Almost Yamabe solitons on LP-Sasakian manifolds with generalized symmetric metric connection of type (α, β)

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Abstract. We classify almost Yamabe and Yamabe solitons on Lorentzian para (briefly, LP) Sasakian manifolds whose potential vector field is torse-forming, admitting a generalized symmetric metric connection of type (α, β). Certain results of such solitons on CR-submanifolds of LP-Sasakian manifolds with respect to a generalized symmetric metric connection are obtained. Finally, a non-trivial example is given to validate our some results.

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1 Introduction

Much progress has been done in recent years in the study of soliton solutions of the Ricci flow, the mean curvature flow and the Yamabe flow. Soliton solutions correspond to self-similar solutions of the corresponding flow. The Yamabe flow,

$$\frac{\partial}{\partial t} g(t) = -R(t)g(t), \tag{1.1}$$

where $R(t)$ is the scalar curvature of the metric $g(t)$, was introduced by Hamilton [10], as an approach to solve the Yamabe problem. In dimension $n(= 2)$, the Yamabe flow is equivalent to the Ricci flow. However, in dimension $n > 2$ the Yamabe and Ricci flow do not agree, since the first one preserves the conformal class of the metric but the Ricci flow does not in general.

A Yamabe soliton on a Riemannian manifold (M, g) of dimension n is a special solution of the Yamabe flow. A triplet structure (g, κ, λ) satisfies

$$\frac{1}{2}\mathfrak{L}_\kappa g(X, Y) = (\hat{\delta} - \lambda)g(X, Y) \tag{1.2}$$

for all X, Y on M is known as a Yamabe soliton, where $\mathfrak{L}_\kappa$ denotes the Lie derivative of the metric g along the vector field κ, $\hat{\delta}$ is the scalar curvature and λ is a real constant.

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The beauty of such soliton depends on the flavor of λ. The soliton is said to be expanding, steady or shrinking according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If $\lambda \in C^\infty(M)$, then the metric satisfying (1.2) is called almost Yamabe soliton [1]. Thus, the almost Yamabe solitons are the generalization of Yamabe solitons. Moreover, if κ is the gradient of some function $\widetilde{\mathfrak{f}}$ on M, then it is known as a gradient Yamabe soliton. In context of geometry, the Yamabe solitons are special solution of Yamabe flow under some regulation. There are several geometers that light up quite extensively on the beauty of Yamabe flow and Yamabe soliton ([6], [7], [9], [11]).

A vector field κ on a Riemannian manifold (M, g) is known as a torse-forming vector field [17] if it satisfies

$$\nabla_X \kappa = \psi X + \theta(X)\kappa, \ \forall \ X \in \chi(M), \tag{1.3}$$

where $\psi \in C^\infty(M)$ and θ is a 1-form. The beauty of such vector field as follows:
i) it is concircular if the 1-form θ vanishes identically [19],
ii) for concurrent, $\psi = 1$ and $\theta = 0$ [18],
iii) it is recurrent if $\psi = 0$,
iv) for parallel if $\psi = \theta = 0$.
In 2017, Chen [5] initiated a new type vector field known as torqued vector field if the vector field κ satisfying (1.2) with $\theta(\kappa) = 0$, where ψ is called the torqued function with the 1-form θ, called the torqued form of κ.

In 1989, Matsumoto [12] introduced the notion of Lorentzian para-Sasakian manifolds. Mihai and Rosca [14] studied the same manifolds independently and they obtained several results on such manifolds. Lorentzian para-Sasakian manifolds have also been studied by Matsumoto and Mihai [13] and Mihai, Shaikh and De [15]. In [16] Perktas, Kilic and Tripathi investigated curvature tensors with respect to semi-symmetric connection in a Lorentzian para-Sasakian manifold. Recently, Chaubey with De [3, 4] characterized the Lorentzian manifolds with quarter-symmetric connections.

The sections of this paper are organized as follows. After introduction, Section 2 contains some definitions and basic results of LP-Sasakian manifolds. In Section 3, we recall generalized symmetric metric connection of type (α, β) for LP-Sasakian manifold. Section 4 is devoted to CR-submanifolds of LP-Sasakian manifolds with respect to a generalized symmetric metric connection of type (α, β). In Section 5, we study Yamabe soliton whose potential vector field is torse-forming vector field on LP-Sasakian manifold with respect to such connection. Section 6 concerns with the study of Yamabe soliton with a torse-forming vector field on CR-submanifolds of LP-Sasakian manifolds. Furthermore, we study almost Yamabe soliton with torse-forming vector field by taking κ^t and κ^n as tangential and normal components of such vector field on CR-submanifolds of LP-Sasakian manifolds in Section 7.

2 LP-Sasakian Manifolds

A differentiable manifold M^n of dimension n is called a *Lorentzian para-Sasakian manifold* (briefly, LP-Sasakian manifold) ([12],[14]), if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and the Lorentzian metric g such that

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y), \tag{2.6}$$

where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, the following relations hold:

$$\phi\xi = 0, \ \ \eta(\phi X) = 0, \ \ \text{rank}\,(\phi) = n - 1. \tag{2.7}$$

If we write

$$\Phi(X, Y) = g(\phi X, Y) \tag{2.8}$$

for any vector field X and Y on M^n, then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [12]. Also, since the vector η is closed in an LP-Sasakian manifold ([12],[15]), we have

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \ \ \Phi(X, \xi) = 0 \tag{2.9}$$

for any vector field X and Y on M^n. Let $M^n(\phi, \xi, \eta, g)$ be an n-dimensional LP-Sasakian manifold, then the following relations hold ([13],[15]):

$$g(R(X, Y)Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.10}$$

$$g(R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.13}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \tag{2.15}$$

for any vector fields X, Y and Z on M^n, where R and S are the curvature tensor and Ricci tensor of $M^n(\phi, \xi, \eta, g)$, respectively.

Let $\grave{M}$ be a submanifold of an LP-Sasakian manifold $M^n(\phi,\xi,\eta,g)$. The Gauss and Weingarten formulas are given by

$$\nabla_X Y = \grave{\nabla}_X Y + h(X,Y),\ \ \forall\ X,Y \in \Gamma(T\grave{M}), \tag{2.16}$$

$$\nabla_X N = -A_N X + \nabla^{\perp}_X N,\ \ \forall\ N \in \Gamma(T^{\perp}\grave{M}), \tag{2.17}$$

where $\nabla_X Y$ and $\{h(X,Y), \nabla^{\perp}_X N\}$ belong to $\Gamma(T\grave{M})$ and $\Gamma(T^{\perp}\grave{M})$, respectively.

3 Generalized Symmetric Metric Connection of type (α,β)

Let $\widetilde{\nabla}$ be a linear connection and ∇ be a Levi-Civita connection of an LP-Sasakian manifold $M^n(\phi,\xi,\eta,g)$.

Lemma 3.1. *[2] In an LP-Sasakian manifold $M^n(\phi,\xi,\eta,g)$, the generalized symmetric metric connection $\widetilde{\nabla}$ of type (α,β) is given by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)X - g(X,Y)\xi\} + \beta\{\eta(Y)\phi X - g(\phi X,Y)\xi\} \tag{3.1}$$

for all X and Y on M^n. If we choose (α,β)=$(1,0)$ and (α,β)=$(0,1)$, the generalized symmetric metric connection reduces to a semi-symmetric metric connection and quarter-symmetric metric connection, respectively. These two connections play important roles in geometry and physics.

Lemma 3.2. *[2] Let $M^n(\phi,\xi,\eta,g)$ be an LP-Sasakian manifold with a generalized symmetric metric connection $\widetilde{\nabla}$. Then we have the following relations:*

$$(\widetilde{\nabla}_X\phi)(Y) = [(1-\beta)\{g(X,Y)+2\eta(X)\eta(Y)\}-\alpha\Phi(X,Y)]\xi+(1-\beta)\eta(Y)X-\alpha\eta(Y)\phi X,$$

$$\widetilde{\nabla}_X\xi = (1-\beta)\phi X - \alpha X - \alpha\eta(X)\xi,$$

$$(\widetilde{\nabla}_X\eta)(Y) = (1-\beta)\Phi(X,Y) - \alpha g(\phi X,\phi Y),$$

$$\widetilde{R}(X,Y)\xi = (1-\beta+\beta^2)\{\eta(Y)X - \eta(X)Y\} + \alpha(1-\beta)\{\eta(X)\phi Y - \eta(Y)\phi X\},$$

$$\widetilde{R}(\xi,Y)\xi = (1-\beta+\beta^2)\{\eta(Y)\xi + Y\} + \alpha(\beta-1)\phi Y,$$

and
$$\begin{aligned}\overline{S}(X,Y) = S(X,Y) &+ \{-\alpha\beta + (n-2)(\alpha\beta-\alpha) + (\beta^2-2\beta)trace\Phi\}\Phi(X,Y)\\ &+\{-2\alpha^2+\beta-\beta^2+n\alpha^2+(\alpha\beta-\alpha)trace\Phi\}g(X,Y)\\ &+\{-2\alpha^2+n(\alpha^2+\beta-\beta^2)\}\eta(X)\eta(Y)\end{aligned} \tag{3.2}$$

for any X, Y $\in (TM)$.

4 CR-Submanifolds of an LP-Sasakian manifold with Generalized Symmetric metric connection of type (α, β)

Definition 4.1. [8] An n-dimensional Riemannian manifold (M, g) of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that
i) D is invariant under ϕ, i.e., $\phi D \subset D$.
ii) The orthogonal complement distribution $D^\perp : x \to D_x^\perp \subset T_x M$ of the distribution D on M is totally real, i.e., $\phi D^\perp \subset T^\perp M$.

Definition 4.2. [8] If the distribution D $(resp., D^\perp)$ is horizontal (resp., vertical), then the pair $(D, D^\perp)$ is called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ (resp., $\xi \in \Gamma(D^\perp)$). The CR-submanifold is also called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ $(resp., \xi \in \Gamma(D^\perp))$.

The orthogonal complement $\phi D^\perp \in T^\perp M$ is given by

$$TM = D \oplus D^\perp, \;\; T^\perp M = \phi D^\perp \oplus \mu, \tag{4.1}$$

where $\phi\mu{=}\mu$.

Let $\grave{M}$ be a CR-submanifold of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$ with a generalized symmetric metric connection $\widetilde{\nabla}$. For any $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we can write

$$X = PX + QX, \;\; PX \in \Gamma(D), \;\; QX \in \Gamma(D^\perp), \tag{4.2}$$

$$\phi N = BN + CN, \;\; BN \in \Gamma(D^\perp), \;\; CN \in \Gamma(\mu). \tag{4.3}$$

The Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ are, respectively, given by

$$\widetilde{\nabla}_X Y = \grave{\widetilde{\nabla}}_X Y + \widetilde{h}(X, Y), \tag{4.4}$$

$$\widetilde{\nabla}_X N = -\widetilde{A}_N X + \widetilde{\nabla}_X^\perp N \tag{4.5}$$

for any $X, Y \in \Gamma(TM)$, where $\widetilde{\nabla}_X Y$, $\widetilde{A}_N X \in \Gamma(TM)$. Here $\grave{\widetilde{\nabla}}$, $\widetilde{h}$ and $\widetilde{A}_N$ are called the induced connection on $\grave{M}$, the second fundamental form and the Weingarten mapping with respect to $\widetilde{\nabla}$, respectively. In view of (2.16), (3.1) and (4.4), we get

$$\begin{aligned}\grave{\widetilde{\nabla}}_X Y + \widetilde{h}(X, Y) = \grave{\nabla}_X Y + h(X, Y) + \alpha\{\eta(Y)X - g(X, Y)\xi\} \\ +\beta\{\eta(Y)\phi X - g(\phi X, Y)\xi\}.\end{aligned} \tag{4.6}$$

Using (4.2) and (4.3) in equation (4.6) and comparing the tangential and normal components on both sides, we obtain

$$P\grave{\widetilde{\nabla}}_X Y = P\grave{\nabla}_X Y + \alpha\eta(Y)PX - \alpha g(X, Y)P\xi + \beta\eta(Y)\phi PX - \beta g(\phi X, Y)P\xi, \tag{4.7}$$

$$\widetilde{h}(X, Y) = h(X, Y) + \beta\eta(Y)\phi QX, \tag{4.8}$$

$$Q\grave{\widetilde{\nabla}}_X Y = Q\grave{\nabla}_X Y + \alpha\eta(Y)QX - \alpha g(X, Y)Q\xi - \beta g(\phi X, Y)Q\xi \tag{4.9}$$

for any $X, Y \in (TM)$.

5 Yamabe solitons whose potential vector field is torse-forming

As per this consequence of our analysis in this section we have the following geometric characterization of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$ admitting a generalized symmetric metric connection $\widetilde{\nabla}$ of type (α, β). Thus, in view of our thought we can state the following:

Theorem 5.1. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, with respect to a generalized symmetric metric connection of type (α, β). If κ is a torse-forming vector field, then (g, κ, λ) is expanding, steady and shrinking according as $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\} \lesseqqgtr 0$, unless $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\}$ is constant.*

Proof. Let (g, κ, λ) be a Yamabe soliton on $M^n(\phi, \xi, \eta, g)$ with respect to a generalized symmetric metric connection of type (α, β). So from (1.2), we have

$$\frac{1}{2}(\widetilde{\mathfrak{L}}_\kappa g)(X, Y) = (\widetilde{\hat{\delta}} - \lambda)g(X, Y). \tag{5.1}$$

From the definition of Lie derivative, equations (1.3) and (3.1), we obtain

$$\begin{aligned}(\widetilde{\mathfrak{L}}_\kappa g)(X, Y) &= g(\widetilde{\nabla}_X \kappa, Y) + g(X, \widetilde{\nabla}_Y \kappa)\\ &= 2\psi g(X, Y) + \theta(X)g(\kappa, Y) + \theta(Y)g(\kappa, X)\\ &\quad +\alpha\{2\eta(\kappa)g(X, Y) - g(X, \kappa)\eta(Y) - g(Y, \kappa)\eta(X)\}\\ &\quad +\beta\{2\eta(\kappa)g(\phi X, Y) - g(\phi X, \kappa)\eta(Y) - g(\phi Y, \kappa)\eta(X)\}\end{aligned} \tag{5.2}$$

for all $X, Y \in \chi(M)$. With the help of (5.1) and (5.2), we get

$$\begin{aligned}(\psi - \widetilde{\hat{\delta}} - \lambda)g(X, Y) &= \frac{1}{n}\{\theta(X)g(\kappa, Y) + \theta(Y)g(\kappa, X)\}\\ &\quad +\alpha\eta(\kappa)g(X, Y) + \beta\eta(\kappa)g(\phi X, Y)\\ &\quad -\frac{\alpha}{2}\{g(X, \kappa)\eta(Y) + g(Y, \kappa)\eta(X)\}\\ &\quad -\frac{\beta}{2}\{g(\phi X, \kappa)\eta(Y) + g(\phi Y, \kappa)\eta(X)\}.\end{aligned} \tag{5.3}$$

Taking an orthonormal frame field and then contracting (5.3), we have

$$\lambda = \psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\}. \tag{5.4}$$

This leads to Theorem 5.1. □

In this sequel, we write the following corollaries.

Corollary 5.2. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, with respect to a generalized symmetric metric connection of type (α, β). Then following relations hold*

κ	Existence condition	Nature of solitons (shrinking, steady or expanding)
torse-forming	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+\alpha(n-1)\eta(\kappa)\}=$ *constant*	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+\alpha(n-1)\eta(\kappa)\}\lesseqqgtr 0$
concircular	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}=$ *constant*	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}\lesseqqgtr 0$
concurrent	$1-\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}=$ *constant*	$1-\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}\lesseqqgtr 0$
recurrent	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+\alpha(n-1)\eta(\kappa)\}=$ *constant*	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+\alpha(n-1)\eta(\kappa)\}\lesseqqgtr 0$
parallel	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}=$ *constant*	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}\lesseqqgtr 0$
torqued	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}=$ *constant*	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\alpha(n-1)\eta(\kappa)\}\lesseqqgtr 0$

Corollary 5.3. *Let (g,κ,λ) be a Yamabe soliton on an n-dimensional LP-Sasakian manifold $M^n(\phi,\xi,\eta,g)$, $n>1$, with respect to a generalized symmetric metric connection of type (α,β)=$(1,0)$. If κ is torse-forming vector field, then (g,κ,λ) is expanding, steady and shrinking according as $\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\}\lesseqqgtr 0$, unless $\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\}$ is constant.*

Corollary 5.4. *Let $M^n(\phi,\xi,\eta,g)$ be an $n(>1)$-dimensional LP-Sasakian manifold endowed with a generalized symmetric metric connection of type (α,β)=$(0,1)$. If (g,κ,λ) be a Yamabe soliton on M^n and κ is a torse-forming vector field, then (g,κ,λ) is expanding, steady and shrinking according as $\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)\}\lesseqqgtr 0$, unless $\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)\}$ is constant.*

Corollary 5.5. *Let (g,κ,λ) be a Yamabe soliton on an n-dimensional LP-Sasakian manifold $M^n(\phi,\xi,\eta,g)$, $n>1$, with respect to a generalized symmetric metric connection of type (α,β)=$(0,1)$. Then the following relations hold*

κ	*Existence condition*	*Nature of solitons*
torse-forming	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)\}$=*constant*	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)\}\lesseqqgtr 0$
concircular	$\psi-\widetilde{\hat{\delta}}$=*constant*	$\psi-\widetilde{\hat{\delta}}\lesseqqgtr 0$
concurrent	$1-\widetilde{\hat{\delta}}$=*constant*	$\widetilde{\hat{\delta}}\lesseqqgtr 1$
recurrent	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)\}$=*constant*	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)\}\lesseqqgtr 0$
parallel	$\widetilde{\hat{\delta}}$=*constant*	$\widetilde{\hat{\delta}}\lesseqqgtr 0$
torqued	$\psi-\widetilde{\hat{\delta}}$=*constant*	$\psi-\widetilde{\hat{\delta}}\lesseqqgtr 0$

Corollary 5.6. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, with respect to a generalized symmetric metric connection of type (α, β)=$(1, 0)$. Then following relations hold*

κ	Existence condition	Nature of solitons (shrinking, steady or expanding)
torse-forming	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$=*constant*	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$
concircular	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}$=*constant*	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$
concurrent	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}$=*constant*	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$
recurrent	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$=*constant*	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$
parallel	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}$=*constant*	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$
torqued	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}$=*constant*	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$

6 Yamabe solitons whose potential vector field is torse-forming on CR-submanifold of an LP-Sasakian manifold

In this section, we study Yamabe soliton whose potential vector field is a torse-forming on CR-submanifolds of an LP-Sasakian manifold with respect to the induced connection $\tilde{\grave{\nabla}}$ of type (α, β). We state the following theorem as:

Theorem 6.1. *Let the CR-submanifold $\grave{M}$ of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, admitting a generalized symmetric metric connection $\tilde{\grave{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\tilde{\grave{\nabla}}$. If (g, κ, λ) be a Yamabe soliton on $\grave{M}$ and κ is a torse-forming vector field, then (g, κ, λ) is expanding, steady or shrinking according as $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$, unless $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\}$ is constant.*

Proof. If M is ξ-horizontal for all X, $Y \in \Gamma(D)$ and D is parallel with respect to $\tilde{\grave{\nabla}}$, then in view of (4.7), we have

$$\tilde{\grave{\nabla}}_X Y = \grave{\nabla}_X Y + \alpha\{\eta(Y)X - g(X, Y)\xi\} + \beta\{\eta(Y)\phi X - g(\phi X, Y)\xi\}. \tag{6.1}$$

With the help of Lemma 3.1, we conclude that the induced connection $\grave{\nabla}$ is also a generalized symmetric metric connection of type (α, β). This leads to the statement of the Theorem 6.1. □

In this sequel, we write the following corollaries.

Corollary 6.2. *Let a CR-submanifold $\grave{M}$ of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, admitting a generalized symmetric metric connection $\grave{\grave{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\grave{\grave{\nabla}}$. If (g, κ, λ) be a Yamabe soliton on $\grave{M}$ and κ is a torse-forming vector field, then the following results hold.*

κ	*Existence condition*	*Nature of solitons (shrinking, steady or expanding)*
torse-forming	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\} = constant$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
concircular	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} = constant$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
concurrent	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} = constant$	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
recurrent	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\} = constant$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + \alpha(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
parallel	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} = constant$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
torqued	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} = constant$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\alpha(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$

Corollary 6.3. *Let a CR-submanifold $\grave{M}$ of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, admitting a generalized symmetric metric connection $\grave{\grave{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\grave{\grave{\nabla}}$ of type $(\alpha, \beta) = (0, 1)$. If (g, κ, λ) be a Yamabe soliton on $\grave{M}$ and κ is a torse-forming vector field, then the following relations hold.*

κ	*Existence condition*	*Nature of solitons (shrinking, steady or expanding)*
torse-forming	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} = constant$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
concircular	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = constant$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
concurrent	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = constant$	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
recurrent	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} = constant$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
parallel	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = constant$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$
torqued	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} = constant$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} \substack{<\\=\\>} 0$

Corollary 6.4. *Let a CR-submanifold $\grave{M}$ of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, admitting a generalized symmetric metric connection $\grave{\grave{\nabla}}$ is ξ-horizontal (resp.*

ξ-vertical) and D is parallel with respect to $\dot{\grave{\nabla}}$ of type $(\alpha, \beta) = (1, 0)$. If (g, κ, λ) be a Yamabe soliton on $\grave{M}$ and κ is a torse-forming vector field, then (g, κ, λ) is expanding, steady or shrinking according as $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} \underset{>}{\leqq} 0$, unless $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$ is constant.

Corollary 6.5. *Let a CR-submanifold $\grave{M}$ of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, admitting a generalized symmetric metric connection $\dot{\grave{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\dot{\grave{\nabla}}$ of type $(\alpha, \beta) = (0, 1)$. If (g, κ, λ) be a Yamabe soliton on $\grave{M}$ and κ is a torse-forming vector field, then (g, κ, λ) is expanding, steady or shrinking according as $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} \underset{>}{\leqq} 0$, unless $\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\}$ is constant.*

Corollary 6.6. *Let (g, κ, λ) be a Yamabe soliton on n-dimensional LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $(n > 1)$, with respect to generalized symmetric metric connection of type (α, β)=$(0, 1)$. Then following relations hold.*

κ	*Existence condition*	*Nature of solitons (shrinking, steady or expanding)*
torse-forming	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\}$=*constant*	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} \underset{>}{\leqq} 0$
concircular	$\psi - \widetilde{\hat{\delta}}$=*constant*	$\psi - \widetilde{\hat{\delta}} \underset{>}{\leqq} 0$
concurrent	$1 - \widetilde{\hat{\delta}}$=*constant*	$\widetilde{\hat{\delta}} \underset{>}{\leqq} 1$
recurrent	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\}$=*constant*	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} \underset{>}{\leqq} 0$
parallel	$\widetilde{\hat{\delta}}$=*constant*	$\widetilde{\hat{\delta}} \underset{>}{\leqq} 0$
torqued	$\psi - \widetilde{\hat{\delta}}$=*constant*	$\psi - \widetilde{\hat{\delta}} \underset{>}{\leqq} 0$

7 Almost Yamabe solitons whose potential vector field is torse-forming on CR-submanifold of an LP-Sasakian manifold

In this section, we classify almost Yamabe solitons whose potential field is torse-forming on CR-submanifold of an LP-Sasakian manifold with respect to a generalized symmetric metric connection of type (α, β). At this stage, we denote κ^t and κ^n as tangential and normal components of such vector field. For almost Yamabe soliton we have the following.

Theorem 7.1. *An almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, with a generalized symmetric metric connection of type (α, β) satisfies*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\}$$
$$+\frac{\beta}{2}\{g(\kappa^n, \phi X)\eta(Y) + g(\phi Y, \kappa^n)\eta(X)\} \tag{7.1}$$

for any vector fields X, Y on $\grave{M}$.

Proof. In view of (1.3), (3.1), (4.4) and (4.5), we have

$$\begin{aligned} \psi X + \theta(P)\kappa = \widetilde{\nabla}_X \kappa = \widetilde{\nabla}_X(\kappa^t + \kappa^n) = \grave{\nabla}_X \kappa^t + h(X, \kappa^t) + \beta\eta(\kappa^t)\phi QX \\ -A_{\kappa^n} X + \nabla^{\perp}_X \kappa^n + \alpha\eta(\kappa^n)X + \beta\eta(\kappa^n)\phi X - \beta g(\phi X, \kappa^n)\xi. \end{aligned} \tag{7.2}$$

On comparing tangential and normal components of (7.2), we obtain

$$\grave{\nabla}_X \kappa^t = \psi X + \theta(P)\kappa + A_{\kappa^n} X - \alpha\eta(\kappa^n)X - \beta\eta(\kappa^n)\phi X + \beta g(\phi X, \kappa^n)\xi \tag{7.3}$$

and

$$h(X, \kappa^t) = -\nabla^{\perp}_X \kappa^n - \beta\eta(\kappa^n)\phi QX. \tag{7.4}$$

From the definition of Lie derivative and (7.4), we have

$$\begin{aligned} \mathfrak{L}_{\kappa^t} g(X, Y) = 2\psi g(X, Y) + 2g(A^n_\kappa X, Y) - 2\eta(\kappa^n)g(X, Y) + \theta(X)g(\kappa, Y) \\ +\theta(Y)g(X, \kappa) + \beta\{g(\kappa^n, \phi X)\eta(Y) + g(\phi Y, \kappa^n)\eta(X)\}. \end{aligned} \tag{7.5}$$

Using (7.5) in (1.2), we yield

$$\begin{aligned} (\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \\ +\frac{\beta}{2}\{g(\kappa^n, \phi X)\eta(Y) + g(\phi Y, \kappa^n)\eta(X)\}. \end{aligned} \tag{7.6}$$

This proves our assertion. □

Corollary 7.2. *If an almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, with a generalized symmetric metric connection of type (α, β) is minimal, then*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))n = \theta(\kappa). \tag{7.7}$$

Corollary 7.3. *Let (g, κ^t, λ) be an almost Yamabe soliton on a CR-submanifold of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, and the distribution is ξ-horizontal (resp. ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\nabla}$ of type (α, β). Then we have*

$$\begin{aligned} (\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \\ +\frac{\beta}{2}\{g(\kappa^n, \phi X)\eta(Y) + g(\phi Y, \kappa^n)\eta(X)\} \end{aligned} \tag{7.8}$$

for any vector fields X, Y on $\grave{M}$.

Corollary 7.4. *If an almost Yamabe soliton (g, κ^t, λ) on CR-submanifold of an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $(n > 1)$ and the distribution is ξ-horizontal (resp. ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\nabla}$ of type (α, β) is minimal, then*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))n = \theta(\kappa) \tag{7.9}$$

8 Example

Example.4.1 Let us consider a 4-dimensional differentiable manifold $M = \{(x, y, z, t) \in \Re^4 : (x, y, z, t) \neq 0\}$, where (x, y, z, t) is the standard coordinate in $\Re^4$. Let (e_1, e_2, e_3, e_4) be a set of linearly independent vector fields at each point of M, and is defined by

$$e_1 = e^{x-t}\frac{\partial}{\partial x}, \quad e_2 = e^{y-t}\frac{\partial}{\partial y}, \quad e_3 = e^{z-t}\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

Define a Lorentzian metric g on M as:

$$g_{ij} = g(e_i, e_j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ -1 & \text{if} \quad i = j = 4 \\ 1 & \text{otherwise} \end{cases}.$$

Let η be the 1-form associated with the Lorentzian metric g by

$$\eta(X) = g(X, e_4)$$

for any $X \in \Gamma(TM)$, where $\Gamma(TM)$ is the set of all smooth vector fields on M. If the $(1, 1)$-tensor field ϕ is defined by

$$\phi(e_1) = e_1, \qquad \phi(e_2) = e_2, \qquad \phi(e_3) = e_3, \qquad \phi(e_4) = 0,$$

then by the linearity properties of ϕ and g, we can easily verify the following relations:

$$\eta(e_4) = -1, \quad \phi^2(X) = X + \eta(X)e_4, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any X, $Y \in \Gamma(TM)$. Thus for $e_4 = \xi$, the structure (ϕ, ξ, η, g) leads to a Lorentzian paracontact structure and the manifold endowed with the Lorentzian paracontact structure is known as the Lorentzian paracontact manifold of dimension 4.
The non-vanishing components of the Lie bracket are calculated as:

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3.$$

If ∇ denotes the Levi-Civita connection with respect to the Lorentzian metric g. Then for $e_4 = \xi$, the Koszul's formula gives

$$\begin{array}{llll} \nabla_{e_1}e_1 = e_4, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = e_4, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = e_4, & \nabla_{e_3}e_4 = e_3, \\ \nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0. \end{array}$$

From the above equations, it can be easily verify that $\nabla_X e_4 = \phi X$ holds for each $X \in \chi(M)$. Thus, the Lorentzian paracontact manifold is an LP-Sasakian manifold of dimension 4. Using (3.1), we calculate the generalized symmetric metric connection $\widetilde{\nabla}$ of type (α, β) as follows.

$$\begin{array}{llll} \widetilde{\nabla}_{e_1}e_1 = (1-\alpha-\beta)\,e_4, & \widetilde{\nabla}_{e_1}e_2 = 0, & \widetilde{\nabla}_{e_1}e_3 = 0, & \widetilde{\nabla}_{e_1}e_4 = (1-\alpha-\beta)e_1, \\ \widetilde{\nabla}_{e_2}e_1 = 0, & \widetilde{\nabla}_{e_2}e_2 = (1-\alpha-\beta)\,e_4, & \widetilde{\nabla}_{e_2}e_3 = 0, & \widetilde{\nabla}_{e_2}e_4 = (1-\alpha-\beta)e_2, \\ \widetilde{\nabla}_{e_3}e_1 = 0, & \widetilde{\nabla}_{e_3}e_2 = 0, & \widetilde{\nabla}_{e_3}e_3 = (1-\alpha-\beta)\,e_4, & \widetilde{\nabla}_{e_3}e_4 = (1-\alpha-\beta)e_3, \\ \widetilde{\nabla}_{e_4}e_1 = 0, & \widetilde{\nabla}_{e_4}e_2 = 0, & \widetilde{\nabla}_{e_4}e_3 = 0, & \widetilde{\nabla}_{e_4}e_4 = 0. \end{array}$$

Again, the relation from Proposition 3.1, i.e., $\widetilde{\nabla}_X e_4 = (1-\beta)\phi X - \alpha X - \alpha\eta(X)e_4$ holds for each $X \in \chi(M)$. Thus, the Lorentzian paracontact manifold is an LP-Sasakian manifold admitting a generalized symmetric metric connection $\widetilde{\nabla}$ of type (α, β).

In light of the above equations, the non-vanishing components of the curvature tensor are given by

$$\begin{aligned}
\widetilde{R}(e_1,e_2)e_1 &= -(1-\alpha-\beta)^2 e_2, & \widetilde{R}(e_1,e_3)e_1 &= -(1-\alpha-\beta)^2 e_3,\\
\widetilde{R}(e_1,e_4)e_1 &= -(1-\alpha-\beta)e_4, & \widetilde{R}(e_1,e_2)e_2 &= (1-\alpha-\beta)^2 e_1,\\
\widetilde{R}(e_2,e_3)e_2 &= -(1-\alpha-\beta)^2 e_3, & \widetilde{R}(e_2,e_4)e_2 &= -(1-\alpha-\beta)e_4,\\
\widetilde{R}(e_1,e_3)e_3 &= (1-\alpha-\beta)^2 e_1, & \widetilde{R}(e_2,e_3)e_3 &= (1-\alpha-\beta)^2 e_2,\\
\widetilde{R}(e_3,e_4)e_3 &= -(1-\alpha-\beta)e_4, & \widetilde{R}(e_1,e_4)e_4 &= -(1-\alpha-\beta)e_1,\\
\widetilde{R}(e_2,e_4)e_4 &= -(1-\alpha-\beta)e_2, & \widetilde{R}(e_3,e_4)e_4 &= -(1-\alpha-\beta)e_3.
\end{aligned}$$

The Ricci tensor $\widetilde{S}$ of M is defined as $\widetilde{S}(X,Y) = \sum_{i=1}^{4} \varepsilon_i\, g(\widetilde{R}(e_i,X)Y, e_i)$, where $\varepsilon_i = g(e_i,e_i)$. Thus we have

$$\widetilde{S}(e_i,e_j) = \begin{bmatrix} 3(1-\alpha-\beta)^2 & 0 & 0 & 0\\ 0 & 3(1-\alpha-\beta)^2 & 0 & 0\\ 0 & 0 & 3(1-\alpha-\beta)^2 & 0\\ 0 & 0 & 0 & -3(1-\alpha-\beta)^2 \end{bmatrix}$$

and the scalar curvature $\widetilde{\hat{\delta}}=\sum_{i=1}^{4} S(e_i,e_j) = 6(1-\alpha-\beta)^2$.
Since $\{e_1,e_2,e_3,e_4\}$ forms a basis for a 4-dimensional LP-Sasakian manifold. Thus any vector fields $X,Y,Z \in \chi(M^4)$ can be written as

$$X = a_1e_1+b_1e_2+c_1e_3+d_1e_4, Y = a_2e_1+b_2e_2+c_2e_3+d_2e_4, Z = a_3e_1+b_3e_2+c_3e_3+d_3e_4,$$

where $a_i, b_i, c_i, d_i \in \Re^+,\ i = 1,2,3,4$ such that

$$\frac{\beta}{2}\left[\frac{\begin{array}{c}2(c_2-d_2)(a_1a_3+b_1b_3+c_1c_3)-(c_3-d_3)(a_1a_2\\ +b_1b_2+c_1c_2)-(c_1-d_1)(a_2a_3+b_2b_3+c_2c_3)\end{array}}{a_1a_3+b_1b_3+c_1c_3-d_1d_3}\right] \neq 0.$$

If we consider the 1-form θ by $\theta(X)=g(X,(1-\alpha-\beta)e_4)$, for any $X \in \chi(M)$ and considering $\psi \in C^\infty(M)$ as

$$\psi = \frac{\beta}{2}\left[\frac{\begin{array}{c}2(c_2-d_2)(a_1a_3+b_1b_3+c_1c_3)\\ -(c_3-d_3)(a_1a_2+b_1b_2+c_1c_2)-(c_1-d_1)(a_2a_3+b_2b_3+c_2c_3)\end{array}}{a_1a_3+b_1b_3+c_1c_3-d_1d_3}\right].$$

So the relation

$$\nabla_X Y = \psi X + \theta(X)Y, \tag{8.1}$$

holds. As per this consequences, Y is a torse-forming vector field. Thus from (5.2), we get

$$\begin{aligned}(\widetilde{\mathfrak{L}}_Y g)(X,Z) &= g(\widetilde{\nabla}_X Y, Z) + g(X, \widetilde{\nabla}_Z Y)\\ &= 2\psi g(X,Z) + \theta(X)g(Y,Z) + \theta(Z)g(Y,X)\\ &+\alpha\{2\eta(Y)g(X,Z) - g(X,Y)\eta(Z) - g(Z,Y)\eta(X)\}\\ &+\beta\{2\eta(Y)g(\phi X,Z) - g(\phi X,Y)\eta(Z) - g(\phi Z,Y)\eta(X)\}.\end{aligned} \tag{8.2}$$

Also, we calculate

$$\begin{aligned}g(X,Z) &= a_1a_3 + b_1b_3 + c_1c_3 - d_1d_3,\\ g(Y,Z) &= a_2a_3 + b_2b_3 + c_2c_3 - d_2d_3,\\ g(Y,X) &= a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2.\end{aligned} \tag{8.3}$$

Also,

$$\begin{aligned}\theta(X) &= -(1-\alpha-\beta)d_1,\\ \theta(Y) &= -(1-\alpha-\beta)d_2,\\ \theta(Z) &= -(1-\alpha-\beta)d_3.\end{aligned} \tag{8.4}$$

With the help of above equations, equation (8.2) reduces to

$$\begin{aligned}\frac{1}{2}(\mathfrak{L}_Y g)(X,Z) &= \psi\{a_1a_3 + b_1b_3 + c_1c_3 - d_1d_3\} - \frac{1}{2}(1-\alpha-\beta)\{d_1(a_2a_3\\ &+b_2b_3 + c_2c_3 - d_2d_3) + d_3(a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2)\} + \frac{\alpha}{2}\{2(c_2-d_2)(a_1a_3\\ &+b_1b_3 + c_1c_3 - d_1d_3) - (c_3-d_3)(a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2)\\ &-(c_1-d_1)(a_2a_3 + b_2b_3 + c_2c_3 - d_2d_3)\} + \frac{\beta}{2}\{2(c_2-d_2)(a_1a_3 + b_1b_3 + c_1c_3)\\ &-(c_3-d_3)(a_1a_2 + b_1b_2 + c_1c_2) - (c_1-d_1)(a_2a_3 + b_2b_3 + c_2c_3)\}.\end{aligned} \tag{8.5}$$

Also,

$$(\widetilde{\hat{\delta}} - \lambda)g(X,Z) = (6(1-\alpha-\beta)^2) - \lambda)\{a_1a_3 + b_1b_3 + c_1c_3 - d_1d_3\} \tag{8.6}$$

We consider that $a_1a_3+b_1b_3+c_1c_3-d_1d_3 \neq 0$, $2d_1(a_2a_3+b_2b_3+c_2c_3-d_2d_3)+2d_3(a_1a_2+b_1b_2+c_1c_2-d_1d_2)+d_2(a_1a_3+b_1b_3+c_1c_3-d_1d_3)=0$ and $-2(c_3-d_3)(a_1a_2+b_1b_2+c_1c_2-d_1d_2)-2(c_1-d_1)(a_2a_3+b_2b_3+c_2c_3-d_2d_3)-3(c_2-d_2)(a_1a_3+b_1b_3+c_1c_3-d_1d_3)=0$. Thus we get (g,Y,λ) is a Yamabe soliton, i.e., $\frac{1}{2}\mathfrak{L}_Y g(X,Z)=(\widetilde{\hat{\delta}}-\lambda)g(X,Z)$ holds, unless

$$\begin{aligned}\lambda &= \psi - 6(1-\alpha-\beta)^2 - \frac{1}{4}\{-(1-\alpha-\beta)d_2 + 3\alpha(c_2-d_2)\}\\ &= \psi - \widetilde{\hat{\delta}} - \frac{1}{4}\{\theta(Y) + 3\alpha\eta(Y)\}=\text{constant}.\end{aligned} \tag{8.7}$$

So the existence of Yamabe soliton (g,Y,λ) on a 4-dimensional LP-Sasakian manifold admitting a generalized symmetric metric connection $\widetilde{\nabla}$ of type (α,β) with potential vector field Y as torse-forming. Thus the Theorems 5.1 and 6.1 are verified.

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