - h^l(Y,\nabla_X Z), \\ (\nabla_X h^s)(Y,Z) &= \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y,Z) - h^s(Y,\nabla_X Z). \end{aligned}$$

In the similar way $\bar{R}^*$ can be obtained, so we get the assertion where, $2S = R + R^*$.
□

By using (4.1) we can compute S for special case. Let $X, Y, Z, W \in \Gamma(TM)$, $U \in \Gamma(S(TM^{\perp}))$, $N \in \Gamma(ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$ from (3.5), (3.6) and (3.8) we derive the following relations.

$$\begin{aligned}2\bar{g}(\bar{S}(X,Y)Z,PW) &= 2g(S(X,Y)Z,PW) + \bar{g}(h'(Y,PW),h^l(X,Z)) \\ &- \bar{g}(h'(X,PW),h^l(Y,Z)) + \bar{g}(h'^*(Y,PW),h^{l*}(X,Z)) \\ &- \bar{g}(h'^*(X,PW),h^{l*}(Y,Z)) + \bar{g}(h^s(Y,PW),h^s(X,Z)) \\ &- \bar{g}(h^s(X,PW),h^s(Y,Z)) + \bar{g}(h^{s*}(Y,PW),h^{s*}(X,Z)) \\ &- \bar{g}(h^{s*}(X,PW),h^{s*}(Y,Z)),\end{aligned} \tag{4.3}$$

$$\begin{aligned}2\bar{g}(\bar{S}(X,Y)Z,U) &= \bar{g}((\nabla_X h^s)(Y,Z),U) - \bar{g}((\nabla_Y h^s)(X,Z),U) \\ &+ \bar{g}((\nabla^*_X h^{s*})(Y,Z),U) - \bar{g}((\nabla^*_Y h^{s*})(X,Z),U) \\ &+ \bar{g}(A_U X,h^l(Y,Z)) - \bar{g}(A_U Y,h^l(X,Z)) \\ &+ \bar{g}(A^*_U X,h^{l*}(Y,Z)) - \bar{g}(A^*_U Y,h^{l*}(X,Z)),\end{aligned} \tag{4.4}$$

$$\begin{aligned}2\bar{g}(\bar{S}(X,Y)Z,N) &= 2\bar{g}(S(X,Y)Z,N) + \bar{g}(A_{h^l(X,Z)}Y,N) - \bar{g}(A_{h^l(Y,Z)}X,N) \\ &+ \bar{g}(A^*_{h^{l*}(X,Z)}Y,N) - \bar{g}(A^*_{h^{l*}(Y,Z)}X,N) + \bar{g}(A_{h^s(X,Z)}Y,N) \\ &- \bar{g}(A_{h^s(Y,Z)}X,N) + \bar{g}(A^*_{h^{s*}(X,Z)}Y,N) - \bar{g}(A^*_{h^{s*}(Y,Z)}X,N)\end{aligned}$$

Proposition 4.2. *Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be a statistical manifold and M be a lightlike submanifold of $\bar{M}$. If M be a totally umbilical submanifold with respect to the $\bar{\nabla}$ in $\bar{M}$ then for all $Z \in \Gamma(Rad(TM))$ and $X, Y, W \in \Gamma(TM)$*

$$\bar{g}(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W).$$

Proof. Since M is a totally umbilical submanifold for all $X, Y, W \in \Gamma(TM)$ and $Z \in \Gamma(Rad(TM))$ we obtain

$$h^l(X,Z) = h^l(Y,Z) = 0,$$

so it is sufficient to prove $\bar{g}(h^l(\nabla_X Z,Y),W) - \bar{g}(h^l(\nabla_Y Z,X),W) = 0$. By using (3.4) in (4.2)

$$\begin{aligned}\bar{g}(h^l(\nabla_X Z,Y),W) - \bar{g}(h^l(\nabla_Y Z,X),W) &= \bar{g}(\nabla_X Z,Y)\bar{g}(H^l,W) \\ - \bar{g}(\nabla_Y Z,X)\bar{g}(H^l,W) &= \bar{g}(H^l,W)(g(A'^*_Z X,Y) - g(A'^*_Z Y,X)) = 0,\end{aligned}$$

which completes the proof, since A'^* is self-adjoint. □

Example 4.1. Let $\bar{M}$ be the statistical manifold defined in Example 2.2. Let $(M = \{(u_1,u_2,u_3) \mid u_i \in \mathbb{R}\}, g = \bar{g}_{|M})$ be a submanifold of $\bar{M}$ where

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = u_1, \quad x_4 = u_2, \quad x_5 = u_3.$$

So we find

$$S(TM) = \{Z = e_5\}, \qquad S(TM^{\perp}) = \varnothing\,,$$

$$Rad(TM) = \{\xi_1 = e_1 + e_3, \xi_2 = e_2 + e_4\},$$

$$ltr(TM) = \{N_1 = \frac{1}{2}(-e_1 + e_3), N_2 = \frac{1}{2}(-e_2 + e_4)\}.$$

where, $\frac{\partial}{\partial x^i} = e_i$. By computing we get

$$\begin{gathered}
\bar{\nabla}_{\xi_1}\xi_1 = \xi_2, \quad \bar{\nabla}_{\xi_2}\xi_2 = -\xi_2, \quad \bar{\nabla}_{\xi_2}\xi_1 = \xi_1, \quad \bar{\nabla}_{\xi_1}\xi_2 = \xi_1,\\
\bar{\nabla}_Z\xi_1 = \bar{\nabla}_{\xi_1}Z = \xi_2, \quad \bar{\nabla}_Z\xi_2 = \bar{\nabla}_{\xi_2}Z = -\xi_1,\\
\bar{\nabla}^*_{\xi_1}\xi_1 = -\xi_2, \quad \bar{\nabla}^*_{\xi_2}\xi_2 = \xi_2, \quad \bar{\nabla}^*_{\xi_2}\xi_1 = -\xi_1, \quad \bar{\nabla}^*_{\xi_1}\xi_2 = -\xi_1,\\
\bar{\nabla}^*_Z\xi_1 = \bar{\nabla}^*_{\xi_1}Z = \xi_2, \quad \bar{\nabla}^*_Z\xi_2 = \bar{\nabla}^*_{\xi_2}Z = -\xi_1.
\end{gathered}$$

Thus we can verify that $h^l = h^{l*} = 0$, and from Theorem 3.6, M is a 2-lightlike statistical submanifold of semi-Riemannian statistical manifold $\bar{M}$.

Example 4.2. Let $\bar{M}$ be a statistical manifold defined in Example 2.2 and submanifold M be ($M = \{(u_1, u_2, u_3) \mid u_i \in \mathbb{R}\}, g = \bar{g}_{|M}$), where

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = u_2, \quad x_4 = u_1, \quad x_5 = u_3.$$

We have the following distributions on submanifold:

$$S(TM) = \{Z = e_5\},$$

$$Rad(TM) = \{\xi_1 = e_1 + e_4, \xi_2 = e_2 + e_3\},$$

$$ltr(TM) = \{N_1 = \frac{1}{2}(-e_1 + e_4), N_2 = \frac{1}{2}(-e_2 + e_3)\}.$$

By direct calculating we can obtain the induced connections ∇, ∇^* and second fundamental forms h^l, h^{l*} as follows

$$\begin{gathered}
\nabla_{\xi_1}\xi_1 = \frac{1}{2}(\xi_2 - \xi_1), \quad \nabla_{\xi_2}\xi_2 = \frac{1}{2}(\xi_1 - \xi_2), \quad \nabla_{\xi_2}\xi_1 = -2Z + \frac{1}{2}(\xi_1 + \xi_2),\\
\nabla_{\xi_1}\xi_2 = 2Z + \frac{1}{2}(\xi_1 + \xi_2), \quad \nabla_{\xi_1}Z = \nabla_Z\xi_1 = \nabla_Z\xi_2 = \nabla_{\xi_2}Z = 0, \quad \nabla_Z Z = 0,\\
\nabla^*_{\xi_1}\xi_1 = \frac{1}{2}(\xi_1 - \xi_2), \quad \nabla^*_{\xi_2}\xi_2 = \frac{1}{2}(\xi_2 - \xi_1), \quad \nabla^*_{\xi_2}\xi_1 = -2Z - \frac{1}{2}(\xi_1 + \xi_2),\\
\nabla^*_{\xi_1}\xi_2 = 2Z - \frac{1}{2}(\xi_1 + \xi_2), \quad \nabla^*_{\xi_1}Z = \nabla^*_Z\xi_1 = \nabla^*_{\xi_2}Z = \nabla^*_Z\xi_2 = 0, \quad \nabla^*_Z Z = 0,
\end{gathered}$$

$$\begin{gathered}
h^l(\xi_1, \xi_1) = -N_1 - N_2 = -h^{l*}(\xi_1, \xi_1), \quad h^l(\xi_2, \xi_2) = N_1 + N_2 = -h^{l*}(\xi_2, \xi_2),\\
h^l(\xi_2, \xi_1) = h^l(\xi_1, \xi_2) = -N_1 + N_2 = -h^{l*}(\xi_2, \xi_1) = -h^{l*}(\xi_1, \xi_2),\\
h^l(\xi_1, Z) = h^l(Z, \xi_1) = h^{l*}(\xi_1, Z) = h^{l*}(Z, \xi_1) = -2N_2,\\
h^l(\xi_2, Z) = h^l(Z, \xi_2) = h^{l*}(\xi_2, Z) = h^{l*}(Z, \xi_2) = 2N_1,\\
h^l(Z, Z) = h^{l*}(Z, Z) = 0,
\end{gathered}$$

and $h^s = h^{s*} = 0$. Thus M is a 2-lightlike submanifold of $\bar{M}$. This example shows that M is not statistical submanifold and (3.10) satisfies. On the other hand, we have

$$\bar{g}(h^l(\xi,\xi'),\xi'') = -\bar{g}(h^{l*}(\xi',\xi),\xi''), \quad \forall \xi,\xi',\xi'' \in \Gamma(Rad(TM)),$$

that shows the part (b) in Proposition 3.10 holds.

Example 4.3. Let $\bar{M}$ be a statistical manifold defined in Example 2.2. Assume M be a 4-dimensional submanifold of $\bar{M}$ defined by $M = \{(u_1,u_2,u_3,u_4) \mid u_i \in \mathbb{R}\}$ such that

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = u_1, \quad x_4 = u_3, \quad x_5 = u_4,$$

and $g = \bar{g}_{|_M}$. We define

$$S(TM) = \{Z_1 = e_2, Z_2 = e_4, Z_3 = e_5\}, \qquad S(TM^\perp) = \varnothing\,,$$

$$Rad(TM) = \{\xi = e_1 + e_3\},$$

$$ltr(TM) = \{N = \frac{1}{2}(-e_1 + e_3)\}.$$

$$\begin{gathered}
\nabla_\xi \xi = Z_1 + Z_2,\ \nabla_{Z_1}\xi = -Z_3 + \frac{1}{2}\xi,\ \nabla_\xi Z_1 = Z_3 + \frac{1}{2}\xi,\ \nabla_{Z_2}\xi = Z_3 + \frac{1}{2}\xi,\\
\nabla_\xi Z_2 = -Z_3 + \frac{1}{2}\xi,\ \nabla_{Z_3}\xi = Z_1 + Z_2,\ \nabla_\xi Z_3 = Z_1 + Z_2,\ \nabla_{Z_2}Z_1 = \nabla_{Z_1}Z_2 = 0,\\
\nabla_{Z_3}Z_1 = \nabla_{Z_1}Z_3 = \frac{-1}{2}\xi, \qquad \nabla_{Z_3}Z_2 = \nabla_{Z_2}Z_3 = \frac{-1}{2}\xi,\\
\nabla_{Z_1}Z_1 = -Z_1, \qquad \nabla_{Z_2}Z_2 = -Z_2, \qquad \nabla_{Z_3}Z_3 = 0,\\
\nabla^*_\xi \xi = -(Z_1 + Z_2),\ \nabla^*_{Z_1}\xi = -Z_3 - \frac{1}{2}\xi,\ \nabla^*_\xi Z_1 = Z_3 - \frac{1}{2}\xi,\ \nabla^*_{Z_2}\xi = Z_3 - \frac{1}{2}\xi,\\
\nabla^*_\xi Z_2 = -Z_3 - \frac{1}{2}\xi,\ \nabla^*_{Z_3}\xi = Z_1 + Z_2,\ \nabla^*_\xi Z_3 = Z_1 + Z_2,\ \nabla^*_{Z_2}Z_1 = \nabla^*_{Z_1}Z_2 = 0,\\
\nabla^*_{Z_3}Z_1 = \nabla^*_{Z_1}Z_3 = \frac{-1}{2}\xi, \qquad \nabla^*_{Z_3}Z_2 = \nabla^*_{Z_2}Z_3 = \frac{-1}{2}\xi,\\
\nabla^*_{Z_1}Z_1 = Z_1, \qquad \nabla^*_{Z_2}Z_2 = Z_2, \qquad \nabla^*_{Z_3}Z_3 = 0,
\end{gathered}$$

$$\begin{gathered}
h^l(\xi,\xi) = h^{l*}(\xi,\xi) = 0,\\
h^l(\xi,Z_1) = h^l(Z_1,\xi) = -N = -h^{l*}(\xi,Z_1) = -h^{l*}(Z_1,\xi),\\
h^l(\xi,Z_2) = h^l(Z_2,\xi) = N = -h^{l*}(\xi,Z_2) = -h^{l*}(Z_2,\xi),\\
h^l(\xi,Z_3) = h^l(Z_3,\xi) = h^{l*}(\xi,Z_3) = h^{l*}(Z_3,\xi) = 0,\\
h^l(Z_1,Z_2) = h^l(Z_2,Z_1) = h^{l*}(Z_1,Z_2) = h^{l*}(Z_2,Z_1) = 0,\\
h^l(Z_1,Z_3) = h^l(Z_3,Z_1) = h^{l*}(Z_1,Z_3) = h^{l*}(Z_3,Z_1) = N,\\
h^l(Z_2,Z_3) = h^l(Z_3,Z_2) = h^{l*}(Z_2,Z_3) = h^{l*}(Z_3,Z_2) = -N,\\
h^l(Z_1,Z_1) = h^l(Z_2,Z_2) = h^{l*}(Z_1,Z_1) = h^{l*}(Z_2,Z_2) = 0,\\
h^l(Z_3,Z_3) = h^{l*}(Z_3,Z_3) = 0,
\end{gathered}$$

Thus M is a 1-lightlike submanifold of $\bar{M}$. One can see that M is not statistical submanifold and Equation (3.10) is satisfied. On the other hand, Corollary 3.8 holds and Equation (2.2) satisfies on $S(TM)$. Moreover, in this example the part (a) of Proposition 3.10 holds.

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