Harmonic vector fields on vertical rescaled generalized Cheeger-Gromoll metrics

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Abstract. In this paper, we introduce the vertical rescaled generalized Cheeger-Gromoll metric on the tangent bundle TM over an m-dimensional Riemannian manifold (M, g), as a natural metric on TM. We establish a necessary and sufficient conditions under which a vector field is harmonic with respect to the vertical rescaled generalized Cheeger-Gromoll metric. We also construct some examples of harmonic vector fields.

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1 Introduction

The geometry of the tangent bundle TM equipped with the Sasaki metric has been studied by many authors such as Sasaki [22], Yano and Ishihara [24], Dombrowski [8], Salimov and Gezer [19], [20], etc. The rigidity of the Sasaki metric has incited some geometers to construct and study other metrics on TM. Musso and Tricerri have introduced the notion of Cheeger-Gromoll metric [17], which has been studied also by many authors (e.g., see [12], [21], [23]).

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds. The energy functional of ϕ is defined by

$$E(\phi) = \int_K e(\phi) dv_g, \tag{1.1}$$

where K is compact subset in M, where

$$e(\phi) = \frac{1}{2} trace_g h(d\phi, d\phi), \tag{1.2}$$

is the energy density of ϕ.

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A map is called *harmonic* if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_K h(\tau(\phi), V)dv_g. \tag{1.3}$$

Here

$$\tau(\phi) = trace_g \nabla d\phi \tag{1.4}$$

is the tension field of ϕ and $\nabla d\phi$ is the second fundamental form of ϕ. Then ϕ is harmonic if and only if $\tau(\phi) = 0$. One can refer to [10], [9] for background on harmonic maps.

The main idea of this note consists of the study of harmonicity with respect to the vertical rescaled generalized Cheeger-Gromoll metric on the tangent bundle TM [2]. We establish necessary and sufficient conditions under which a vector field is harmonic (Theorem 4.3 and Theorem 4.4). We also construct examples of harmonic vector fields and we give a formula for the construction of non trivial examples of vector fields (Theorem 4.7 and Corollary 4.9). We further study the harmonicity of the map $\sigma : (M, g) \longrightarrow (TN, h^f)$ (Theorem 4.11 and Theorem 4.12) and the map $\phi : (TM, g^f) \longrightarrow (N, h)$ (Theorem 4.14 and Theorem 4.15).

2 Basic notions and definitions on TM

Let (M^m, g) be an m-dimensional Riemannian manifold and let (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and let $\Gamma(TM)$ be the module over $C^\infty(M)$ of C^∞-vector fields on M.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by:

$$\begin{aligned}
\mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)};\quad a^i \in \mathbb{R}\},\\
\mathcal{H}_{(x,u)} &= \left\{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)};\quad a^i \in \mathbb{R}\right\},
\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \left\{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\right\}. \tag{2.2}$$

Consequently, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$, $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, and $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

If $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial x^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}, \tag{2.3}$$

$$w^v = (\overline{w}^k + w^i u^j \Gamma^k_{ij}) \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}. \tag{2.4}$$

Lemma 2.1. *[24] Let (M, g) be a Riemannian manifold and R its curvature tensor. Then for all vector fields $X, Y \in \Gamma(TM)$, we have:*

1. $[X^H, Y^H]_p = [X, Y]^H_p - (R_x(X, Y)u)^V$,

2. $[X^H, Y^V]_p = (\nabla_X Y)^V_p$,

3. $[X^V, Y^V]_p = 0$,

where $p = (x, u) \in TM$.

3 Vertical rescaled generalized Cheeger-Gromoll metric

Definition 3.1. [2] Let (M, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function. We define a vertical rescaled generalized Cheeger-Gromoll metric g^f on the tangent bundle TM by

1. $g^f(X^H, Y^H)_{(x,u)} = g_x(X, Y)$,

2. $g^f(X^H, Y^V)_{(x,u)} = 0$,

3. $g^f(X^V, Y^V)_{(x,u)} = f(x)\omega^p \big[g_x(X, Y) + q g_x(X, u) g_x(Y, u)\big]$,

for all vector fields $X, Y \in \Gamma(TM)$, $(x, u) \in TM$ and $r = \|u\| = \sqrt{g(u, u)}$, where $\omega = \dfrac{1}{1 + \|u\|^2}$, $p, q \in \mathbb{R}$, and q positive ensures non-degeneracy.

Theorem 3.1. *[2] Let (M, g) be a Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If ∇ (resp. $\bar{\nabla}$) denote the Levi-Civita connections of (M, g) (resp (TM, g^f)), then we* *have:*

$$
\begin{aligned}
1.\,(\bar{\nabla}_{X^H}Y^H) &= (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V,\\
2.\,(\bar{\nabla}_{X^H}Y^V) &= \frac{f}{2}\omega^p(R(u,Y)X)^H + \frac{X(f)}{2f}Y^V + (\nabla_X Y)^V,\\
3.\,(\bar{\nabla}_{X^V}Y^H) &= \frac{f}{2}\omega^p(R(u,X)Y)^H + \frac{Y(f)}{2f}X^V,\\
4.\,(\bar{\nabla}_{X^V}Y^V) &= -\frac{p\omega^{(-p+1)}}{f(1+qr^2)}\Big[g^f(X^V,U^V)Y^V + g^f(Y^V,U^V)X^V\Big]\\
&\quad +\frac{(p\omega+q)\omega^{-p}}{f(1+qr^2)}g^f(X^V,Y^V)U^V\\
&\quad -\frac{q^2\omega^{-2p}}{f^2(1+qr^2)^3}g^f(X^V,U^V)g^f(Y^V,U^V)U^V\\
&\quad -\frac{1}{2f}g^f(X^V,Y^V)(grad\, f)^H,
\end{aligned}
$$

for all vector fields $X, Y, U \in \Gamma(TM)$, $U_x = u \in T_xM$ and $(x,u) \in TM$, where R denotes the curvature tensor of (M,g).

4 Vertical rescaled generalized Cheeger-Gromoll metric and harmonicity.

4.1 Harmonicity of a vector field $X : (M,g) \longrightarrow (TM, g^f)$

Lemma 4.1. *Let (M,g) be a Riemannian manifold. If $X, Y \in \Gamma(TM)$ are vector fields on M and $(x,u) \in TM$ such that $Y_x = u$, then we have:*

$$d_xY(X_x) = X^H_{(x,u)} + (\nabla_X Y)^V_{(x,u)}.$$

Proof. Let (U, x^i) be a local chart on M in $x \in M$ and $\pi^{-1}(U), x^i, y^j)$ be the induced chart on TM. If $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x = Y^i(x)\frac{\partial}{\partial x^i}|_x = u$, then

$$d_xY(X_x) = X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)} + X^i(x)\frac{\partial Y^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,u)},$$

and thus the horizontal part is given by:

$$
\begin{aligned}
(d_xY(X_x))^h &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)} - X^i(x)Y^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,u)}\\
&= X^h_{(x,u)},
\end{aligned}
$$

and the vertical part, by:

$$
\begin{aligned}
(d_xY(X_x))^v &= \left\{X^i(x)\frac{\partial Y^k}{\partial x^i}(x) + X^i(x)Y^j(x)\Gamma^k_{ij}(x)\right\}\frac{\partial}{\partial y^k}|_{(x,u)}\\
&= (\nabla_X Y)^v_{(x,u)}.
\end{aligned}
$$

□

Lemma 4.2. *Let (M^m, g) be an m-dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then the energy density associated to X is given by:*

$$e(X) = \frac{m}{2} + \frac{f\omega^p}{2} trace_g\big[g(\nabla X, \nabla X) + qg(\nabla X, X)^2\big], \tag{4.1}$$

where $\omega = \dfrac{1}{1 + \|X\|^2}$ *and* $\|X\|^2 = g(X, X)$.

Proof. Let $(x, u) \in TM$, $X \in \Gamma(TM)$, $X_x = u$ and let $(E_1, \cdots, E_m)$ be a local orthonormal frame on M. Then:

$$\begin{aligned} e(X)_x &= \frac{1}{2} trace_g g^f(dX, dX)_{(x,u)} \\ &= \frac{1}{2}\sum_{i=1}^{m} g^f(dX(E_i), dX(E_i))_{(x,u)}. \end{aligned}$$

Using Lemma 4.1, we obtain:

$$\begin{aligned} e(X) &= \frac{1}{2}\sum_{i=1}^{m} g^f(E_i^H + (\nabla_{E_i}X)^V, E_i^H + (\nabla_{E_i}X)^V) \\ &= \frac{1}{2}\sum_{i=1}^{m} \big[(g^f(E_i^H, E_i^H) + g^f((\nabla_{E_i}X)^V, (\nabla_{E_i}X)^V))\big] \\ &= \frac{1}{2}\sum_{i=1}^{m} \big[g(E_i, E_i) + f\omega^p[g(\nabla_{E_i}X, \nabla_{E_i}X) + qg(\nabla_{E_i}X, X)^2]\big] \\ &= \frac{m}{2} + \frac{f\omega^p}{2} trace_g\big[g(\nabla X, \nabla X) + g(\nabla X, X)^2\big]. \end{aligned}$$

□

Theorem 4.3. *Let (M^m, g) be an m-dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then the tension field associated to X is given by:*

$$\tau(X) = \big[trace_g A(X)\big]^H + \big[trace_g B(X)\big]^V, \tag{4.2}$$

where $A(X)$ and $B(X)$ are the bilinear maps defined by:

$$\begin{aligned} A(X) &= f\omega^p R(X, \nabla X)* - \frac{\omega^p}{2}\big[g(\nabla X, \nabla X) + qg(\nabla X, X)^2\big] grad\, f, \\ B(X) &= \nabla^2 X + \big[\frac{1}{f} df(*) - 2p\omega g(\nabla X, X)\big]\nabla X \\ &\quad + \big[\frac{p\omega + q}{1 + q\|X\|^2} g(\nabla X, \nabla X) + \frac{p\omega}{1 + q\|X\|^2} g(\nabla X, X)^2\big]X, \end{aligned}$$

where $\omega = \dfrac{1}{1 + \|X\|^2}$ *and* $\|X\|^2 = g(X, X)$.

Proof. Let $(x, u) \in TM$, $X \in \Gamma(TM)$, $X_x = u$ and let $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i} E_i)_x = 0$. Then

$$\begin{aligned}
\tau(X)_x &= \sum_{i=1}^{m}\{(\nabla^X_{E_i} dX(E_i))_x - dX(\nabla^M_{E_i}E_i)_x\} \\
&= \sum_{i=1}^{m}\{\bar{\nabla}_{dX(E_i)} dX(E_i)\}_{(x,u)} \\
&= \sum_{i=1}^{m}\{\bar{\nabla}_{(E_i^H+(\nabla_{E_i}X)^V)}(E_i^H + (\nabla_{E_i}X)^V)\}_{(x,u)} \\
&= \sum_{i=1}^{m}\{\bar{\nabla}_{E_i^H}E_i^H + \bar{\nabla}_{E_i^H}(\nabla_{E_i}X)^V + \bar{\nabla}_{(\nabla_{E_i}X)^V}(E_i)^H \\
&\quad +\bar{\nabla}_{(\nabla_{E_i}X)^V}(\nabla_{E_i}X)^V\}_{(x,u)}.
\end{aligned}$$

Using Theorem 3.1, we obtain

$$\begin{aligned}
\tau(X) &= \sum_{i=1}^{m}\Big[(\nabla_{E_i}E_i)^H - \frac{1}{2}(R(E_i,E_i)X)^V + \frac{f}{2}\omega^p(R(X,\nabla_{E_i}X)E_i)^H \\
&\quad +\frac{1}{2f}E_i(f)(\nabla_{E_i}X)^V + (\nabla_{E_i}\nabla_{E_i}X)^V + \frac{f}{2}\omega^p(R(X,\nabla_{E_i}X)E_i)^H \\
&\quad +\frac{1}{2f}E_i(f)(\nabla_{E_i}X)^V - \frac{2p\,\omega^{(-p+1)}}{f(1+q\|X\|^2)}g^f((\nabla_{E_i}X)^V, X^V)(\nabla_{E_i}X)^V \\
&\quad +\frac{(p\omega+q)\omega^{-p}}{f(1+q\|X\|^2)}g^f((\nabla_{E_i}X)^V,(\nabla_{E_i}X)^V)X^V \\
&\quad -\frac{q^2\omega^{-2p}}{f^2(1+q\|X\|^2)^3}g^f((\nabla_{E_i}X)^V, X^V)^2X^V \\
&\quad -\frac{1}{2f}g^f((\nabla_{E_i}X)^V,(\nabla_{E_i}X)^V)(grad\,f)^H\Big] \\
&= \sum_{i=1}^{m}\Big[f\omega^p(R(X,\nabla_{E_i}X)E_i)^H + \frac{1}{f}E_i(f)(\nabla_{E_i}X)^V + (\nabla_{E_i}\nabla_{E_i}X)^V \\
&\quad -2p\omega g(\nabla_{E_i}X,X)(\nabla_{E_i}X)^V + \frac{p\omega+q}{1+q\|X\|^2}g(\nabla_{E_i}X,\nabla_{E_i}X)X^V \\
&\quad +\frac{(p\omega+q)q}{1+q\|X\|^2}g(\nabla_{E_i}X,X)^2X^V - \frac{q^2}{1+q\|X\|^2}g(\nabla_{E_i}X,X)^2X^V \\
&\quad -\frac{\omega^p}{2}\big[g(\nabla_{E_i}X,\nabla_{E_i}X) + qg(\nabla_{E_i}X,X)^2\big](grad\,f)^H\Big] \\
&= \Bigg[trace_g\Big[f\omega^pR(X,\nabla X)* - \frac{\omega^p}{2}\big[g(\nabla X,\nabla X) + qg(\nabla X,X)^2\big]grad\,f\Big]\Bigg]^H \\
&\quad +\Bigg[trace_g\Big[\nabla^2X + \big[\frac{1}{f}df(*) - 2p\omega g(\nabla X,X)\big]\nabla X \\
&\quad +\big[\frac{p\omega+q}{1+q\|X\|^2}g(\nabla X,\nabla X) + \frac{p\omega}{1+q\|X\|^2}g(\nabla X,X)^2\big]X\Big]\Bigg]^V.
\end{aligned}$$

□

Theorem 4.4. *Let (M^m, g) be an m-dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$, then X is a harmonic vector field if and only the following conditions are verified:*

$$(4.3)\, trace_g\Big(f\omega^p R(X,\nabla X)* - \frac{\omega^p}{2}\big[g(\nabla X,\nabla X) + qg(\nabla X, X)^2\big] grad\, f\Big) = 0,$$

and

$$\begin{aligned} & trace_g\Big(\nabla^2 X + \big[\frac{1}{f}df(*) - 2p\omega g(\nabla X, X)\big]\nabla X \\ (4.4) \qquad & +\big[\frac{p\omega + q}{1+q\|X\|^2} g(\nabla X,\nabla X) + \frac{p\omega}{1+q\|X\|^2} g(\nabla X, X)^2\big] X\Big) = 0, \end{aligned}$$

where $\omega = \dfrac{1}{1+\|X\|^2}$ *and* $\|X\|^2 = g(X,X)$.

Proof. The statement is a direct consequence of Theorem 4.3. □

Corollary 4.5. *Let (M^m, g) be an m-dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X \in \Gamma(TM)$ such that X is a parallel vector field (i.e, $\nabla X = 0$), then X is harmonic.*

Theorem 4.6. *Let (M^m, g) be a compact m-dimensional Riemannian manifold and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. A vector field $X \in \Gamma(TM)$ is harmonic if and only if X is parallel (i.e, $\nabla X = 0$).*

Proof. If X is parallel, from Corollary 4.5, we infer that X is a harmonic vector field. Conversely, let φ_t be a compactly supported variation of X, defined by:

$$\begin{array}{rcl} \varphi : \mathbb{R}\times M & \longrightarrow & T_x M \\ (t,x) & \longmapsto & \varphi(t,x) = \varphi_t(x) = (t+1)X_x \end{array}$$

From Lemma 4.2, we have:

$$e(\varphi_t) = \frac{m}{2} + \frac{(1+t)^2}{2} f\omega^p trace_g g(\nabla X,\nabla X) + \frac{(1+t)^4}{2} f\omega^p q\, trace_g g(\nabla X, X)^2$$

$$\begin{aligned} E(\varphi_t) &= \frac{m}{2}Vol(M) + \frac{(1+t)^2}{2}\int_M f\omega^p trace_g g(\nabla X,\nabla X) dv_g \\ &\quad + \frac{(1+t)^4}{2}\int_M f\omega^p q\, trace_g g(\nabla X, X)^2 dv_g \end{aligned}$$

If X is a critical point of the energy functional, then we have:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big[\frac{m}{2} Vol(M) + \frac{(1+t)^2}{2}\int_M f\omega^p trace_g g(\nabla X, \nabla X) dv_g\Big]_{t=0} \\ &+ \frac{\partial}{\partial t}\Big[+ \frac{(1+t)^4}{2}\int_M f\omega^p q\, trace_g g(\nabla X, X)^2 dv_g\Big]_{t=0} \\ &= \int_M f\omega^p trace_g g(\nabla X, \nabla X) dv_g + 2\int_M f\omega^p q\, trace_g g(\nabla X, X)^2 dv_g \\ &= \int_M f\omega^p trace_g \big(g(\nabla X, \nabla X) + 2qg(\nabla X, X)^2\big) dv_g \end{aligned}$$

which gives

$$g(\nabla X, \nabla X) + +2qg(\nabla X, X)^2 = 0.$$

Hence $\nabla X = 0$. □

Example 4.1. The Riemannian compact manifold $\mathbb{S}^1$ can be equipped with the metric:

$$g = e^x dx^2.$$

The only Christoffel symbol of the Levi-Civita connection is given by:

$$\Gamma_{11}^1 = \frac{1}{2} g^{11}\Big(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1}\Big) = \frac{1}{2}.$$

The vector field $X = h(x)\frac{d}{dx}$, with $h \in \mathcal{C}^\infty(\mathbb{S}^1)$, is harmonic if and only if X is parallel,

$$\begin{aligned} \nabla X = 0 \quad &\Leftrightarrow \quad h'(x) + \frac{1}{2}h(x) = 0 \\ &\Leftrightarrow \quad h(x) = k\exp(-\frac{x}{2})\ ,\ k \in \mathbb{R} \\ &\Leftrightarrow \quad X = k\exp(-\frac{x}{2})\frac{d}{dx}\ ,\ k \in \mathbb{R}. \end{aligned}$$

Example 4.2. Let $\mathbb{R}^3$ be endowed with the cylindrical Riemannian metric given by:

$$g = dr^2 + r^2 d\theta^2 + dt^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r},\ \Gamma_{22}^1 = -r.$$

Then, we have

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0,\ \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta},\ \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial t} = 0,\ \nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial \theta},\ \nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial \theta} = -r\frac{\partial}{\partial r},$$

$$\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial t}=0,\ \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial r}=0,\ \nabla_{\partial t}\partial\theta=0,\ \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=0,$$

the vector field $X=\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}+\frac{\partial}{\partial t}$ is harmnic because X is parallel, indeed,

$$\begin{aligned}\nabla_{\frac{\partial}{\partial r}}X &= \sin\theta\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}-\frac{1}{r^2}\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r}\cos\theta\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\theta}+\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial t}=0,\\ \nabla_{\frac{\partial}{\partial\theta}}X &= \cos\theta\frac{\partial}{\partial r}+\sin\theta\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r}\sin\theta\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta}+\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial t}=0,\\ \nabla_{\frac{\partial}{\partial t}}\omega &= \sin\theta\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial\theta}+\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=0,\end{aligned}$$

i.e, $\nabla X=0$, which yields that X is harmonic.

Remark 4.3. In general, using Corollary 4.5 and Theorem 4.6, we can construct numerous examples of harmonic vector fields.

Theorem 4.7. *Let $(\mathbb{R}^m, g_0)$ be the real Euclidean space and let $(T\mathbb{R}^m, g^f)$ be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $X=(X^1,\cdots,X^m)\in\Gamma(T\mathbb{R}^m)$, then X is harmonic if and only if the following conditions are verified*

$$(4.5)\quad X=constant\ \ or\ \ f=constant,$$

and

$$\begin{aligned}&\sum_{i=1}^m\Big[\frac{\partial^2X^k}{\partial(x^i)^2}+\frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial X^k}{\partial x^i}\Big]+\frac{p\omega X^k}{1+q\|X\|^2}\sum_{i=1}^m\big(\sum_{j=1}^m X^j\frac{\partial X^j}{\partial x^i}\big)^2\\ &+\sum_{i,j=1}^m\Big[-2p\omega X^j\frac{\partial X^j}{\partial x^i}\frac{\partial X^k}{\partial x^i}+\frac{(p\omega+q)X^k}{1+q\|X\|^2}\big(\frac{\partial X^j}{\partial x^i}\big)^2\Big]=0.\end{aligned}\tag{4.6}$$

for all $k=\overline{1,m}$, where $\{\frac{\partial}{\partial x_i}\}_{i=\overline{1,m}}$ is the canonical frame on $\mathbb{R}^m$, $\omega=\dfrac{1}{1+\|X\|^2}$ and $\|X\|^2=g(X,X)$.

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,m}}$ be the canonical frame on $\mathbb{R}^m$. Using Theorem 4.4, we have: $\tau(X)=0$ holds true, iff the following conditions are satisfied:

$$\begin{aligned}(4.3)\quad&\Leftrightarrow\quad trace_g\Big(-\frac{\omega^p}{2}\big[g(\nabla X,\nabla X)+qg(\nabla X,X)^2\big]grad\,f\Big)=0\\ &\Leftrightarrow\quad\sum_{i=1}^m\big[g(\nabla_{\frac{\partial}{\partial x^i}}X,\nabla_{\frac{\partial}{\partial x^i}}X)+qg(\nabla_{\frac{\partial}{\partial x^i}}X,X)^2\big]=0\ \ or\ \ grad\,f=0\\ &\Leftrightarrow\quad\sum_{i=1}^m\big[\sum_{j=1}^m\big(\frac{\partial X^j}{\partial x^i}\big)^2+q\big(\sum_{j=1}^m\big(\frac{\partial X^j}{\partial x^i}X^j\big)\big)^2\big]=0\ \ or\ \ f=constant\\ &\Leftrightarrow\quad\frac{\partial X^j}{\partial x^i}=0\ ,\ \forall\, i,j=\overline{1,m}\ \ or\ \ f=constant\\ &\Leftrightarrow\quad X=constant\ \ or\ \ f=constant.\end{aligned}$$

$$(4.4) \quad \Leftrightarrow \quad trace_g\Big[\nabla^2 X + \big[\frac{1}{f}df(*) - 2p\omega g(\nabla X, X)\big]\nabla X + \big[\frac{p\omega+q}{1+q\|X\|^2}g(\nabla X,\nabla X) + \frac{p\omega}{1+q\|X\|^2}g(\nabla X, X)^2\big]X\Big] = 0$$

$$\Leftrightarrow \quad \sum_{i=1}^{m}\Big[\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^i}}X + \big[\frac{1}{f}df(\frac{\partial}{\partial x^i}) - 2p\omega g(\nabla_{\frac{\partial}{\partial x^i}}X, X)\big](\nabla_{\frac{\partial}{\partial x^i}}X) + \big[\frac{p\omega+q}{1+q\|X\|^2}g(\nabla_{\frac{\partial}{\partial x^i}}X,\nabla_{\frac{\partial}{\partial x^i}}X) + \frac{p\omega}{1+q\|X\|^2}g(\nabla_{\frac{\partial}{\partial x^i}}X, X)^2\big]X\Big] = 0$$

$$\Leftrightarrow \quad \sum_{i=1}^{m}\Bigg\{\sum_{k=1}^{m}(\frac{\partial^2 X^k}{\partial(x^i)^2}\frac{\partial}{\partial x^k}) + \frac{1}{f}\frac{\partial f}{\partial x^i}\sum_{k=1}^{m}(\frac{\partial X^k}{\partial x^i}\frac{\partial}{\partial x^k}) - 2p\omega\sum_{j=1}^{m}(\frac{\partial X^j}{\partial x^i}X^j)\sum_{k=1}^{m}(\frac{\partial X^k}{\partial x^i}\frac{\partial}{\partial x^k})$$

$$+\frac{p\omega+q}{1+q\|X\|^2}\sum_{j=1}^{m}(\frac{\partial X^j}{\partial x^i})^2\sum_{i=1}^{k}(X^k\frac{\partial}{\partial x^k}) + \frac{p\omega}{1+q\|X\|^2}(\sum_{j=1}^{m}X^j\frac{\partial X^j}{\partial x^i})^2\sum_{i=1}^{k}(X^k\frac{\partial}{\partial x^k})\Bigg\} = 0$$

$$\Leftrightarrow \quad \sum_{i=1}^{m}\Big[\frac{\partial^2 X^k}{\partial(x^i)^2} + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial X^k}{\partial x^i}\Big] + \frac{p\omega X^k}{1+q\|X\|^2}\sum_{i=1}^{m}(\sum_{j=1}^{m}X^j\frac{\partial X^j}{\partial x^i})^2 + \sum_{i,j=1}^{m}\Big[-2p\omega X^j\frac{\partial X^j}{\partial x^i}\frac{\partial X^k}{\partial x^i} + \frac{(p\omega+q)X^k}{1+q\|X\|^2}(\frac{\partial X^j}{\partial x^i})^2\Big] = 0.$$

for all $k = \overline{1,m}$. □

Corollary 4.8. *Let $(\mathbb{R}^m, g_0)$ be the real Euclidean space, let $(T\mathbb{R}^m, g_0^f)$ be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric and let f be a constant function. If $X = (X^1, \cdots, X^m)$ is a vector field on $\mathbb{R}^m$, then X is harmonic on $\mathbb{R}^m$ if and only if X satisfies the following system of equations:*

$$\sum_{i=1}^{m}\frac{\partial^2 X^k}{\partial(x^i)^2} + \frac{p\omega X^k}{1+q\|X\|^2}\sum_{i=1}^{m}(\sum_{j=1}^{m}X^j\frac{\partial X^j}{\partial x^i})^2$$

$$(4.7) \quad +\sum_{i,j=1}^{m}\Big[-2p\omega X^j\frac{\partial X^j}{\partial x^i}\frac{\partial X^k}{\partial x^i} + \frac{(p\omega+q)X^k}{1+q\|X\|^2}(\frac{\partial X^j}{\partial x^i})^2\Big] = 0.$$

for all $k = \overline{1,m}$, where $\{\frac{\partial}{\partial x_i}\}_{i=\overline{1,m}}$ is the canonical frame on $\mathbb{R}^m$, $\omega = \dfrac{1}{1+\|X\|^2}$ and $\|X\|^2 = g(X,X)$.

Corollary 4.9. *Let $(\mathbb{R}^m, g_0)$ the real Euclidean space, let $(T\mathbb{R}^m, g_0^f)$ be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric and let $X = (X^1, \cdots, X^m) \in \Gamma(T\mathbb{R}^m)$. If $f \neq$ constant, then X is a harmonic if and only if X is constant.*

Remark 4.4. Using Corollary 4.8, we can construct many examples of nontrivial harmonic vector fields.

Example 4.5. If $\mathbb{R}^m$ is endowed with the canonical metric and $T\mathbb{R}^m$ is its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric, such that f is a constant function, from Corollary 4.8, we infer that $X = (y(x_1), 0, \cdots, 0)$ is a harmonic vector field if and only the function y is solution of differential equation:

$$y'' + \frac{q - p + (p + q - 2pq)y^2}{(1 + y^2)(1 + qy^2)}(y')^2 y = 0. \tag{4.8}$$

In the case $p = q = 1$, (the case of Cheeger-Gromoll metric), the solution of the differential equation (4.8) is given by $y(x) = ax_1 + b$, $a, b \in \mathbb{R}$.

4.2 Harmonicity of the map $\sigma : (M, g) \longrightarrow (TN, h^f)$

Lemma 4.10. *Let (M^m, g), (N^n, h) be two Riemannian manifolds and let $\varphi : M \to N$ be a smooth map. If*

$$\begin{array}{rcl} \sigma : M & \longrightarrow & TN \\ x & \longmapsto & (\varphi(x), v) \end{array}$$

is a smooth map such that $\varphi = \pi_N \circ \sigma$, where $v \in T_{\varphi(x)}N$ and $\pi_N : TN \to N$ is the canonical projection, then

$$d\sigma(X) = (d\varphi(X))^H + (\nabla^{\varphi}_X \sigma)^V, \tag{4.9}$$

for all $X \in \Gamma(TM)$.

Proof. Let $x \in M$, $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$, such that $Y_{\varphi(x)} = v \in T_{\varphi(x)}N$. Using Lemma 4.1, we obtain:

$$\begin{aligned} d_x\sigma(X_x) &= d_x(Y \circ \varphi)(X_x) \\ &= d_{\varphi(x)}Y(d_x\varphi(X_x)) \\ &= (d\varphi(X))^H_{(\varphi(x),v)} + (\nabla_{d\varphi(X)}Y)^V_{(\varphi(x),v)} \\ &= (d\varphi(X))^H_{(\varphi(x),v)} + (\nabla^{\varphi}_X \sigma)^V_{(\varphi(x),v)}. \end{aligned}$$

□

Theorem 4.11. *Let (M^m, g), (N^n, h) be two Riemannian manifolds and let f be a strictly positive smooth function on N. Let (TN, h^f) be the tangent bundle of N, equipped with vertical rescaled generalized Cheeger-Gromoll metric.*
Let $\varphi : M \to N$ be a smooth map and let

$$\begin{array}{rcl} \sigma : M & \longrightarrow & TN \\ x & \longmapsto & (\varphi(x), v) \end{array}$$

be a smooth map such that $\varphi = \pi_N \circ \sigma$ and $v \in T_{\varphi(x)}N$. The tension field of σ is given by

$$\tau(\sigma) \;=\; \big[\tau(\varphi) + trace_g A(\sigma)\big]^H + \big[trace_g(B(\sigma)\big]^V, \tag{4.10}$$

where $A(\sigma)$ and $B(\sigma)$ are the bilinear maps defined by

$$\begin{aligned}
A(\sigma) &= f\omega^p R^N(\sigma,\nabla^\varphi\sigma)d\varphi(*) - \frac{\omega^p}{2}[h(\nabla^\varphi\sigma,\nabla^\varphi\sigma)+qh(\nabla^\varphi\sigma,\sigma)^2]grad\,f \\
B(\sigma) &= (\nabla^\varphi)^2\sigma + [\frac{1}{f}d\varphi(*)(f) - 2p\omega h(\nabla^\varphi\sigma,\sigma)]\nabla^\varphi\sigma \\
&\quad +[\frac{p\omega+q}{1+q\|\sigma\|^2}h(\nabla^\varphi\sigma,\nabla^\varphi\sigma)+\frac{p\omega}{1+q\|\sigma\|^2}h(\nabla^\varphi\sigma,\sigma)^2]\sigma,
\end{aligned}$$

where $\omega = \dfrac{1}{1+\|\sigma\|^2}$ *and* $\|\sigma\|^2 = h(\sigma,\sigma)$.

Proof. Let $x \in M$ and let $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i}E_i)_x = 0$ and $\sigma(x) = (\varphi(x), v), v \in T_{\varphi(x)}N$. Using Lemma 4.10, we have

$$\begin{aligned}
\tau(\sigma)_x &= trace_g(\nabla d\sigma)_x \\
&= \sum_{i=1}^m \{(\nabla^\sigma_{E_i}d\sigma(E_i))_x - d\sigma(\nabla^M_{E_i}E_i)_x\} \\
&= \sum_{i=1}^m \{\nabla^{TN}_{d\sigma(E_i)}d\sigma(E_i)\}_{(\varphi(x),v)} \\
&= \sum_{i=1}^m \{\nabla^{TN}_{(d\varphi(E_i))^H}(d\varphi(E_i))^H + \nabla^{TN}_{(d\varphi(E_i))^H}(\nabla^\varphi_{E_i}\sigma)^V \\
&\quad +\nabla^{TN}_{(\nabla^\varphi_{E_i}\sigma)^V}(d\varphi(E_i))^H + \nabla^{TN}_{(\nabla^\varphi_{E_i}\sigma)^V}(\nabla^\varphi_{E_i}\sigma)^V\}_{(\varphi(x),v)}
\end{aligned}$$

From Theorem 3.1, we obtain:

$$\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^m \Big[(\nabla^N_{d\varphi(E_i)}d\varphi(E_i))^H - \frac{1}{2}(R^N(d\varphi(E_i),d\varphi(E_i))\sigma)^V \\
&\quad +\frac{f}{2}\omega^p(R^N(\sigma,\nabla^\varphi_{E_i}\sigma)d\varphi(E_i))^H + \frac{1}{2f}d\varphi(E_i)(f)(\nabla^\varphi_{E_i}\sigma)^V \\
&\quad +(\nabla^N_{d\varphi(E_i)}\nabla^\varphi_{E_i}\sigma)^V + \frac{f}{2}\omega^p(R^N(\sigma,\nabla^\varphi_{E_i}\sigma)d\varphi(E_i))^H \\
&\quad +\frac{1}{2f}d\varphi(E_i)(f)(\nabla^\varphi_{E_i}\sigma)^V - \frac{2p\omega^{(-p+1)}}{f(1+q\|\sigma\|^2)}h^f((\nabla^\varphi_{E_i}\sigma)^V,\sigma^V)(\nabla^\varphi_{E_i}\sigma)^V \\
&\quad +\frac{(p\omega+q)\omega^{-p}}{f(1+q\|\sigma\|^2)}h^f((\nabla^\varphi_{E_i}\sigma)^V,(\nabla^\varphi_{E_i}\sigma)^V)\sigma^V \\
&\quad -\frac{q^2\omega^{-2p}}{f(1+q\|\sigma\|^2)^3}h^f((\nabla^\varphi_{E_i}\sigma)^V,\sigma^V)^2\sigma^V \\
&\quad -\frac{1}{2f}h^f((\nabla^\varphi_{E_i}\sigma)^V,(\nabla^\varphi_{E_i}\sigma)^V)(grad\,f)^H\Big]
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{i=1}^{m}\Big[(\nabla^{\varphi}_{E_i} d\varphi(E_i))^H + f\omega^p(R^N(\sigma,\nabla^{\varphi}_{E_i}\sigma)d\varphi(E_i))^H \\
& +\frac{1}{f}d\varphi(E_i)(f)(\nabla^{\varphi}_{E_i}\sigma)^V + (\nabla^{\varphi}_{E_i}\nabla^{\varphi}_{E_i}\sigma)^V \\
& -2p\omega h(\nabla^{\varphi}_{E_i}\sigma,\sigma)(\nabla^{\varphi}_{E_i}\sigma)^V + \frac{p\omega+q}{1+q\|\sigma\|^2}h(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)\sigma^V \\
& +\frac{p\omega}{1+q\|\sigma\|^2}h(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\sigma^V \\
& -\frac{\omega^p}{2}\big[h(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+qh(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\big](grad\, f)^H\Big]
\end{aligned}
$$

This implies:

$$
\begin{aligned}
\tau(\sigma) = & \Big(\tau(\varphi) + trace_g\big[f\omega^p R^N(\sigma,\nabla^{\varphi}\sigma)d\varphi(*) \\
& -\frac{\omega^p}{2}[h(\nabla^{\varphi}\sigma,\nabla^{\varphi}\sigma)+qh(\nabla^{\varphi}\sigma,\sigma)^2]grad\, f\big]\Big)^H \\
& +\Big(trace_g\big[(\nabla^{\varphi})^2\sigma + [\frac{1}{f}d\varphi(*)(f) - 2p\omega h(\nabla^{\varphi}\sigma,\sigma)]\nabla^{\varphi}\sigma \\
& +[\frac{p\omega+q}{1+q\|\sigma\|^2}h(\nabla^{\varphi}\sigma,\nabla^{\varphi}\sigma)+\frac{p\omega}{1+q\|\sigma\|^2}h(\nabla^{\varphi}\sigma,\sigma)^2]\sigma\big]\Big)^V.
\end{aligned}
$$

□

From Theorem 4.11 we consequently get the following

Theorem 4.12. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and let f be a strictly positive smooth function on N. Let (TN, h^f) be the tangent bundle of N, equipped with vertical rescaled generalized Cheeger-Gromoll metric.*
Let $\varphi : M \to N$ be a smooth map and let

$$
\begin{aligned}
\sigma : (M,g) & \longrightarrow (TN, h^f) \\
x & \longmapsto (\varphi(x), v)
\end{aligned}
$$

be a smooth map such that $\varphi = \pi_N \circ \sigma$ and $v \in T_{\varphi(x)}N$. Then σ is a harmonic if and only if the following conditions are satisfied

$$
\begin{aligned}
\tau(\varphi) = & \; trace_g\big[-f\omega^p R^N(\sigma,\nabla^{\varphi}\sigma)d\varphi(*) \\
& +\frac{\omega^p}{2}[h(\nabla^{\varphi}\sigma,\nabla^{\varphi}\sigma)+qh(\nabla^{\varphi}\sigma,\sigma)^2]grad\, f\big],
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
0 = & \; trace_g\big[(\nabla^{\varphi})^2\sigma + [\frac{1}{f}d\varphi(*)(f) - 2p\omega h(\nabla^{\varphi}\sigma,\sigma)]\nabla^{\varphi}\sigma \\
& +[\frac{p\omega+q}{1+q\|\sigma\|^2}h(\nabla^{\varphi}\sigma,\nabla^{\varphi}\sigma)+\frac{p\omega}{1+q\|\sigma\|^2}h(\nabla^{\varphi}\sigma,\sigma)^2]\sigma\big].
\end{aligned}
\tag{4.12}
$$

4.3 Harmonicity of the map $\phi : (TM, g^f) \longrightarrow (N, h)$

Lemma 4.13. *Let (M^m, g) be an m-dimensional Riemannian manifold, let f be a strictly positive smooth function on M and let (TM, g^f) be its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. Then the tension field of the canonical projection*

$$\begin{array}{rcl} \pi : (TM, g^f) & \longrightarrow & (M, g) \\ (x, u) & \longmapsto & x \end{array}$$

is given by:

$$\tau(\pi) = \frac{m}{2f}(\,grad\, f) \circ \pi. \tag{4.13}$$

Proof. Let $(x, u) \in TM$ and let $\{E_i\}_{i=\overline{1,m}}$, such that $E_1 = \frac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\{E_i^H, \frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V, \frac{1}{\sqrt{f\omega^p}}E_j^V\}_{i=\overline{1,m}, j=\overline{2,m}}$ is an orthonormal basis of TM at (x, u).

$$\begin{aligned}
\tau(\pi) &= trace_{g^f} \nabla d\pi \\
&= \sum_{i=1}^m \Big\{ \nabla^\pi_{E_i^H} d\pi(E_i^H) - d\pi(\nabla^{TM}_{E_i^H} E_i^H) \Big\} \\
&\quad + \nabla^\pi_{(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V)} d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V) \\
&\quad - d\pi(\nabla^{TM}_{(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V)}(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V)) \\
&\quad + \sum_{j=2}^m \Big\{ \nabla^\pi_{(\frac{1}{\sqrt{f\omega^p}}E_j^V)} d\pi(\frac{1}{\sqrt{f\omega^p}}E_j^V) - d\pi(\nabla^{TM}_{(\frac{1}{\sqrt{f\omega^p}}E_j^V)}(\frac{1}{\sqrt{f\omega^p}}E_j^V)) \Big\} \\
&= \sum_{i=1}^m \Big\{ \nabla^M_{d\pi(E_i^H)} d\pi(E_i^H) - d\pi(\nabla^{TM}_{E_i^H} E_i^H) \Big\} \\
&\quad + \nabla^M_{d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V)} d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V) \\
&\quad - \frac{1}{\sqrt{f\omega^p(1+qr^2)}} d\pi(\nabla^{TM}_{E_1^V}(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V)) \\
&\quad + \sum_{j=2}^m \Big\{ \nabla^M_{d(\frac{1}{\sqrt{f\omega^p}}E_j^V)} d\pi(\frac{1}{\sqrt{f\omega^p}}E_j^V) - \frac{1}{\sqrt{f\omega^p}} d\pi(\nabla^{TM}_{E_j^V}(\frac{1}{\sqrt{f\omega^p}}E_j^V)) \Big\}
\end{aligned}$$

as $d\pi(E_i^V) = 0$ and $d\pi(E_i^H) = E_i \circ \pi$ then:

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{m}\Big\{(\nabla^M_{E_i}E_i)\circ\pi - d\pi(\nabla^M_{E_i}E_i)^H\Big\} \\
&\quad -\frac{1}{f\omega^p(1+qr^2)}d\pi(\nabla^{TM}_{E_1^V}E_1^V) - \sum_{j=2}^{m}\Big\{\frac{1}{f\omega^p}d\pi(\nabla^{TM}_{E_j^V}E_j^V)\Big\} \\
&= \frac{1}{2f}(grad\, f)\circ\pi + \frac{m-1}{2f}(grad\, f)\circ\pi \\
&= \frac{m}{2f}(\,grad\, f)\circ\pi.
\end{aligned}$$

□

Theorem 4.14. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds, let f be a strictly positive smooth function on M and let (TM, g^f) be the tangent bundle of M equipped with the vertical rescaled generalized Cheeger-Gromoll metric.*

If $\varphi : (M, g) \longrightarrow (N, h)$ a smooth map, then the tension field of the map

$$\begin{aligned}
\phi : (TM, g^f) &\longrightarrow (N, h) \\
(x, u) &\longmapsto \varphi(x)
\end{aligned}$$

is given by:

$$\tau(\phi) = \big[\tau(\varphi) + \frac{m}{2f}d\varphi(grad\, f)\big]\circ\pi. \tag{4.14}$$

Proof. Let $(x, u) \in TM$ and let $\{E_i\}_{i=\overline{1,m}}$ with $E_1 = \frac{u}{\|u\|}$ be an orthonormal basis on M at x. Then $\{E_i^H, \frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V, \frac{1}{\sqrt{f\omega^p}}E_j^V\}_{i=\overline{1,m}, j=\overline{2,m}}$ is an orthonormal basis of TM at (x, u). For *phi* defined by:

$$\begin{aligned}
\phi : (TM, g^f) &\xrightarrow{\pi} (M, g) \xrightarrow{\varphi} (N, h) \\
(x, y) &\longmapsto x \longmapsto \varphi(x)
\end{aligned}$$

i.e., $\phi = \varphi\circ\pi$, we have:

$$\begin{aligned}
\tau(\phi) &= \tau(\varphi\circ\pi) \\
&= d\varphi(\tau(\pi)) + trace_{g^f}\nabla d\varphi(d\pi, d\pi)
\end{aligned}$$

$$
\begin{aligned}
trace_{g^f}\nabla d\varphi(d\pi,d\pi) &= \sum_{i=1}^{m}\Big\{\nabla^{\varphi}_{d\pi(E_i^H)}d\varphi(d\pi(E_i^H))-d\varphi(\nabla^{M}_{d\pi(E_i^H)}d\pi(E_i^H))\Big\}\\
&\quad+\nabla^{\varphi}_{d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V)}d\varphi(d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V))\\
&\quad-d\varphi(\nabla^{M}_{d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V)}d\pi(\frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V))\\
&\quad+\sum_{j=2}^{m}\Big\{\nabla^{\varphi}_{d\pi(\frac{1}{\sqrt{f\omega^p}}E_j^V)}d\varphi(d\pi(\frac{1}{\sqrt{f\omega^p}}E_j^V))\\
&\quad-d\varphi(\nabla^{M}_{d\pi(\frac{1}{\sqrt{f\omega^p}}E_j^V)}d\pi(\frac{1}{\sqrt{f\omega^p}}E_j^V))\Big\}\\
&= \sum_{i=1}^{m}\Big\{(\nabla^{\varphi}_{E_i\circ\pi}d\varphi(E_i\circ\pi))-d\varphi(\nabla^{M}_{E_i\circ\pi}E_i\circ\pi)\Big\}\\
&= \sum_{i=1}^{m}\Big\{\nabla^{\varphi}_{E_i}d\varphi(E_i)-d\varphi(\nabla^{M}_{E_i}E_i)\Big\}\circ\pi\\
&= \tau(\varphi)\circ\pi
\end{aligned}
$$

Using Lemma 4.13, we obtain:

$$
\tau(\phi) \quad = \quad \big[\tau(\varphi)+\frac{m}{2f}d\varphi(grad\, f)\big]\circ\pi
$$

□

Theorem 4.15. *Let (M^m,g) and (N^n,h) be two Riemannian manifolds, let f be a strictly positive smooth function on M and let (TM,g^f) the tangent bundle of M equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $\varphi:(M,g)\longrightarrow(N,h)$ is a smooth map, then the map*

$$
\begin{aligned}
\phi:(TM,g^f) &\longrightarrow (N,h)\\
(x,u) &\longmapsto \varphi(x)
\end{aligned}
$$

is a harmonic if and only if

$$
\tau(\varphi)=-\frac{m}{2f}d\varphi(grad\, f).
$$

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