# The dual logarithmic Aleksandrov-Fenchel inequality

## Chang-Jian Zhao

**Abstract.** In this paper, we establish a dual logarithmic Aleksandrov-Fenchel inequality involving logarithms by introducing new geometric measures and using the newly published $L_p$-dual Aleksandrov-Fenchel inequality. The dual logarithmic Aleksandrov-Fenchel inequality is also derived. This new dual logarithmic Aleksandrov-Fenchel type inequality in special cases yields $L_p$-dual logarithmic Minkowski's inequality, the classical dual Aleksandrov-Fenchel inequality and related dual logarithmic Minkowski type inequalities.

## 1 Introduction

In 2012, a logarithmic Minkowski inequality for origin-symmetric convex bodies was conjectured by Böröczky, Lutwak, and et al [1].

**The conjectured logarithmic Minkowski inequality.**

*If $K$ and $L$ are convex bodies in $\mathbb{R}^n$ which are symmetric with respect to the origin, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{V}_L \geq \frac{1}{n} \ln\left(\frac{V(K)}{V(L)}\right), \tag{1.1}$$

*where $dv_L = \frac{1}{n} h_L dS(L, \cdot)$ is the cone-volume measure of $L$, and $d\overline{V}_L = \frac{1}{V(L)} dv_L$ is its normalization, and $S(L, \cdot)$ is the mixed surface area measure of $L$.*

The functions are the support functions. If $K$ is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h_K = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defines the support function $h_K$ of $K$. A nonempty closed convex set is uniquely determined by its support function.

Recently, the conjectured logarithmic Minkowski inequality and its dual form have attracted extensive attention and research. The recent research on the logarithmic Minkowski type inequalities and its dual can be found in the references [2, 3, 5, 6, 7, 11, 12, 13, 15, 17, 18, 19, 20, 21, 23].

The dual mixed volume of star bodies $K_1, \ldots, K_n$, $\widetilde{V}(K_1, \ldots, K_n)$ defined by Lutwak (see [10])

$$\widetilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \tag{1.2}$$

Here, $\rho(K, \cdot)$ denotes the radial function of star body $K$. The radial function of star body $K$ is defined by

$$\rho(K, u) = \max\{c \geq 0 : cu \in K\},$$

for $u \in S^{n-1}$. If $\rho(K, \cdot)$ is positive and continuous, $K$ will be called a star body. In the following, let $\mathcal{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$. Moreover, Lutwak's dual Aleksandrov-Fenchel inequality is the following: If $K_1, \cdots, K_n \in \mathcal{S}^n$ and $1 \leq r \leq n$, then

$$\widetilde{V}(K_1, \cdots, K_n) \leq \prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{1}{r}}, \tag{1.3}$$

with equality if and only if $K_1, \ldots, K_r$ are all dilations of each other (see [10]).

It is well known that in dual Brunn-Minkowski theory, dual Minkowski inequality and dual Aleksandrov-Fenchel inequality appear at the same time, and the latter is a generalization of the former. So a natural question is raised: is there a dual logarithmic Aleksandrov-Fenchel inequality relative to a dual logarithmic Minkowski inequality? The main purpose of this article is to answer the above questions perfectly and obtain a dual logarithmic Aleksandrov-Fenchel inequality involving logarithms by introducing two new concepts of mixed dual volume measure and $L_p$-multiple dual mixed volume measure, and using the $L_p$-dual Aleksandrov-Fenchel inequality for the $L_p$-multiple dual mixed volume. The dual logarithmic Aleksandrov-Fenchel inequality is also derived. The new dual logarithmic Aleksandrov-Fenchel inequality involving logarithms in special cases yields $L_p$-dual logarithmic Minkowski's inequality, the classical dual Aleksandrov-Fenchel inequality, and some new dual logarithmic Minkowski type inequalities. Our main result is given in the following inequality.

**The dual logarithmic Aleksandrov-Fenchel inequality involving logarithms.**

*If $L_1, K_1, \ldots, K_n \in \mathcal{S}^n$, $1 \leq r \leq n$ and $p \geq 1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n) \geq \ln\left(\frac{\widetilde{V}(L_1, K_2, \cdots, K_n)}{\prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{1}{r}}}\right), \tag{1.4}$$

*with equality if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.*

Here, $d\widetilde{V}_{-p}(L_1, K_1 \cdots, K_n)$ denotes a new probability measure call it $L_p$-multiple dual mixed volume probability measure of star bodies $L_1, K_1, \ldots, K_n$, defined by

$$d\widetilde{V}_{-p}(L_1, K_1 \cdots, K_n) = \frac{1}{\widetilde{V}_{-p}(L_1, K_1 \cdots, K_n)} d\tilde{v}_{-p}(L_1, K_1 \cdots, K_n), \tag{1.5}$$

where $p \geq 1$, and

$$d\tilde{v}_{-p}(L_1, K_1 \cdots, K_n) = \frac{1}{n}\rho(K_1, u)^{-p}\rho(L_1, u)^{1+p}\rho(K_2, u) \cdots \rho(K_n, u)dS(u),$$

and $\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)$ is the $L_p$-multiple dual mixed volume of star bodies $L_1, K_1, \ldots, K_n$, defined by ([22])

$$\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right)^p \rho(L_1, u)\rho(K_2, u) \cdots \rho(K_n, u)dS(u). \quad (1.6)$$

Obviously, putting $p = 1$, $L_1 = L$, $K_1 = K$ and $K_2 = \ldots = K_n = L$ in (1.6), then $\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)$ becomes the well-known dual mixed volume $\widetilde{V}_{-1}(L, K)$, defined by (see [8])

$$\widetilde{V}_{-1}(L, K) = \frac{1}{n}\int_{S^{n-1}} \rho(L, u)^{n+1}\rho(K, u)^{-1}dS(u).$$

**Remark.** When $L_1 = K_1$, inequality (1.4) becomes the classical dual Aleksandrov-Fenchel inequality as follows: If $K_1, \cdots, K_n \in \mathcal{S}^n$ and $1 \leq r \leq n$, then

$$\widetilde{V}(K_1, \cdots, K_n) \leq \prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{1}{r}}, \quad (1.7)$$

with equality if and only if $K_1, \ldots, K_r$ are all dilations of each other.

If putting $p = 1$, $r = n - 1$, $L_1 = L$, $K_1 = K$ and $K_2 = \ldots = K_n = L$ in (1.4), and noting that

$$\widetilde{V}_{-p}(L, K, \underbrace{L, \ldots, L}_{n-1}) = \widetilde{V}_{-1}(L, K),$$

$$d\tilde{v}_{-1}(L, K, \underbrace{L, \ldots, L}_{n-1}) = \frac{1}{n}\rho(L, u)^{n+1}\rho(K, u)^{-1},$$

$$d\widetilde{V}_{-1}(L, K, \underbrace{L, \ldots, L}_{n-1}) = \frac{1}{\widetilde{V}_{-1}(L, K)}d\tilde{v}_{-1}(L, K, \underbrace{L, \ldots, L}_{n-1}),$$

and in view of

$$\prod_{i=1}^{n-1} \widetilde{V}(K_i, \ldots, K_i, K_n)^{1/(n-1)} = \left(\widetilde{V}_1(K, L)V(L)^{n-2}\right)^{1/(n-1)}$$

$$\leq V(K)^{1/n}V(L)^{(n-1)/n},$$

with equality if and only if $K$ and $L$ are dilates, then (1.4) becomes the following dual logarithmic Minkowski inequality.

**The dual logarithmic Minkowski inequality.**

*If $K$ and $L$ are star bodies in $\mathbb{R}^n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(L, u)}{\rho(K, u)}\right) d\widetilde{V}_{-1}(L, K) \geq \frac{1}{n}\ln\left(\frac{V(L)}{V(K)}\right). \quad (1.8)$$

*with equality if and only if $K$ and $L$ are dilates, where $d\widetilde{V}_{-1}(L, K) = d\widetilde{V}_{-1}(L, K, \underbrace{L, \dots, L}_{n-1})$*

*denotes the dual mixed volume probability measure of $K$ and $L$.*

Obviously, a special case of (1.4) is the following dual logarithmic Aleksandrov-Fenchel inequality.

**The dual logarithmic Aleksandrov-Fenchel inequality.**

*If $L_1, K_1, \dots, K_n \in \mathcal{S}^n$, $1 \le r \le n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\widetilde{V}_{-1}(L_1, K_1, \dots, K_n) \ge \ln\left(\frac{\widetilde{V}(L_1, K_2, \cdots, K_n)}{\prod_{i=1}^r \widetilde{V}(K_i \dots, K_i, K_{r+1}, \dots, K_n)^{\frac{1}{r}}}\right),$$
(1.9)

*with equality if and only if $L_1, K_1, \dots, K_r$ are all dilations of each other, where $d\widetilde{V}_{-1}(L_1, K_1, \dots, K_n)$ is as in (1.5).*

# 2    Notations and preliminaries

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. For a compact set $K \subset \mathbb{R}^n$, we write $V(K)$ for the ($n$-dimensional) Lebesgue measure of $K$ and call this the volume of $K$. The unit ball in $\mathbb{R}^n$ and its surface are denoted by $B$ and $S^{n-1}$, respectively. Let $\mathcal{K}^n$ denote the class of nonempty compact convex subsets containing the origin in their interiors in $\mathbb{R}^n$. Associated with a compact subset $K$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K, \cdot) : S^{n-1} \to [0, \infty)$, defined by

$$\rho(K, u) = \max\{\lambda \ge 0 : \lambda u \in K\}.$$

Two star bodies $K$ and $L$ are dilates if $\rho(K, u)/\rho(L, u)$ is independent of $u \in S^{n-1}$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \mathcal{S}^n$, then (see e.g. [14])

$$\tilde{\delta}(K, L) = |\rho(K, u) - \rho(L, u)|_{\infty}.$$

*2.1  Dual mixed volumes*

The polar coordinate formula for volume of a compact set $K$ is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u).$$
(2.1)

The first dual mixed volume, $\widetilde{V}_1(K, L)$, defined by

$$\widetilde{V}_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+} \varepsilon \cdot L) - V(K)}{\varepsilon},$$

where $K, L \in \mathcal{S}^n$. The integral representation for first dual mixed volume is proved: For $K, L \in \mathcal{S}^n$,

$$\widetilde{V}_1(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-1} \rho(L, u) dS(u).$$
(2.2)

The Minkowski inequality for first dual mixed volume is the following: If $K, L \in \mathcal{S}^n$, then

$$\widetilde{V}_1(K, L)^n \leq V(K)^{n-1} V(L), \tag{2.3}$$

with equality if and only if $K$ and $L$ are dilates (see [10]).

If $K_1, \ldots, K_n \in \mathcal{S}^n$, $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ is written as $\widetilde{V}_i(K, L)$. If $L = B$, the dual mixed volume $\widetilde{V}_i(K, L) = \widetilde{V}_i(K, B)$ is written as $\widetilde{W}_i(K)$ and called dual quermassintegral of $K$. For $K \in \mathcal{S}^n$ and $0 \leq i < n$,

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \tag{2.4}$$

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$ and $K_n = L$, the dual mixed volume $\widetilde{V}(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i}, L)$ is written as $\widetilde{W}_i(K, L)$ and called dual mixed quermassintegral of $K$ and $L$. For $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, it is easy that

$$\widetilde{W}_i(K, L) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K \widetilde{+} \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i-1} \rho(L, u) dS(u). \tag{2.5}$$

The fundamental inequality for dual mixed quermassintegral stated that: If $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, then

$$\widetilde{W}_i(K, L)^{n-i} \leq \widetilde{W}_i(K)^{n-1-i} \widetilde{W}_i(L), \tag{2.6}$$

with equality if and only if $K$ and $L$ are dilates. The Brunn-Minkowski inequality for dual quermassintegral is the following: If $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, then

$$\widetilde{W}_i(K \widetilde{+} L)^{1/(n-i)} \leq \widetilde{W}_i(K)^{1/(n-i)} + \widetilde{W}_i(L)^{1/(n-i)}, \tag{2.7}$$

with equality if and only if $K$ and $L$ are dilates.

### 2.2 $L_p$-dual mixed volume

The dual mixed volume $\widetilde{V}_{-1}(K, L)$ of star bodies $K$ and $L$ is defined by ([8])

$$\widetilde{V}_{-1}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K) - V(K \widehat{+} \varepsilon \cdot L)}{\varepsilon}, \tag{2.8}$$

where $\widehat{+}$ is the harmonic addition. The following is a integral representation for the dual mixed volume $\widetilde{V}_{-1}(K, L)$:

$$\widetilde{V}_{-1}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+1} \rho(L, u)^{-1} dS(u). \tag{2.9}$$

The dual Minkowski inequality for the dual mixed volume states that

$$\widetilde{V}_{-1}(K, L)^n \geq V(K)^{n+1} V(L)^{-1}, \tag{2.10}$$

with equality if and only if $K$ and $L$ are dilates (see [9]).

The dual Brunn-Minkowski inequality for the harmonic addition states that

$$V(K\widehat{+}L)^{-1/n} \geq V(K)^{-1/n} + V(L)^{-1/n}, \tag{2.11}$$

with equality if and only if $K$ and $L$ are dilates (this inequality is due to Firey [4]).

The $L_p$ dual mixed volume $\widetilde{V}_{-p}(K, L)$ of $K$ and $L$ is defined by [8])

$$\widetilde{V}_{-p}(K, L) = -\frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K\widehat{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon}, \tag{2.12}$$

where $K, L \in \mathcal{S}^n$ and $p \geq 1$.

The following is an integral representation for the $L_p$ dual mixed volume: For $K, L \in \mathcal{S}^n$ and $p \geq 1$,

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} dS(u). \tag{2.13}$$

$L_p$-dual Minkowski and Brunn-Minkowski inequalities were established by Lutwak [8]: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{2.14}$$

with equality if and only if $K$ and $L$ are dilates, and

$$V(K\widehat{+}_p L)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{2.15}$$

with equality if and only if $K$ and $L$ are dilates.

### 2.3 Mixed p-harmonic quermassintegral

In 1996, $L_p$-harmonic radial addition for star bodies was defined by Lutwak [8]: If $K, L$ are star bodies, for $p \geq 1$, the $L_p$-harmonic radial addition defined by

$$\rho(K\widehat{+}_p L, x)^{-p} = \rho(K, x)^{-p} + \rho(L, x)^{-p}, \tag{2.16}$$

for $x \in \mathbb{R}^n$. For convex bodies, $L_p$-harmonic addition was first investigated by Firey [4]. The operations of the $L_p$-radial addition, $L_p$-harmonic radial addition and the $L_p$-dual Minkowski, Brunn-Minkwski inequalities are fundamental notions and inequalities from the $L_p$-dual Brunn-Minkowski theory.

From (2.16), it is easy to see that if $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$-\frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K\widehat{+}_p \varepsilon \cdot L) - \widetilde{W}_i(L)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(K.u)^{n-i+p} \rho(L.u)^{-p} dS(u). \tag{2.17}$$

Let $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, the mixed $p$-harmonic quermassintegral of star $K$ and $L$, denoted by $\widetilde{W}_{-p,i}(K, L)$, defined by (see [16])

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i+p} \rho(L, u)^{-p} dS(u). \tag{2.18}$$

Obviously, when $K = L$, the $p$-harmonic quermassintegral $\widetilde{W}_{-p,i}(K, L)$ becomes the dual quermassintegral $\widetilde{W}_i(K)$. The Minkowski and Brunn-Minkowski inequalities

for the mixed $p$-harmonic quermassintegral are following (see [16]): If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \geq \widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}, \tag{2.19}$$

with equality if and only if $K$ and $L$ are dilates. If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$\widetilde{W}_i(K \widehat{+}_p L)^{-p/(n-i)} \geq \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)}, \tag{2.20}$$

with equality if and only if $K$ and $L$ are dilates.

# 3 $L_p$-multiple dual mixed volumes

In [22], the $L_p$-multiple mixed volume was introduced as follows:

**Definition 3.1** For $p \geq 1$, the $L_p$-multiple dual mixed volume of star bodies $L_1, K_1, \ldots, K_n$, denoted by

$$\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \left( \frac{\rho(L_1, u)}{\rho(K_1, u)} \right)^p \rho(L_1, u)\rho(K_2, u) \cdots \rho(K_n, u) dS(u). \tag{3.1}$$

Putting $L_1 = K_1$ in (3.1), the $L_p$ multiple dual mixed volume $\widetilde{V}_{-p}(L_1, K_1, \cdots, K_n)$ becomes the usual dual mixed volume $\widetilde{V}(K_1, \cdots, K_n)$. Putting $K_1 = L$ and $L_1 = K_2 = \cdots = K_n = K$ in (3.1), $\widetilde{V}_{-p}(L_1, K_1, \cdots, K_n)$ becomes the $L_p$ dual mixed volume $\widetilde{V}_{-p}(K, L)$. Putting $K_1 = L$ and $L_1 = K_2 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = B$ in (3.1), $\widetilde{V}_{-p}(L_1, K_1, \cdots, K_n)$ becomes the harmonic mixed $p$-quermassintegral, $\widetilde{W}_{-p,i}(K, L)$.

**$L_p$-dual Aleksandrov-Fenchel inequality for $L_p$-multiple dual mixed volumes.**

If $L_1, K_1, \cdots, K_n \in \mathcal{S}^n$, $p \geq 1$ and $1 \leq r \leq n$, then

$$\widetilde{V}_{-p}(L_1, K_1, K_2, \cdots, K_n) \geq \frac{\widetilde{V}(L_1, K_2, \cdots, K_n)^{p+1}}{\prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{p}{r}}}. \tag{3.2}$$

*with equality if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.*

The following inequality follows immediately from (3.2). If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \geq \widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}, \tag{3.3}$$

with equality if and only if $K$ and $L$ are dilates. Taking $i = 0$ in (3.3), this yields the $L_p$-dual Minkowski inequality: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K,L)^n \geq V(K)^{n+p}V(L)^{-p}, \tag{3.4}$$

with equality if and only if $K$ and $L$ are dilates.

A limit of representation of the $L_p$-multiple dual mixed volume was found,

$$\frac{1}{-p}\widetilde{V}_{-p}(L_1, K_1, \cdots, K_n) = \lim_{\varepsilon \to 0^+} \frac{\widetilde{V}(L_1 \widehat{+}_p \varepsilon \cdot K_1, K_2, \cdots, K_n) - \widetilde{V}(L_1, K_2, \cdots, K_n)}{\varepsilon}. \tag{3.5}$$

# 4   The dual logarithmic Aleksandrov-Fenchel inequality

In the section, in order to derive a dual logarithmic Aleksandrov-Fenchel inequality involving logarithms, we need to define some new mixed volume measures.

If $K_1, \ldots, K_n \in \mathcal{S}^n$, the dual mixed volume of star bodies $K_1, \ldots, K_n$, $\widetilde{V}(K_1, \ldots, K_n)$ defined by

$$\widetilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \tag{4.1}$$

From (4.1), we introduce the dual mixed volume measure of star bodies $L_1, K_2, \ldots, K_n$.

**Definition 4.1**   (dual mixed volume measure) For $L_1, K_2, \cdots, K_n \in \mathcal{S}^n$, the $L_p$-dual mixed volume measure of $L_1, K_2, \ldots, K_n$, denoted by $d\tilde{v}(L_1, K_2, \ldots, K_n)$, defined by

$$d\tilde{v}(L_1, K_2, \ldots, K_n) = \frac{1}{n} \rho(L_1, u) \rho(K_2, u) \cdots \rho(K_n, u) dS(u). \tag{4.2}$$

From Definition 4.1, we get the following mixed volume probability measure.

$$d\widetilde{V}(L_1, K_2, \ldots, K_n) = \frac{1}{\widetilde{V}(L_1, K_2, \ldots, K_n)} d\tilde{v}(L_1, K_2, \ldots, K_n). \tag{4.3}$$

For $p \geq 1$, $L_p$-multiple dual mixed volume of $L_1, K_1 \cdots, K_n$, denoted by $\widetilde{V}_{-p}(L_1, K_1, \cdots, K_n)$, defined by

$$\widetilde{V}_{-p}(L_1, K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u)^{-p} \rho(L_1, u)^{1+p} \rho(K_2, u) \cdots \rho(K_n, u) dS(u). \tag{4.4}$$

From (4.4), we introduce $L_p$-multiple dual mixed volume measure of star bodies $L_1, K_1 \cdots, K_n$ as follows:

**Definition 4.2**   ($L_p$-multiple dual mixed volume measure). For $L_1, K_1, \ldots, K_n \in \mathcal{S}^n$, the dual mixed volume measure of $L_1, K_1 \ldots, K_n$, denoted by $d\tilde{v}_{-p}(L_1, K_1 \cdots, K_n)$, defined by

$$d\tilde{v}_{-p}(L_1, K_1 \cdots, K_n) = \frac{1}{n} \rho(K_1, u)^{-p} \rho(L_1, u)^{1+p} \rho(K_2, u) \cdots \rho(K_n, u) dS(u). \tag{4.5}$$

From Definition 4.2, $L_p$-multiple dual mixed volume probability measure is defined by

$$d\widetilde{V}_{-p}(L_1, K_1 \cdots, K_n) = \frac{1}{\widetilde{V}_{-p}(L_1, K_1 \cdots, K_n)} d\tilde{v}_{-p}(L_1, K_1 \cdots, K_n). \tag{4.6}$$

Obviously, the dual mixed volume measure $d\tilde{v}(L_1, K_2, \ldots, K_n)$ is special case of the $L_p$-multiple dual mixed volume measure. When $K_1 = L_1$, we have

$$d\tilde{v}_{-p}(L_1, L_1, K_2, \cdots, K_n) = d\tilde{v}(L_1, K_2, \cdots, K_n), \tag{4.7}$$

and

$$d\widetilde{V}_{-p}(L_1, L_1, K_2 \cdots, K_n) = \frac{1}{\widetilde{V}(L_1, K_2 \cdots, K_n)} d\tilde{v}(L_1, K_2 \cdots, K_n). \tag{4.8}$$

**Theorem 4.1** (The dual logarithmic Aleksandrov-Fenchel inequality involving logarithms). *If* $L_1, K_1, \ldots, K_n \in \mathcal{S}^n$, $1 \le r \le n$ *and* $p \ge 1$, *then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n) \ge \frac{1}{p} \ln\left(\frac{\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)}{\widetilde{V}(L_1, K_2, \ldots, K_n)}\right)$$

$$\ge \ln\left(\frac{\widetilde{V}(L_1, K_2, \cdots, K_n)}{\prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{1}{r}}}\right), \; (4.9)$$

*the left inequality in* (4.9) *with equality if and only if* $L_1$ *and* $K_1$ *are dilates, and the right inequality with equality if and only if* $L_1, K_1, \ldots, K_r$ *are all dilations of each other.*

*Proof.* From (4.2), (4.5) and (4.6), we have

$$\int_{S^{n-1}} \left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right)^p \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\tilde{v}(L_1, K_2, \ldots, L_n) = \int_{S^{n-1}} \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\tilde{v}_{-p}(L_1, K_1, \ldots, K_n).$$
$$(4.10)$$

Noting that

$$\widetilde{V}_{-p}(L_1, K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u)^{-p} \rho(L_1, u)^{1+p} \rho(K_2, u) \cdots \rho(K_n, u) dS(u),$$

and from Lebesgues dominated convergence theorem, we obtain

$$\int_{S^{n-1}} \left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right)^{\frac{pq}{q+n}} d\tilde{v}(L_1, K_2 \ldots, K_n) \to \widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)$$

as $q \to \infty$, and

$$\int_{S^{n-1}} \left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right)^{\frac{pq}{q+n}} \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\tilde{v}(L_1, K_2, \ldots, K_n) \to \int_{S^{n-1}} \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\tilde{v}_{-p}(L_1, K_1, \ldots, K_n)$$

as $q \to \infty$. Considering the function $g_{L_1, K_1, \ldots, K_n} : [1, \infty] \to \mathbb{R}$, defined by

$$g_{L_1, K_1, \ldots, K_n}(q) = \frac{1}{\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)} \int_{S^{n-1}} \left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right)^{\frac{pq}{q+n}} d\tilde{v}(L_1, K_2, \ldots, K_n). \quad (4.11)$$

By calculating the derivative and limit of this function, we have

$$\frac{dg_{L_1, K_1, \ldots, K_n}(q)}{dq} = \frac{pn}{(q+n)^2} \cdot \frac{1}{\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)}$$

$$\times \int_{S^{n-1}} \left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right)^{\frac{pq}{q+n}} \ln\left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right) d\tilde{v}(L_1, K_2, \ldots, K_n). \quad (4.12)$$

and

$$\lim_{q \to \infty} g_{L_1, K_1, \ldots, K_n}(q) = 1. \tag{4.13}$$

From (4.11), (4.12) and (4.13), and by using L'Hôpital's rule, we have

$$
\begin{aligned}
\lim_{q\to\infty} \ln\left(g_{L_1,K_1,\ldots,K_n}(q)\right)^{q+n} &= -(q+n)^2 \lim_{q\to\infty} \frac{\dfrac{dg_{L_1,K_1,\ldots,K_n}(q)}{dq}}{g_{L_1,K_1,\ldots,K_n}(q)} \\
&= -\frac{pn}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)} \\
&\quad\times \lim_{q\to\infty} \frac{\int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{\frac{pq}{q+n}} \ln\left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right) d\tilde{v}(L_1,K_2,\ldots,K_n)}{g_{L_1,K_1,\ldots,K_n}(q)} \\
&= -\frac{pn}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)} \\
&\quad\times \int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{p} \ln\left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right) d\tilde{v}(L_1,K_2,\ldots,K_n).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\exp\left(-\frac{pn}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)} \int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{p} \ln\left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right) d\tilde{v}(L_1,K_2,\ldots,K_n)\right) \\
&= \lim_{q\to\infty}(g_{L_1,K_1,\ldots,K_n})^{q+n} \\
&= \lim_{q\to\infty}\left(\frac{1}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)} \int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{\frac{pq}{q+n}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{q+n}.
\end{aligned}
\tag{4.14}
$$

On the other hand, from Hölder's inequality

$$
\begin{aligned}
&\left(\int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{\frac{pq}{q+n}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{(q+n)/q} \left(\int_{S^{n-1}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{-n/q} \\
&\leq \int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{p} d\tilde{v}(L_1,K_2,\ldots,K_n) \\
&= \widetilde{V}_{-p}(L_1,K_1,\ldots,K_n).
\end{aligned}
\tag{4.15}
$$

From the equality condition of Hölder's inequality, it follows the equality in (4.15) holds if and only if $\rho(K_1,u)$ and $\rho(L_1,u)$ are proportional. This yiels equality in (4.15) holds if and only if $K_1$ and $L_1$ are dilates. Namely,

$$
\left(\frac{1}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)} \int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{\frac{pq}{q+n}} d\tilde{v}(L_1,K_2,\ldots,K_n)\right)^{q+n} \leq \left(\frac{\widetilde{V}(L_1,K_2,\ldots,K_n)}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)}\right)^{n},
$$

with equality if and only if $K_1$ and $L_1$ are dilates. Hence

$$
\begin{aligned}
&\exp\left(-\frac{pn}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)} \int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{p} \ln\left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right) d\tilde{v}(L_1,K_2,\ldots,K_n)\right) \\
&\qquad\qquad\leq \left(\frac{\widetilde{V}(L_1,K_2,\ldots,K_n)}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)}\right)^{n},
\end{aligned}
$$

with equality if and only if $K_1$ and $L_1$ are dilates. That is

$$
\frac{p}{\widetilde{V}_{-p}(L_1,K_1,\ldots,K_n)} \int_{S^{n-1}} \left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right)^{p} \ln\left(\frac{\rho(L_1,u)}{\rho(K_1,u)}\right) d\tilde{v}(L_1,K_2,\ldots,K_n)
$$

$$\geq \ln \left( \frac{\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)}{\widetilde{V}(L_1, K_2, \ldots, K_n)} \right),$$

with equality if and only if $K_1$ and $L_1$ are dilates. Therefore

$$\int_{S^{n-1}} \ln \left( \frac{\rho(L_1, u)}{\rho(K_1, u)} \right) d\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n) \geq \frac{1}{p} \ln \left( \frac{\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n)}{\widetilde{V}(L_1, K_2, \ldots, K_n)} \right), \quad (4.16)$$

with equality if and only if $K_1$ and $L_1$ are dilates. The completes proof of the first inequality in (4.9).

Further, by using the $L_p$-dual Aleksandrov-Fenchel inequality (3.2), we obtain

$$
\begin{aligned}
\int_{S^{n-1}} \ln \left( \frac{\rho(L_1, u)}{\rho(K_1, u)} \right) d\widetilde{V}_{-p}(L_1, K_1, \ldots, K_n) \quad &\geq \quad \frac{1}{p} \ln \left( \frac{1}{\widetilde{V}(L_1, K_2, \ldots, K_n)} \times \right. \\
&\qquad \times \quad \left. \frac{\widetilde{V}(L_1, K_2, \cdots, K_n)^{p+1}}{\prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{p}{r}}} \right) \\
&= \quad \ln \left( \frac{\widetilde{V}(L_1, K_2, \cdots, K_n)}{\prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{\frac{1}{r}}} \right),
\end{aligned}
$$

with equality if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

This completes the proof. $\qquad\square$

**Theorem 4.2** *If $K$ and $L$ are star bodies in $\mathbb{R}^n$, and $0 \leq i < n$ and $p \geq 1$, then*

$$\int_{S^{n-1}} \ln \left( \frac{\rho(L, u)}{\rho(K, u)} \right) d\widetilde{W}_{-p,i}(L, K) \geq \frac{1}{p} \ln \left( \frac{\widetilde{W}_{-p,i}(L, K)}{\widetilde{W}_i(L)} \right) \geq \frac{1}{n-i} \ln \left( \frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)} \right). \quad (4.17)$$

*each equality holds if and only if $K$ and $L$ are dilates, and where*

$$d\tilde{w}_{-p,i}(L, K) = d\tilde{v}_{-p}(L, K, \underbrace{L, \ldots, L}_{n-1-i}, \underbrace{B, \ldots, B}_{i}) = \frac{1}{n} \rho(K, u)^{-p} \rho(L, u)^{n-i+p} dS(u),$$

$$(4.18)$$

*and*

$$d\widetilde{W}_{-p,i}(L, K) = \frac{1}{\widetilde{W}_{-p,i}(L, K)} d\tilde{w}_{-p,i}(L, K), \quad (4.19)$$

*denotes its normalization.*

*Proof.* Putting $L_1 = L$, $K_1 = K$, $K_2 = \cdots = K_{n-i-1} = L$, $K_{n-i} = \ldots = K_n = B$ in (4.4), (4.5) and (4.6), we obtain

$$\widetilde{V}_{-p}(L, K, \underbrace{L, \ldots, L}_{n-i-1}, \underbrace{B, \ldots, B}_{i}) = \widetilde{W}_{-p,i}(L, K), \quad (4.20)$$

$$d\tilde{v}_{-p}(L, K, \underbrace{L, \ldots, L}_{n-i-1}, \underbrace{B, \ldots, B}_{i}) = d\tilde{w}_{-p,i}(L, K). \quad (4.21)$$

and

$$d\widetilde{V}_{-p}(L, K, \underbrace{L, \ldots, L}_{n-i-1}, \underbrace{B, \ldots, B}_{i}) = \frac{1}{\widetilde{W}_{-p,i}(L, K)} d\tilde{w}_{-p,i}(L, K). \quad (4.22)$$

In view of (4.16) and (4.20)-(4.22), we have

$$\int_{S^{n-1}} \ln\left(\frac{\rho(L,u)}{\rho(K,u)}\right) d\widetilde{W}_{-p,i}(L,K) \geq \frac{1}{p}\ln\left(\frac{\widetilde{W}_{-p,i}(L,K)}{\widetilde{W}_i(L)}\right) \tag{4.23}$$

From (2.19) and (4.23), (4.17) easy follows. This completes the proof.    $\square$

A special case of (4.17) is the following dual logarithmic Minkowski type inequality.

**Corollary 4.1**  *If $K$ and $L$ are star bodies in $\mathbb{R}^n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(L,u)}{\rho(K,u)}\right) d\widetilde{V}_{-1}(L,K) \geq \ln\left(\frac{\widetilde{V}_{-1}(L,K)}{V(L)}\right) \geq \frac{1}{n}\ln\left(\frac{V(L)}{V(K)}\right). \tag{4.24}$$

*each equality holds if and only if $K$ and $L$ are dilates.*

*Proof.* This yields immediately from Theorem 4.2 with $p = 1$ and $i = 0$.

Another special case is the following logarithmic $L_p$-dual Minkowski type inequality.

**Corollary 4.2**  *If $K$ and $L$ are star bodies in $\mathbb{R}^n$ and $p \geq 1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho(L,u)}{\rho(K,u)}\right) d\widetilde{V}_{-p}(L,K) \geq \frac{1}{p}\ln\left(\frac{\widetilde{V}_{-p}(L,K)}{V(L)}\right) \geq \frac{1}{n}\ln\left(\frac{V(L)}{V(K)}\right). \tag{4.25}$$

*each equality holds if and only if $K$ and $L$ are dilates. Here*

$$d\tilde{v}_{-p}(L,K) = d\tilde{v}_{-p}(L,K,\underbrace{L,\ldots,L}_{n-1}) = \frac{1}{n}\rho(K,u)^{-p}\rho(L,u)^{n+p}dS(u),$$

*and*

$$d\widetilde{V}_{-p}(L,K) = \frac{1}{\widetilde{V}_{-p}(L,K)}d\tilde{v}_{-p}(L,K),$$

*denotes its normalization.*

*Proof.* This yields immediately from (4.17) with $i = 0$.

# References

[1] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *The log-Brunn-Minkowski inequality*, Adv. Math., 231 (2012), 1974-1997.

[2] A. Colesanti, P. Cuoghi, *The Brunn-Minkowski inequality for the n-dimensional logarithmic capacity of convex bodies*, Potential Math., 22 (2005), 289-304.

[3] M. Fathi, B. Nelson, *Free Stein kernels and an improvement of the free logarithmic Sobolev inequality*, Adv. Math., 317 (2017), 193-223.

[4] W. J. Firey, *Polar means of convex bodies and a dual to the Brunn-Minkowski theorem*, Canad. J. Math., 13 (1961), 444-453.

[5] M. Henk, H. Pollehn, *On the log-Minkowski inequality for simplices and parallelepipeds*, Acta Math. Hungarica, 155 (2018), 141-157.

[6] S. Hou, J. Xiao, *A mixed volumetry for the anisotropic logarithmic potential*, J. Geom Anal., 28 (2018), 2018-2049.

[7] C. Li, W. Wang, *Log-Minkowski inequalities for the $L_p$-mixed quermassintegrals*, J. Inequal. Appl., 2019 (2019): 85.

[8] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc., 60 (1990), 365-391.

[9] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math., 71 (1988), 232-261.

[10] E. Lutwak, *Dual mixed volumes*, Pacific J. Math., 58 (1975), 531-538.

[11] S.-J. Lv, *The $\varphi$-Brunn-Minkowski inequality*, Acta Math. Hungarica, 156 (2018), 226-239.

[12] L. Ma, *A new proof of the Log-Brunn-Minkowski inequality*, Geom. Dedicata, 177 (2015), 75-82.

[13] C. Saroglou, *Remarks on the conjectured log-Brunn-Minkowski inequality*, Geom. Dedicata, 177 (2015), 353-365.

[14] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, 1993.

[15] W. Wang, M. Feng, *The log-Minkowski inequalities for quermassintegrals*, J. Math. Inequal., 11, (2017), 983995

[16] W. Wang, G. Leng, *$L_p$-dual mixed quermassintegrals*, Indian J. Pure Appl. Math., 36 (2005), 177-188.

[17] W. Wang, L. Liu, *The dual Log-Brunn-Minkowski inequalities*, Taiwanese J. Math., 20 (2016), 909-919.

[18] X. Wang, W. Xu, J. Zhou, *Some logarithmic Minkowski inequalities for nonsymmetric convex bodies* , Sci. China, 60 (2017), 1857-1872.

[19] C.-J. Zhao, *Orlicz-Aleksandrov-Fenchel inequality for Orlicz multiple mixed volumes*, J. Func. Spaces, 2018, Article ID 9752178, 16 pages.

[20] C.-J. Zhao, *On the Orlicz-Brunn-Minkowski theory*, Balkan J. Geom. Appl., 22 (2017), 98-121.

[21] C.-J. Zhao, *Inequalities for Orlicz mixed quermassintegrals*, J. Convex Anal, 26 (1) (2019), 129-151.

[22] C.-J. Zhao, *The dual Orlicz-Aleksandrov-Fenchel inequality*, arXiv:2002.11112v1 [math.MG] 25 Feb 2020. https://arxiv.org/abs/2002.11112.

[23] G. Zhu, *The logarithmic Minkowski problem for polytopes*, Adv. Math., 262 (2014), 909-931.

*Author's address:*

Chang-Jian Zhao
Department of Mathematics, China Jiliang University,
Hangzhou 310018, P. R. China.
E-mail: chjzhao@163.com, chjzhao@cjlu.edu.cn