Nonexistence proper biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n

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Abstract. In this paper we studied biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n. Indeed, we proved that the biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n are minimal. Actually, it showed that the Weingarten operator A of the biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n has exactly three distinct eigenvalues at each point.

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1 Introduction

A harmonic map $f : M \to N$ between two Riemannian manifolds is known as critical point of the energy functional $E(f) = \frac{1}{2}\int_M |df|^2 dv$. By taking the similar idea, the problem was proposed to investigate the k-harmonic maps as the critical point of the k-energy functional (see [7, 6]). In case $k = 2$, the bienergy of f defined by $E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 dv$, where $\tau(f) = \mathsf{trace}\nabla df$ is the tension field of f. In [12, 13] the authors showed a new definition of the biharmonic function in point of the variational formulas and the Euler-Lagrange equation correlated to E_2, written as:

$$\tau_2(f) \quad = \quad -\Delta\tau(f) - \mathsf{trace}R^N(df(.), \tau(f))df(.) = 0, \tag{1.1}$$

where $\tau_2(f) = 0$ is named the biharmonic equation and it is the fourth order elliptic semilinear PDE. Obviously, every harmonic map is biharmonic the interesting is in the non harmonic biharmonic maps which are called proper biharmonic. The first ambient spaces to investigate the proper biharmonic submanifolds are spaces of the constant sectional curvature. In this case, the biharmonic concept of submanifold in the Euclidean space with the harmonic mean curvature vector was established by B. Y. Chen. Indeed, the well known conjecture was posted: any biharmonic submanifold in Euclidean space is harmonic (see [3]). Also, the first class of submanifolds to be studied is that of the hypersurfaces. Up to now, the following classification results reached.

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- biharmonic surfaces in R^n, n=3, 4, 5 are minimal [4, 11, 10];
- biharmonic hypersurfaces in 4-dimensional space forms R^4 and H^4 are minimal [2];
- biharmonic hypersurfaces with three distinct principal curvatures in R^n and S^n are minimal [8, 9];
- biharmonic submanifold with constant mean curvature is minimal in Euclidean space [5].

Recently, in the Euclidean case with regarding the idea about the number of distinct principal curvatures, in [14, 15] the authors showed the biharmonic Hopf hypersurfaces in the complex Euclidean spaces and in the odd dimensional spheres are minimal. Furthermore, the nonexistence result of the proper biharmonic Ricci Soliton hypersurfaces obtained in the Euclidean space E^{n+1}.

After all, the conjecture was persuasive enough that to be considered on the certain hypersurfaces in the quaternionic Kählerian manifold. The objectives of the present article is to give an affirmative answer to the conjecture about the biharmonic Hopf QR- hypersurfaces in the quaternionic Euclidean space Q^n, where the ambient manifold admits a vector bundle V consists of the tensors of type (1, 1). We investigate on the biharmonic QR-hypersurface of the quaternionic Euclidean space Q^n, where the structural vector fields are eigenvectors of the Weingarten operator. Finally, with respect to the number of the distinct principal curvatures at each point of these hypersurfaces, we obtain the biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n are minimal.

The paper is organized as following. Section 2, is devoted to establish the fundamental definitions and formulas about the biharmonic condition in the Euclidean space, the quaternionic Kählarien manifold and its special QR-hypersurfaces, which required in the following sections. Next, we illustrate some examples of the Hopf hypersurfaces in the quatenionic Eulidean space R^4 at the end of this section. Section 3 is consisted the main computation and the principal theorem, where we prove that " biharmonic Hopf QR-hypersurfaces in the quatenionic Eulidean space Q^n are minimal".

2 Preliminaries

In this section, we recall some fundamental definitions for the principal theorem about the biharmonic Hopf QR-hypersurfaces, which are immersed isometrically in the quaternionic Euclidean space Q^n. At first, we put the biharmonic concept of a hypersurface in the Euclidean space E^n.

Let $x : M \longrightarrow E^{n+1}$ be an isometric immersion of an n-dimensional hypersurface M into the Euclidean space E^{n+1}. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections on M and E^{n+1}, respectively. Let N be a local unit normal vector field to M in E^{n+1}. Assume that $\overrightarrow{H} = HN$ and $\overrightarrow{H}$ implies the mean curvature vector field. One of the considerable equation in differential geometry is $\triangle x = -n\overrightarrow{H}$, where $\triangle$ the Laplacian-Beltrami operator is defined $\triangle = -\ \mathrm{trace}\ \nabla^2$. The expressions assumed by the tension and bitension fields are

$$\tau(x) = n\overrightarrow{H}, \quad \tau_2(x) = -n\Delta\overrightarrow{H}, \tag{2.1}$$

then, the immersion x is called biharmonic if and only if $\triangle\overrightarrow{H} = 0$, where written as:

$$0 = \triangle\overrightarrow{H} = 2A(\mathsf{grad}H) + nH\mathsf{grad}H + (\triangle^{\perp}H + H\mathsf{trace}A^2),$$

by identifying the bitension field in its normal and tangent components, the main tool is obtained in the study of the proper biharmonic hypersurfaces in the Euclidean spaces.

Theorem 2.1. *[4] Let $x : M \longrightarrow E^{n+1}$ be an isometric immersion of an n-dimensional hypersurface M into the Euclidean space E^{n+1}. Then M is a biharmonic hypersurface if and only if*

$$\begin{cases} \triangle^{\perp}H + H\mathsf{trace}A^2 = 0; \\ 2A(\mathsf{grad}H) + nH\mathsf{grad}H = 0, \end{cases} \tag{2.2}$$

where A denotes the Weingarten operator and $\Delta^{\perp}$ the Laplacian in the normal bundle of M in E^{n+1}.

Suppose that, an n-dimension differentiable manifold M^n admits a 3-dimensional vector bundle V including tensors of type $(1,1)$, that satisfies:

1. In any coordinate neighborhood U on M^n there exists a local basis $\{J_1, J_2, J_3\}$ of V such that

$$\begin{aligned} J_sJ_t + J_tJ_s &= -2\delta_{st}I \qquad s,t = 1,2,3, \\ J_1J_2 &= J_3, \end{aligned} \tag{2.3}$$

the local base $\{J_1, J_2, J_3\}$ is named a canonical local base of the bundle V in the coordinate U. The bundle V is called an almost quaternionic structure and (M^n, V) is an almost quaternionic manifold. Essentially, the almost quaternionic manifolds are of dimension $4m$.

2. With respect to the Riemannian metric g, which is the Hermitian metric, the Levi-Civita connection $\overline{\nabla}$ of (M^n, g, V) and the canonical local basis $\{J_1, J_2, J_3\}$ of V in the coordinate U we have

$$\begin{aligned} \overline{\nabla}_X J_1 &= r(X)J_2 - q(X)J_3, \\ \overline{\nabla}_X J_2 &= -r(X)J_1 \quad + p(X)J_3, \\ \overline{\nabla}_x J_3 &= q(X)J_1 - p(X)J_2, \end{aligned} \tag{2.4}$$

for all $X \in T(M^n)$, where r, p and q are special local 1-forms define in U. Now, by taking into account the above conditions, M^n and V are called quaternionic Kählerian manifold and quaternionic Kählerian structure, respectively.

Also, a real submanifold M of real codimension p of a quaternionic Kählerian manifold is called a QR-submanifold of QR-dimension of r, provided that there is a r-dimensional normal distribution ν of the normal bundle $TM^{\perp}$ satisfies

$$\begin{aligned} J_s\nu_x \subset \nu_x, &\quad s = 1,2,3 \\ J_s\nu_x^{\perp} \subset T_xM, &\quad s = 1,2,3 \end{aligned}$$

at each point $x \in M$, where $\nu^\perp$ denotes the complementary orthogonal distribution to ν in $TM^\perp$. We recall a real hypersurface M is called the Hopf hypersurface, provided that $-J_sN$ is the eigenvector of the Weingarten operator A. Indeed, we concentrate on the biharmonic Hopf QR-hypersurfaces M, which are immersed in the quaternionic Euclidean space Q^n isometrically and equipped with the induced almost quaternionic structure too. So, we takeing into account the second term of the biharmonic condition (2.2), which yields the $\mathsf{grad}H$ is the eigenvector of the Weingarten operator corresponding to the eigenvalue $-\frac{4n+3}{2}H$. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections on M and Q^n, respectively. Also, there is a local unit normal vector field N on M such that $\nu^\perp = \mathsf{Span}\{N\}$ and $\xi_s = -J_sN \in T(M)$, where $s = 1, 2, 3$. In this way, the Hopf hypersurface is a QR-submanifold with the tangent bundle

$$T_x(M) = D_x \oplus D_x^\perp, \tag{2.5}$$

where D_x denotes the quaternionic bundle and $D_x^\perp = \mathsf{Span}\{\xi_1, \xi_2, \xi_3\}$ in $T_x(M)$. Also, suppose that the Weingarten operator A keeps D_x, that is, $AD_x = D_x$. Now, for any $X \in \chi(M)$ we have

$$J_sX = \varphi_sX + \eta_s(X)N, \qquad s = 1, 2, 3, \tag{2.6}$$

where φ_s is a $(1, 1)$ tensor, which acts on T_xM, and the one-form $\eta_s(X) = g(-J_sN, X)$ on M. Also, by taking the covariant derivative of both sides of the equation $\xi_s = -J_sN$, where $s = 1, 2, 3$, and applying the equations (2.4), then comparing the tangent and the normal parts written as:

$$\begin{aligned} \nabla_Y\xi_1 &= r(Y)\xi_2 - q(Y)\xi_3 + J_1AY, \\ \nabla_Y\xi_2 &= -r(Y)\xi_1 + p(Y)\xi_3 + J_2AY, \\ \nabla_Y\xi_3 &= q(Y)\xi_1 - p(Y)\xi_2 + J_3AY, \end{aligned} \tag{2.7}$$

where $Y \in \chi(M)$. We end this section with examples of the Hopf hypersurfaces in the Euclidean quaternionic space Q^n.

Example 2.1. Hypercylinder in R^4. Let

$$M = \{(x_1, x_2, x_3, x_4) \in R^4;\;\; x_1^2 + x_3^2 = 1\} \approx S^1 \times R^2,$$

we consider the quaternionic Kählerian structure $\{J_s\}$ for $s = 1, 2, 3$ as following

$$J_1 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},\; J_2 = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix},\; J_3 = \begin{pmatrix} -\sigma & 0 \\ 0 & \sigma \end{pmatrix},$$

where

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we consider a local unit normal vector field $N = (x_1, 0, x_3, 0)^t$ on M, as respect

$$-J_1N = \xi_1 = \begin{pmatrix} x_3 \\ 0 \\ -x_1 \\ 0 \end{pmatrix},\;\; -J_2N = \xi_2 = \begin{pmatrix} 0 \\ -x_3 \\ 0 \\ -x_1 \end{pmatrix},\;\; -J_3N = \xi_3 = \begin{pmatrix} 0 \\ x_1 \\ 0 \\ -x_3 \end{pmatrix},$$

then with respect to the Weingarten formula we have

$$\overline{\nabla}_{\xi_1} N = \xi_1, \quad \overline{\nabla}_{\xi_2} N = 0, \quad \overline{\nabla}_{\xi_3} N_3 = 0,$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on R^4. Hence, the above equations shows that ξ_s for $s = 1, 2, 3$ are the eigenvectors of the Weingarten operator A.

Furthermore, the hypersphere and the hyperplane are the Hopf hypersurfaces in the Euclidean quaternionic space R^4, obviously.

3 QR-hypersurfaces in the quaternionic Euclidean space

In this section we show that the biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n are minimal. More precisely, the biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n have three distinct principal curvatures at each point. So, according to the result in [9], M is a minimal QR-hypersurface. The goal will be obtained through the following investigation.

Lemma 3.1. *Let M be a Hopf biharmonic QR-hypersurface in the quaternionic Euclidean space Q^n. Let the Weingarten operator A satisfies $AX = \lambda X$, where $X \in D_x$ for $x \in M$. Then we have*

$$AJ_sX = \frac{\alpha_s \lambda}{2\lambda - \alpha_s} J_sX \qquad s = 1, 2, 3, \tag{3.1}$$

where α_s is an eigenvalue of the Weingarten operator corresponding to the eigenvector $\xi_s \in D^{\perp}$ and $\{J_1, J_2, J_3\}$ is the almost quaternionic structure.

Proof. Let $Y, Z \in \chi(M)$ and ξ_1 be an eigenvector of A corresponding to the eigenvalue α_1. Taking the covariant derivative of both sides of $A\xi_1 = \alpha_1 \xi_1$, which yields

$$\nabla_Y A\xi_1 \quad = \quad Y(\alpha_1)\xi_1 + \alpha_1 \nabla_Y \xi_1,$$

from the equation (2.7) we have

$$\begin{aligned} &(\nabla_Y A)\xi_1 + \alpha_2 r(Y)\xi_2 - \alpha_3 q(Y)\xi_3 + AJ_1AY \quad = \\ &Y(\alpha_1)\xi_1 + \alpha_1 r(Y)\xi_2 - \alpha_1 q(Y)\xi_3 + \alpha_1 J_1AY, \end{aligned}$$

and because the Weingarten operator is self adjoint it yields

$$\begin{aligned} &g((\nabla_Y A)Z, \xi_1) + \alpha_2 r(Y) g(\xi_2, Z) - \alpha_3 q(Y) g(\xi_3, Z) + g(AJ_1AY, Z) = \\ &\quad Y(\alpha_1) g(\xi_1, Z) + \alpha_1 r(Y) g(\xi_2, Z) - \alpha_1 q(Y) g(\xi_3, Z) + \alpha_1 g(J_1AY, Z), \end{aligned}$$

thus

$$\begin{aligned} g((\nabla_Y A)Z &- (\nabla_Z A)Y, \xi_1) \\ &+ 2g(AJ_1AY, Z) + \alpha_2 r(Y) g(\xi_2, Z) \\ &- \alpha_3 q(Y) g(\xi_3, Z) - \alpha_2 r(Z) g(\xi_2, Y) \\ &+ \alpha_3 q(Z) g(\xi_3, Y) = \\ & Y(\alpha_1) g(\xi_1, Z) + \alpha_1 r(Y) g(\xi_2, Z) \\ &- \alpha_1 q(Y) g(\xi_3, Z) - Z(\alpha_1) g(\xi_1, Y) \\ &- \alpha_1 r(Z) g(\xi_2, Y) + \alpha_1 q(Z) g(\xi_3, Y) \\ &+ \alpha_1 g(J_1AY, Z) - \alpha_1 g(J_1AZ, Y). \end{aligned}$$

Consequently, by applying the Codazzi equation we have

$$\begin{aligned} 2g(AJ_1AY, Z) &+ \alpha_2 r(Y) g(\xi_2, Z) \\ &- \alpha_3 q(Y) g(\xi_3, Z) - \alpha_2 r(Z) g(\xi_2, Y) \\ &+ \alpha_3 q(Z) g(\xi_3, Y) = \\ & Y(\alpha_1) g(\xi_1, Z) + \alpha_1 r(Y) g(\xi_2, Z) \\ &- \alpha_1 q(Y) g(\xi_3, Z) - Z(\alpha_1) g(\xi_1, Y) \\ &- \alpha_1 r(Z) g(\xi_2, Y) + \alpha_1 q(Z) g(\xi_3, Y) \\ &+ \alpha_1 g(J_1AY, Z) + \alpha_1 g(Z, AJ_1Y). \end{aligned} \tag{3.2}$$

Now according to the assumption, where $AX = \lambda X$ for $Z \in D$ from the equation (3.2) we get

$$\begin{aligned} 2g(AJ_1AX, Z) &= \alpha_1 g(J_1AX, Z) + \alpha_1 g(AJ_1X, Z), \\ (2\lambda - \alpha_1) g(AJ_1X, Z) &= \alpha_1 \lambda g(J_1X, Z), \\ AJ_1X &= \frac{\alpha_1 \lambda}{2\lambda - \alpha_1} J_1X. \end{aligned} \tag{3.3}$$

Similarly, the equation (3.3) can be hold for J_2 and J_3 as well. Hence, the result obtain as it is claimed. □

By applying the equation (3.2), where $Y \in D$, putting $Z = \xi_2$ or $Z = \xi_3$, and taking into account that $A\xi_t = \alpha_t \xi_t$ we get the following results.

Corollary 3.2. *Let M be a biharmonic Hopf QR-hypersurface in the quaternionic Euclidean space Q^n. Then the eigenvalues α_s, corresponding to the eigenvectors $\xi_s \in D^\perp$, where $s = 1, 2, 3$, of the Weingarten operator are equal, that is, $\alpha_1 = \alpha_2 = \alpha_3$.*

By summarizing the above information and taking into account that the eigenvalue corresponding to the eigenvector $\mathsf{grad}H$ of the Weingarten operator is unique [9], we have:

Corollary 3.3. *Let M be a biharmonic Hopf QR-hypersurface in the quaternionic Euclidean space Q^n. Then the eigenvector $\mathsf{grad}H$ can not be in $D^\perp$, that is, $\mathsf{grad}H$ is not in the direction of $\xi_s \in D^\perp$, where $s = 1, 2, 3$.*

Now we consider a biharmonic Hopf QR-hypersurface M in the quaternionic Euclidean space Q^n. Suppose that in the appropriate orthogonal frame field $\{X_1, ..., X_i, J_sX_1, ..., J_sX_i, \xi_t\}$, for $s,t = 1,2,3$, the Weingarten operator A satisfies $AX_i = \lambda_i X_i$, $A\xi_t = \alpha\xi_t$ and $AJ_sX_i = \lambda_{i_s} J_sX_i$, where $\lambda_{i_s} = \frac{\lambda_i\alpha}{2\lambda_i-\alpha}$ with respect to the Lemma 3.1. Also, we have

$$\nabla_{\xi_r} X_i = \sum_{j=1}^{n} \omega_{ri}^{j} X_j + \sum_{s=1,2,3} \sum_{j=1}^{n} \omega_{ri}^{j_s} J_s X_j + \sum_{t=1}^{3} \omega_{ri}^{t} \xi_t, \tag{3.4}$$

$$\nabla_{X_i} \xi_r = \sum_{j=1}^{n} \omega_{ir}^{j} X_j + \sum_{s=1,2,3} \sum_{j=1}^{n} \omega_{ir}^{j_s} J_s X_j + \sum_{t=1}^{3} \omega_{ir}^{t} \xi_t, \tag{3.5}$$

where $\omega_{ri}^{j_s}$, ω_{ri}^{j},..., are the smooth functions on M for $1 \le i,j \le n$ and $r,s,t = 1,2,3$. Morevere, with respect to the equations (3.4) and (3.5), then the Codazzi equation implies

$$(\lambda_i - \lambda_j)\omega_{ri}^{j} = (\alpha - \lambda_j)\omega_{ir}^{j}, \tag{3.6}$$

$$(\lambda_i - \lambda_{j_s})\omega_{ri}^{j_s} = (\alpha - \lambda_{j_s})\omega_{ir}^{j_s}, \tag{3.7}$$

for distinct i and j, where $r,s,t = 1,2,3$ and $i,j = 1,...,n$.

Putting all the above information together and summarizing them, we are ready to prove the principal theorem about the Hopf biharmonic QR-hypersurface in the quternionic Euclidean space Q^n.

Theorem 3.4. *The biharmonic Hopf QR-hypersurfaces in the quaternionic Euclidean space Q^n are minimal.*

Proof. With respect to the proceeding Lemma, there is an appropriate orthogonal frame $\{X_1, ..., X_n, J_sX_1, ..., J_sX_n, \xi_1, \xi_2, \xi_3\}$, for $s = 1,2,3$ such that X_1 is parallel to the grad$H$, where the grad$H \neq 0$, and the shape operator A of M is taken the following form:

$$\begin{array}{rcll} AX_i &=& \lambda_i X_i \quad ; & 1 \le i \le n \\ A\xi_t &=& \alpha\xi_t \quad ; & t = 1,2,3 \\ AJ_sX_i &=& \lambda_{i_s} J_sX_i \quad ; & s = 1,2,3, \quad 1 \le i \le n, \end{array} \tag{3.8}$$

where λ_i and $\lambda_{i_s} = \frac{\lambda_i\alpha}{2\lambda_i-\alpha}$ are the eigenvalues corresponding to the eigenvectors X_i and J_sX_i, that $s = 1,2,3$, respectively. Also, α is the eigenvalue corresponding to the eigenvectors $\xi_t \in D^\perp$ and $\lambda_1 = -\frac{4n+3}{2}H$ too. Let X, Y and Z be in $\chi(M)$. Then the Codazzi equation yields

$$g(\nabla_X AY, Z) - g(\nabla_Y AX, Z) \;=\; g(A\nabla_X Y, Z) - g(A\nabla_Y X, Z), \tag{3.9}$$

let $Y = \xi_1$ and pay attention to the assumption $AX_i = \lambda_i X_i$, where $1 < i \le n$. Then from the equation (3.9) we get

$$g(\nabla_{X_i} A\xi_1, Z) - g(\nabla_{\xi_1} \lambda_i X_i, Z) = g(A\nabla_{X_i}\xi_1, Z) - g(A\nabla_{\xi_1} X_i, Z), \tag{3.10}$$

now the equations (3.10) and (2.7) follow

$$\begin{aligned} X_i(\alpha)g(\xi_1, Z) &+ \alpha g(\nabla_{X_i}\xi_1, Z) \\ &- \xi_1(\lambda_i)g(X_i, Z) + g(\lambda_i \nabla_{\xi_1} X_i, Z) = \\ &\alpha r(X_i)g(\xi_2, Z) - \alpha q(X_i)g(\xi_3, Z) \\ &+ \lambda_i g(AJ_1X_i, Z) - g(A\nabla_{\xi_1}X_i, Z), \end{aligned}$$

then we have

$$\begin{aligned} X_i(\alpha)g(\xi_1, Z) &+ \alpha r(X_i)g(\xi_2, Z) \\ &- \alpha q(X_i)g(\xi_3, Z) + \alpha\lambda_i g(J_1X_i, Z) \\ &- \xi_1(\lambda_i)g(X_i, Z) + \lambda_i g(\nabla_{\xi_1}X_i, Z) = \\ &\alpha r(X_i)g(\xi_2, Z) - \alpha q(X_i)g(\xi_3, Z) \\ &+ \lambda_i g(AJ_1X_i, Z) - g(A\nabla_{\xi_1}X_i, Z). \end{aligned} \tag{3.11}$$

Suppose that $Z = X_j$ is an eigenvector of the Weingarten operator corresponding to the eigenvalue λ_j, where $i \neq j$. Then from the equation (3.11) we have:

$$-\lambda_i g(\nabla_{\xi_1}X_i, X_j) = -\lambda_j g(\nabla_{\xi_1}X_i, X_j), \tag{3.12}$$

where $j \neq 1$. Now, one more time we suppose that $Z = J_sX_j$, where $s = 1, 2, 3$ and utilize the equation (3.11) we get

$$\lambda_i g(\nabla_{\xi_1}X_i, J_sX_j) = \lambda_{j_s} g(\nabla_{\xi_1}X_i, J_sX_j). \tag{3.13}$$

Finally, we show that the equations (3.12) and (3.13), follow $\lambda_i = \lambda_j = \alpha$.
Indeed, from the equations (3.4), (3.5) and (3.12) we have

$$(\lambda_i - \lambda_j)\omega_{1i}^j g(X_j, X_j) = 0, \tag{3.14}$$

where ω_{1i}^j satisfies at the equation (3.6). Now, taking into account the equations (3.6) and (3.14), then the results written as:

1. $\lambda_i = \lambda_j$, provided that $\omega_{1i}^j \neq 0$ for distinct i and j, where $1 < i, j \leq n$.

2. If $\omega_{1i}^j = 0$, either $\alpha = \lambda_j$ or $\lambda_i = \lambda_j$ for distinct i and j, where $1 < i, j \leq n$.

Similarly, from (3.4), (3.5) and (3.13) we have

$$(\lambda_i - \lambda_{j_s})\omega_{1i}^{j_s} g(J_sX_j, J_sX_j) = 0, \tag{3.15}$$

where $\lambda_{j_s} = \frac{\lambda_j \alpha}{2\lambda_j - \alpha}$ is the eigenvalue corresponding to the eigenvector J_sX_j that $\omega_{1i}^{j_s}$ satisfies the (3.7). After all, by taking the equations (3.7) and (3.15) we arrive at

1. $\lambda_i = \lambda_{j_s}$ if $\omega_{1i}^{j_s} \neq 0$, where $s = 1, 2, 3$ and $1 < i, j \leq n$.

2. If $\omega_{i1}^{j_s} = 0$, either $\lambda_j = \alpha$ or $\lambda_i = \lambda_{j_s}$, where $s = 1, 2, 3$ and $1 < i, j \leq n$.

Consequently, with respect to the results 1 and 2 of the both above cases, we obtain $\lambda_i = \lambda_j = \alpha$, where $(i \neq j \neq 1)$ as it is claimed.

Furthermore, we point that the $\mathsf{grad}H$ is an eigenvector of the Weingarten operator corresponding to the unique eigenvalue $-\frac{4n+3}{2}H$. Also, the Lemma 3.1 shows $J_s(\mathsf{grad}H)$ are the eigenvector with respect to the same eigenvalue $\frac{(4n+3)H\alpha}{2(\alpha-(4n+3)H)}$, where $s = 1, 2, 3$. After all, the above computation follows the Weingarten operator has three distinct eigenvalues, at follow the Hopf biharmonic QR-hypersurfaces in the Euclidean space Q^n have three distinct principal curvatures. Then, according to the result in [9], these hypersurfaces are minimal. □

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