On $\epsilon-$Kenmotsu 3-manifolds admitting $*-$conformal $\eta-$Ricci solitons

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Abstract. In the present paper we study $\epsilon-$Kenmotsu 3-manifolds admitting $*-$conformal $\eta-$Ricci solitons. Besides, we study gradient $*-$conformal $\eta-$ Ricci solitons on $\epsilon-$Kenmotsu 3-manifolds and prove that a gradient $*-$conformal $\eta-$ Ricci soliton on an $\epsilon-$Kenmotsu 3-manifold is $*-$conformal $\eta-$Einstein if and only if $\xi f = 0$. Finally, the existence of $*-$conformal $\eta-$Ricci soliton in an $\epsilon-$Kenmotsu 3-manifold has been proved by a concrete example.

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1 Introduction

The study of manifolds with indefinite metrics is of high interest in physics and relativity theory. In 1993, the concept of $\epsilon-$Sasakian manifolds was introduced by Bejancu and Duggal [2]. Later, it was shown by Xufeng and Xiaoli [22] that every $\epsilon-$Sasakian manifolds are real hypersurfaces of indefinite Kahlerian manifolds. In 1972, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [13]. We call it Kenmotsu manifold. The concept of $\epsilon-$Kenmotsu manifold was introduced by De and Sarkar [5] who showed that the existence of new structure on an indefinite metric influences the curvatures. Recently, $\epsilon-$Kenmotsu manifolds have also been studied by various authors such as ([9], [10], [11], [15], [21]) and many others.

In 2004, the concept of conformal Ricci flow was developed by Fischer [6] as a variation of the classical Ricci flow equation. The conformal Ricci flow on a smooth closed connected oriented $n-$manifold M is defined by the equation

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \tag{1.1}$$

and $r = -1$, where p is a time dependent non-dynamical scalar field, S and r are the Ricci tensor and the scalar curvature, respectively on M.

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The equations of a conformal Ricci soliton and of a conformal $\eta-$Ricci soliton are given respectively by ([1], [18])

$$\pounds_V g + 2S = (2\lambda - (p + \frac{2}{n}))g, \tag{1.2}$$

$$\pounds_V g + 2S + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta\otimes\eta = 0, \tag{1.3}$$

where λ and μ are constants.

The notion of $*-$Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [19]. Later, Hamada [8] studied $*-$Ricci flat real hypersurfaces of complex space forms and Blair [3] defined $*-$Ricci tensor in contact metric manifolds given by

$$S^*(X,Y) = g(Q^*X,Y) = \text{Trace}\,\{\phi\circ R(X,\phi Y)\} \tag{1.4}$$

for any vector fields X, Y on M, where Q^* is the (1,1) $*-$Ricci operator and S^* is a tensor field of type $(0,2)$.

Definition 1.1. [12] A Riemannian ($or\ semi-Riemannian$) metric g on M is called a $*-$Ricci soliton, if

$$(\pounds_V g)(X,Y) + 2S^*(X,Y) + 2\lambda g(X,Y) = 0 \tag{1.5}$$

for all vector fields X, Y on M and λ is a constant.

Definition 1.2. [17] A Riemannian ($or\ semi-Riemannian$) metric g on M is called a $*-$conformal $\eta-$Ricci soliton, if

$$\pounds_V g + 2S^* + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta\otimes\eta = 0, \tag{1.6}$$

where $\pounds_V$ is the Lie derivative along the vector field V, S^* is the $*-$Ricci tensor and λ, μ are constants.

Definition 1.3. A Riemannian ($or\ semi-Riemannian$) metric g on M is called a gradient $*-$conformal $\eta-$Ricci soliton, if

$$Hessf + S^* + (\lambda - \frac{1}{2}(p + \frac{2}{n}))g + \mu\eta\otimes\eta = 0, \tag{1.7}$$

where $Hessf$ denotes the Hessian of a smooth function f on M and defined by $Hessf = \nabla\nabla f$.

If $S^*(X,Y) = (\lambda - \frac{1}{2}(p + \frac{2}{n}))g(X,Y) + \mu\eta(X)\eta(Y)$ for all vector fields X, Y and λ, μ are smooth functions on M, then the manifold is called $*-$conformal $\eta-$Einstein manifold. Further if $\mu = 0$, that is, $S^*(X,Y) = (\lambda - \frac{1}{2}(p + \frac{2}{n}))g(X,Y)$ for all vector fields X, Y, then the manifold becomes $*-$conformal Einstein manifold.

If an $\epsilon-$ Kenmotsu manifold satisfies (1.6), then we say that M admits a $*-$conformal $\eta-$Ricci soliton. Recently, De et al. [4] studied $*-$Ricci solitons in an $\epsilon-$Kenmotsu 3-manifold and provide the condition for a $*-$Ricci soliton in an $\epsilon-$Kenmotsu 3-manifold with constant scalar curvature to be steady. The $*-$Ricci solitons have also been studied by various authors in several ways to a different extent such as ([4], [7], [14], [16], [20]) and many others.

2 Preliminaries

An $n-$dimensional smooth manifold (M, g) is said to be an $\epsilon-$almost contact metric manifold [2], if it admits a $(1, 1)$ tensor field ϕ, a structure vector field ξ, a $1-$form η and an indefinite metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\xi, \xi) = \epsilon, \quad \eta(X) = \epsilon g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M, where ϵ is 1 or -1 according as ξ is spacelike or timelike vector fields and rank ϕ is $(n - 1)$. If

$$d\eta(X, Y) = g(X, \phi Y) \tag{2.4}$$

for every $X, Y \in \chi(M)$, then we say that M is an $\epsilon-$contact metric manifold. Also, we have

$$\phi\xi = 0, \ \eta(\phi X) = 0. \tag{2.5}$$

If an ϵ-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X, \tag{2.6}$$

where ∇ denotes the Levi-Civita connection with respect to g, then M is called an $\epsilon-$Kenmotsu manifold [5].
An $\epsilon-$almost contact metric manifold is an $\epsilon-$Kenmotsu if and only if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{2.7}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in an $\epsilon-$Kenmotsu manifold M with respect to the Levi-Civita connection satisfies

$$(\nabla_X \eta)Y = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.8}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.10}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.11}$$

$$\eta(R(X, Y)Z) = \epsilon(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)), \tag{2.12}$$

$$(i) \ S(X, \xi) = -(n - 1)\eta(X), \quad (ii) \ S(\xi, \xi) = -(n - 1), \tag{2.13}$$

$$Q\xi = -\epsilon(n - 1)\xi \tag{2.14}$$

for any X, Y, Z on M, where $g(QX, Y) = S(X, Y)$. We note that if $\epsilon = 1$ and the structure vector field ξ is spacelike, then an $\epsilon-$Kenmotsu manifold is usual Kenmotsu manifold.

Lemma 2.1. *In an $\epsilon-$Kenmotsu n-manifold $(M,\phi,\xi,\eta,g,\epsilon)$, we have* [11]

$$\begin{aligned} \bar{R}(X,Y,\phi Z,\phi W) &= \bar{R}(X,Y,Z,W) \\ &+\epsilon\Phi(X,Z)\Phi(Y,W)-\epsilon\Phi(Y,Z)\Phi(X,W) \\ &+\epsilon g(Y,Z)g(X,W)-\epsilon g(X,Z)g(Y,W) \end{aligned} \tag{2.15}$$

for any X,Y,Z,W on M, where $\bar{R}(X,Y,Z,W)=g(R(X,Y)Z,W)$ and Φ is the fundamental 2-form of M defined by $\Phi(X,Y)=g(X,\phi Y)$.

The curvature tensor of an $\epsilon-$Kenmotsu $3-$manifold is given by

$$\begin{aligned} R(X,Y)Z &= S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY \\ &-\frac{r}{2}(g(Y,Z)X-g(X,Z)Y) \end{aligned} \tag{2.16}$$

for any $X,Y,Z\in\chi(M)$ and r is the scalar curvature of the manifold. Putting $Z=\xi$ in (2.16) and using (2.2), (2.9) and (2.13)(i), we find

$$\eta(Y)QX-\eta(X)QY=\frac{r}{2}(\eta(Y)X-\eta(X)Y). \tag{2.17}$$

Again putting $Y=\xi$ in (2.17) and using (2.1) and (2.14), we get

$$QX=(\frac{r}{2}+\epsilon)X-(\frac{r}{2}+3\epsilon)\eta(X)\xi. \tag{2.18}$$

From (2.18), we find

$$S(X,Y)=(\frac{r}{2}+\epsilon)g(X,Y)-(\frac{\epsilon r}{2}+3)\eta(X)\eta(Y). \tag{2.19}$$

Now we prove the following Lemma :

Lemma 2.2. *In an $\epsilon-$Kenmotsu $3-$manifold $(M,\phi,\xi,\eta,g,\epsilon)$, the $*-$Ricci tensor is given by*

$$S^*(Y,Z) = S(Y,Z)+\epsilon g(Y,Z)+\eta(Y)\eta(Z) \tag{2.20}$$

for any $Y,Z\in\chi(M)$, where S and S^* are the Ricci tensor and the $*-$Ricci tensor of type $(0,2)$, respectively on M.

Proof. Let $\{e_i\}, i=1,2,3$ be an orthonormal basis of the tangent space at each point of the manifold. From the equations (2.15) and (1.4), we have

$$\begin{aligned} S^*(Y,Z) &= \sum_{i=1}^{3}\bar{R}(e_i,Y,\phi Z,\phi e_i) \\ &= \sum_{i=1}^{3}[\bar{R}(e_i,Y,Z,e_i)+\epsilon\Phi(e_i,Z)\Phi(Y,e_i)-\epsilon\Phi(Y,Z)\Phi(e_i,e_i) \\ &+ \epsilon g(Y,Z)g(e_i,e_i)-\epsilon g(e_i,Z)g(Y,e_i)]. \end{aligned}$$

By using (2.3) and $\Phi(X,Y)=g(X,\phi Y)$ in the above equation, Lemma 2.2 follows. □

3 $\epsilon-$Kenmotsu $3-$manifolds admitting $*-$conformal $\eta-$Ricci solitons

In this section we prove the following theorem:

Theorem 3.1. *If an $\epsilon-$Kenmotsu $3-$manifold with a constant scalar curvature admits a $*-$conformal $\eta-$Ricci soliton, then $\lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{3})$.*

Proof. By using (2.19) in (2.20), the $*-$Ricci tensor S^* is given by

$$S^*(X,Y) = (\frac{r}{2} + 2\epsilon)g(X,Y) - (\frac{\epsilon r}{2} + 2)\eta(X)\eta(Y). \tag{3.1}$$

From the definition of a $*-$conformal $\eta-$Ricci soliton, we have

$$\begin{aligned}(\pounds_V g)(X,Y) &= -2S^*(X,Y) - (2\lambda - (p + \frac{2}{3}))g(X,Y) - 2\mu\eta(X)\eta(Y) \\ &= -(r + 4\epsilon + 2\lambda - (p + \frac{2}{3}))g(X,Y) + (\epsilon r + 4 - 2\mu)\eta(X)\eta(Y). \end{aligned}\tag{3.2}$$

Now taking covariant differentiation of (3.2) with respect to Z, we get

$$\begin{aligned}(\nabla_Z \pounds_V g)(X,Y) &= -(Zr)(g(X,Y) - \epsilon\eta(X)\eta(Y)) \\ &\quad +(\epsilon r + 4 - 2\mu)(g(X,Z) - \epsilon\eta(X)\eta(Z))\eta(Y) \\ &\quad +(\epsilon r + 4 - 2\mu)(g(Y,Z) - \epsilon\eta(Y)\eta(Z))\eta(X).\end{aligned}\tag{3.3}$$

Following Yano [23], the following formula

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y)$$

is well known for any vector fields X, Y, Z on M. As g is parallel with respect to the Levi-Civita connection ∇, the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y) \tag{3.4}$$

for any vector fields X, Y, Z. Since $\pounds_V \nabla$ is a symmetric tensor of type $(1,2)$, that is, $(\pounds_V \nabla)(X,Y) = (\pounds_V \nabla)(Y,X)$, then it follows from (3.4) that

$$\begin{aligned}g((\pounds_V \nabla)(X,Y),Z) &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) \\ &\quad -\frac{1}{2}(\nabla_Z \pounds_V g)(X,Y).\end{aligned}\tag{3.5}$$

Using (3.3) in (3.5), we have

$$\begin{aligned}2g((\pounds_V \nabla)(X,Y),Z) &= -(Xr)g(\phi Y,\phi Z) \\ &\quad +(\epsilon r + 4 - 2\mu)(g(\phi Y,\phi X)\eta(Z) + g(\phi Z,\phi X)\eta(Y)) \\ &\quad -(Yr)g(\phi X,\phi Z) \\ &\quad +(\epsilon r + 4) - 2\mu(g(\phi X,\phi Y)\eta(Z) + g(\phi Z,\phi Y)\eta(X)) \\ &\quad +(Zr)g(\phi X,\phi Y) \\ &\quad -(\epsilon r + 4 - 2\mu)(g(\phi X,\phi Z)\eta(Y) + g(\phi Y,\phi Z)\eta(X)).\end{aligned}$$

By removing Z from the last equation, it follows that

$$
\begin{aligned}
2(\pounds_V\nabla)(X,Y) &= -(Xr)(Y-\eta(Y)\xi) \\
&+(\epsilon r+4-2\mu)(\epsilon g(\phi Y,\phi X)\xi+(X-\eta(X)\xi)\eta(Y)) \\
&-(Yr)(X-\eta(X)\xi) \\
&+(\epsilon r+4-2\mu)(\epsilon g(\phi X,\phi Y)\xi+(Y-\eta(Y)\xi)\eta(X)) \\
&+(Dr)g(\phi X,\phi Y) \\
&-(\epsilon r+4-2\mu)((X-\eta(X)\xi)\eta(Y)+(Y-\eta(Y)\xi)\eta(X)),
\end{aligned}
\tag{3.6}
$$

where $X\alpha = g(D\alpha, X)$, D denotes the gradient operator with respect to g. Putting $Y = \xi$ in (3.6) and using $r = constant$ (hence $(Dr) = 0$ and $(\xi r = 0)$), we find

$$(\pounds_V\nabla)(X,\xi) = 0. \tag{3.7}$$

Taking the covariant derivative of (3.7) with respect to Y, we have

$$(\nabla_Y\pounds_V\nabla)(X,\xi) = 0. \tag{3.8}$$

Again from [23], we have

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z).$$

Thus the last two equations give

$$(\pounds_V R)(X,Y,\xi) = 0. \tag{3.9}$$

Taking the Lie-derivative of $R(X,\xi)\xi = \eta(X)\xi - X$ along V, we have

$$(\pounds_V R)(X,\xi)\xi - 2\eta(\pounds_V\xi)X + \epsilon g(X,\pounds_V\xi)\xi = (\pounds_V\eta)(X)\xi$$

which by using (3.9) reduces to

$$(\pounds_V\eta)(X)\xi = -2\eta(\pounds_V\xi)X + \epsilon g(X,\pounds_V\xi)\xi. \tag{3.10}$$

Now taking the Lie derivative of $\eta(X) = g(X,\xi)$, we find

$$(\pounds_V\eta)X = \epsilon(\pounds_V g)(X,\xi) + \epsilon g(X,\pounds_V\xi). \tag{3.11}$$

Taking $Y = \xi$ in (3.2) leads to

$$(\pounds_V g)(X,\xi) = -2\epsilon(\lambda+\epsilon\mu-\frac{1}{2}(p+\frac{2}{3}))\eta(X). \tag{3.12}$$

Putting $X = \xi$ in (3.12) yields

$$\eta(\pounds_V\xi) = \lambda+\epsilon\mu-\frac{1}{2}(p+\frac{2}{3}). \tag{3.13}$$

By making use of (3.11) – (3.13), we get from (3.10) that

$$(\lambda+\epsilon\mu-\frac{1}{2}(p+\frac{2}{n}))\phi^2X = 0 \tag{3.14}$$

from which it follows that

$$\lambda+\epsilon\mu = \frac{1}{2}(p+\frac{2}{3}), \tag{3.15}$$

where $\phi^2X \neq 0$. This completes the proof of the Theorem 3.1. □

4 Gradient $*-$conformal $\eta-$Ricci solitons on $\epsilon-$Kenmotsu 3-manifolds

Let M be an $\epsilon-$Kenmotsu $3-$manifold with g as a gradient $*-$conformal $\eta-$ Ricci soliton. Then equation (1.7) can be written as

$$\nabla_Y Df + Q^*Y + (\lambda - \frac{1}{2}(p + \frac{2}{3}))Y + \epsilon\mu\eta(Y)\xi = 0 \tag{4.1}$$

for all vector fields Y on M, where D denotes the gradient operator of g. First we prove the following Lemmas for later use:

Lemma 4.1. *In an $\epsilon-$Kenmotsu $3-$manifold, we have*

$$(\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y = -(\frac{\epsilon r}{2} + \frac{\xi r}{2} + 2)(Y - \eta(Y)\xi). \tag{4.2}$$

for all vector fields Y on M.

Proof. From (3.1), we can write

$$Q^*X = (\frac{r}{2} + 2\epsilon)(X - \eta(X)\xi). \tag{4.3}$$

Differentiating (4.3) covariantly with respect to Y, we get

$$\begin{aligned} \nabla_Y Q^*X &= \frac{Yr}{2}(X - \eta(X)\xi) + (\frac{r}{2} + 2\epsilon)[\nabla_Y X - (\nabla_Y \eta)(X)\xi \\ &\quad -\eta(\nabla_Y X)\xi - \eta(X)\nabla_Y \xi]. \end{aligned} \tag{4.4}$$

By using (4.3) and (4.4), we find

$$(\nabla_Y Q^*)X = \frac{Yr}{2}(X - \eta(X)\xi) - (\frac{r}{2} + 2\epsilon)[(\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi] \tag{4.5}$$

which by replacing X by ξ and using (2.1), (2.7), (2.8) reduces to

$$(\nabla_Y Q^*)\xi = -(\frac{\epsilon r}{2} + 2)(Y - \eta(Y)\xi). \tag{4.6}$$

Again replacing Y by ξ in (4.5) and using (2.7) and (2.8), we find

$$(\nabla_\xi Q^*)Y = \frac{\xi r}{2}(Y - \eta(Y)\xi). \tag{4.7}$$

By substracting (4.7) from (4.6), (4.2) follows. □

Lemma 4.2. *In an $\epsilon-$Kenmotsu $3-$manifold, we have*

$$R(X,Y)Df = (\nabla_Y Q^*)X - (\nabla_X Q^*)Y + \mu(\eta(X)Y - \eta(Y)X). \tag{4.8}$$

for all vector fields X, Y on M.

Proof. Differentiating (4.1) covariantly along the vector field X, we have

$$\nabla_X \nabla_Y Df + \nabla_X Q^* Y + (\lambda - \frac{1}{2}(p + \frac{2}{3}))\nabla_X Y + \epsilon\mu \nabla_X(\eta(Y)\xi) = 0. \tag{4.9}$$

Interchanging X and Y in (4.9), we have

$$\nabla_Y \nabla_X Df + \nabla_Y Q^* X + (\lambda - \frac{1}{2}(p + \frac{2}{3}))\nabla_Y X + \epsilon\mu \nabla_Y(\eta(X)\xi) = 0. \tag{4.10}$$

Also from (4.1), we find

$$\begin{aligned} &\nabla_{[X,Y]} Df + Q^*(\nabla_X Y - \nabla_Y X) \\ &+(\lambda - \frac{1}{2}(p + \frac{2}{3}))(\nabla_X Y - \nabla_Y X) + \epsilon\mu\eta([X,Y])\xi = 0. \end{aligned} \tag{4.11}$$

By using $(4.9) - (4.11)$ in $R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df$, Lemma 4.2 follows. This completes the proof. □

Theorem 4.3. *A gradient $*-$conformal $\eta-$Ricci soliton on an $\epsilon-$Kenmotsu 3$-$manifold is $*-$conformal $\eta-$Einstein if and only if $\xi f = 0$.*

Proof. Putting $X = \xi$ in (4.8), we have

$$R(\xi, Y)Df = (\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y + \mu(Y - \eta(Y)\xi)$$

which by taking the inner product with ξ and using the Lemma 4.1 gives

$$g(R(\xi, Y)Df, \xi) = 0. \tag{4.12}$$

By using (2.9), we have

$$g(R(\xi, Y)Df, \xi) = \eta(Y)(\xi f) - \epsilon(Yf). \tag{4.13}$$

From (4.12) and (4.13), we find

$$(Yf) = \epsilon\eta(Y)(\xi f) \tag{4.14}$$

for any $Y \in \chi(M)$. Therefore, $Df = (\xi f)\xi$. Thus $Df = 0$ if $\xi f = 0$. Therefore, it follows from (1.7) that $S^*(X,Y) = -(\lambda - \frac{1}{2}(p + \frac{2}{3}))g(X,Y) - \mu\eta(X)\eta(Y)$. This completes the proof. □

Example: We consider the three dimensional manifold $M = [(x, y, z) \in R^3 \mid z \neq 0]$, where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^z \frac{\partial}{\partial y}, \quad e_3 = -\epsilon \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the indefinite Riemannian metric defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = \epsilon,$$

where $\epsilon = \pm 1$. Let η be the 1-form on M defined by $\eta(X) = \epsilon g(X, e_3) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field on M defined by

$$\phi(e_1) = e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = 0.$$

Then by the linearity property of ϕ and g, we have

$$\phi^2 X = -X + \eta(X)e_3, \qquad \eta(e_3) = 1 \qquad \text{and} \qquad g(\phi X, \phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y)$$

for any vector fields $X, Y \in \chi(M)$. Thus for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the indefinite metric g. Then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = \epsilon e_1, \qquad [e_2, e_3] = \epsilon e_2.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z)$$
$$-g([Y,Z],X) + g([Z,X],Y).$$

From above equation which is known as Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -e_3, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_3 = \epsilon e_1,$$
$$\nabla_{e_2} e_1 = 0, \qquad \nabla_{e_2} e_2 = -e_3, \qquad \nabla_{e_2} e_3 = \epsilon e_2,$$
$$\nabla_{e_3} e_1 = 0, \qquad \nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_3 = 0.$$

Using the above relations, it follows that

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi)$$

for $\xi = e_3$. Hence the manifold is an $\epsilon-$Kenmotsu manifold of dimension three. It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

By using the above results, one can easily obtain the components of the curvature tensor as follows:

$$R(e_1, e_2)e_1 = \epsilon e_2, \quad R(e_1, e_2)e_2 = -\epsilon e_1, \quad R(e_1, e_2)e_3 = 0,$$
$$R(e_2, e_3)e_1 = 0, \qquad R(e_2, e_3)e_2 = \epsilon e_3, \quad R(e_2, e_3)e_3 = -e_2$$
$$R(e_1, e_3)e_1 = \epsilon e_3, \qquad R(e_1, e_3)e_2 = 0, \qquad R(e_1, e_3)e_3 = -e_1.$$

From these curvature tensors, we calculate the components of Ricci tensor as follows:

$$S(e_1, e_1) = S(e_2, e_2) = -2\epsilon, \quad S(e_3, e_3) = -2. \tag{4.15}$$

In [11], the authors proved that an $\epsilon-$Kenmotsu $3-$manifold admitting a $*-$conformal $\eta-$Ricci soliton is an $\eta-$Einstein manifold of the form $S(Y,Z) = -(\lambda + 2\epsilon - \frac{1}{2}(p + \frac{2}{3}))g(Y,Z) - \mu\eta(Y)\eta(Z)$. From this equation, we have $S(e_3, e_3) = -\epsilon\lambda - \mu - 2 + \frac{\epsilon}{2}(p + \frac{2}{3})$. By equating both the values of $S(e_3, e_3)$, we obtain

$$\lambda + \epsilon\mu = \frac{1}{2}\left(p + \frac{2}{3}\right).$$

Hence λ and μ satisfies the equation (3.15) and so g defines a $*-$conformal $\eta-$Ricci soliton on the $3-$dimensional $\epsilon-$Kenmotsu manifold.

5 Conclusions

In recent years, the study of $*-$Ricci solitons and gradient $*-\eta-$Ricci solitons on Riemannian (as well as, semi-Riemannian) manifolds became of major importance in the area of differential geometry, physics and relativity as well. The problem of studying $*-$Ricci solitons in a Kaehler manifold was initiated by Kaimakamis and Panagiotidou. Recently, S. Roy with other geometers introduced the notion of a special type of metric on Sasakian manifold, called $*-$conformal $\eta-$Ricci soliton. As a continuation of this study, we made an effort to study $*-$conformal $\eta-$Ricci solitons in the frame-work of $\epsilon-$Kenmotsu geometry.

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