On a class of Finsler metrics of scalar flag curvature

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Abstract. An (α, β)-metric is defined by a Riemannian metric α and 1-form β. In this paper, we study a class of (α, β)-metrics $F = \alpha\phi(\beta/\alpha)$ with $\phi(s)$ satisfying a known ODE. For any metric F in such a class, we show that in dimension $n \geq 3$, F is of scalar flag curvature if and only if F is locally projectively flat, if β is closed. While for a subclass with F being a general square metric type, we prove that in dimension $n \geq 3$, F is of scalar flag curvature if and only if F is locally projectively flat.

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1 Introduction

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, and every two-dimensional Finsler metric is of scalar flag curvature. It is the Hilbert's Fourth Problem to study and classify projectively flat metrics. The Beltrami Theorem states that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. It is known that every locally projectively flat Finsler metric is of scalar flag curvature ([6] [12]). However, the converse is not true, due to the existence of Finsler metrics of constant flag curvature which are not locally projectively flat ([1]). Therefore, it is an interesting point to study and classify Finsler metrics of scalar flag curvature. This problem is far from being solved for general Finsler metrics. Recent studies on this problem are concentrated on Randers metrics and square metrics.

Randers metrics are among the simplest Finsler metrics in the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form satisfying $\|\beta\|_\alpha < 1$. Bao-Robles-Shen classify Randers metrics of constant flag curvature by using the navigation method ([1]). Further, Shen-Yildirim classify Randers metrics of weakly isotropic flag curvature ([11]). There are Randers metrics of scalar flag curvature which are neither of weakly isotropic flag curvature nor locally projectively flat ([2]). So far, the problem of classifying Randers metrics of scalar flag curvature still remains open.

A square metric is defined in the form $F = (\alpha + \beta)^2/\alpha$ with $\|\beta\|_\alpha < 1$. In [10], Shen-Yildirim determine the local structure of locally projectively flat square

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metrics of constant flag curvature. L. Zhou shows that a square metric of constant flag curvature must be locally projectively flat ([16]). Later on, we further prove that a square metric in dimension $n \geq 3$ is of scalar flag curvature if and only if it is locally projectively flat ([9]).

Let $F = \alpha\phi(\beta/\alpha)$ be a (regular) (α, β)-metric (see its regular condition in Section 2). Two (α, β)-metrics F and $\widetilde{F}$ are called of the same metric type if

$$F = \alpha\phi(\beta/\alpha), \ \widetilde{F} = \widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha}) : \quad \widetilde{\alpha} = \sqrt{\alpha^2 + \epsilon\beta^2}, \ \widetilde{\beta} = k\beta,$$

where ϵ, k are constant. In this paper, we consider a class of (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ with $\phi(s)$ being defined by

(1.1)
$$\big\{1 + (k_1 + k_3)s^2 + k_2 s^4\big\}\phi''(s) = (k_1 + k_2 s^2)\big\{\phi(s) - s\phi'(s)\big\}, \quad \big(\phi(0) = 1, \ k_2 \neq k_1 k_3\big),$$

where k_1, k_2, k_3 are constant. If $k_2 = k_1 k_3$, then F is of Randers metric type. The ODE (1.1) appears in characterizing an (α, β)-metric which is Douglasian or locally projectively flat ([4] [8] [13]). An important special metric type of (1.1), called general square metric type, is

$$F = \alpha + \epsilon\beta \pm \frac{\beta^2}{\alpha}, \quad (\epsilon = constant). \tag{1.2}$$

If $\epsilon = 2$ in (1.2), then $F = (\alpha + \beta)^2/\alpha$ is a square metric.

Theorem 1.1. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an $n(\geq 3)$-dimensional manifold M, where $\phi(s)$ satisfies (1.1). Assume β is closed if F is not of the metric type (1.2). Then F is of scalar flag curvature if and only if F is locally projectively flat.*

Theorem 1.1 generalized a known result proved in [9] for square metrics. In Theorem 1.1, the flag curvature K can be determined (Theorem 4.1 below). Theorem 1.1 might hold without the condition that β is closed when F is not of the metric type (1.2), but we have not found a way to prove the general case. Possibly it even might be true that β is closed for any (α, β)-metric (not of Randers type) of scalar flag curvature in dimension $n \geq 3$.

After proving Theorem 1.1 in Section 3, we further characterize locally projectively flat (α, β)-metrics determined by (1.1) in dimension $n \geq 3$ in terns of the covariant derivatives $b_{i|j}$ and the Riemann curvature $\bar{R}^i{}_k$ of α (Theorem 4.1 below). This characterization is different from that given in [8]. In the final section, we add an appendix to show an application of Theorem 4.1 in two aspects: the local structure of locally projectively flat (α, β)-metrics (see Section 5.1 below) (cf. [8] [15]), and the classification for (α, β)-metrics which are locally projectively flat with constant flag curvature (see Section 5.2 below) (cf. [5] [14]).

2 Preliminaries

For a Finsler metric F, the Riemann curvature $R_y = R^i{}_k(y)\frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}, \tag{2.1}$$

where the spray coefficients G^i are given by

$$G^i := \frac{1}{4} g^{il}\big\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\big\}. \tag{2.2}$$

The Ricci curvature Ric is the trace of the Riemann curvature, that is, $Ric := R^m_{\ m}$. A Finsler metric is said to be of scalar flag curvature if there is a function $K = K(x, y)$ such that

$$R^i_{\ k} = KF^2(\delta^i_k - F^{-2} y^i y_k), \quad y_k := (F^2/2)_{y^i y^k} y^i. \tag{2.3}$$

If K is a constant, F is said to be of constant flag curvature. A Finsler metric F is said to be projectively flat in U, if there is a local coordinate system (U, x^i) such that $G^i = Py^i$, where $P = P(x, y)$ is called the projective factor satisfying $P(x, \lambda y) = \lambda P(x, y)$ for $\lambda > 0$.

The Weyl curvature $W^i_{\ k}$ and the Douglas curvature $D_h{}^i{}_{jk}$ are two important projectively invariant tensors which are defined respectively by

$$W^i_{\ k} := R^i_{\ k} - \frac{R^m_{\ m}}{n-1}\delta^i_k - \frac{1}{n+1}\frac{\partial}{\partial y^m}\big(R^m_{\ k} - \frac{R^h_{\ h}}{n-1}\delta^m_k\big) y^i, \tag{2.4}$$
$$D_h{}^i{}_{jk} := \frac{\partial^3}{\partial y^h \partial y^j \partial y^k}\big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\big).$$

A Finsler metric is called a Douglas metric if $D_h{}^i{}_{jk} = 0$. A Finsler metric is of scalar flag curvature if and only if $W^i_{\ k} = 0$ ([12]). An $n(\geq 3)$-dimensional Finsler metric is locally projectively flat if and only if: $W^i_{\ k} = 0$ and $D_h{}^i{}_{jk} = 0$ ([6]).

For a Riemannian $\alpha = \sqrt{a_{ij} y^i y^j}$ and a 1-form $\beta = b_i y^i$, let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r^i_{\ j} := a^{ik} r_{kj}, \quad s^i_{\ j} := a^{ik} s_{kj},$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad t_{ij} := s_{im} s^m_{\ j}, \quad t_j := b^i t_{ij},$$

where we define $b^i := a^{ij} b_j$, (a^{ij}) is the inverse of (a_{ij}), and $\nabla\beta = b_{i|j} y^i dx^j$ denotes the covariant derivatives of β with respect to α. Here are some of our conventions in the whole paper. For a general tensor T_{ij} as an example, we define $T_{i0} := T_{ij} y^j$ and $T_{00} := T_{ij} y^i y^j$, etc. We use a_{ij} to raise or lower the indices of a tensor.

An (α, β)-metric is a Finsler metric defined by a Riemann metric $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and a 1-form $\beta = b_i(x) y^i$ as follows:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s) > 0$ is a C^∞ function on $(-b_o, b_o)$. It is proved in [7] that an (α, β)-metric is regular (positively definite on $TM - 0$) if and only if

$$\phi(s) - s\phi'(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad \big(|\beta/\alpha| = |s| \leq b < b_o\big), \tag{2.5}$$

where b is defined by $b := ||\beta||_\alpha$. By (2.2), the spray coefficients G^i of an (α, β)-metric F are given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \alpha^{-1}\Theta(-2\alpha Q s_0 + r_{00}) y^i + \Psi(-2\alpha Q s_0 + r_{00}) b^i, \tag{2.6}$$

where G^i_α denote the spray coefficents of α and

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{Q - sQ'}{2\Delta}, \quad \Psi := \frac{Q'}{2\Delta}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q'.$$

For an (α, β)-metric, we can use (2.6), (2.1) and (2.4) to get the expression of the Weyl curvature ${W^i}_k$. We have given a Maple program in [9] to compute the Weyl curvature for any (α, β)-metric. However, the expression of ${W^i}_k$ is very lengthy (cf. [3]). So for the briefness, we will not write out the whole expression of ${W^i}_k$ in this paper, but some key terms will be given.

3 Proof of Theorem 1.1

The following Lemma is already known.

Lemma 3.1. *([4]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $n(\geq 3)$-dimensional (α, β)-metric, where $\phi = \phi(s)$ satisfies the ODE (1.1). Then F is a Douglas metric if and only if β satisfies*

$$b_{i|j} = \tau\big\{(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\big\}, \tag{3.1}$$

where $\tau = \tau(x)$ is a scalar function.

In this section, we will show that, in dimension $n \geq 3$, if the metric F in Theorem 1.1 is of scalar flag curvature in two cases, then F satisfies (3.1) and it must be of Douglas type. Thus F is locally projectively flat.

3.1 The case of β being closed

In this subsection, we assume β is closed for the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ in Theorem 1.1. We will show (3.1) holds.

Lemma 3.2. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric, where $\phi(s)$ is given by (1.1). Then we have*

$$1 + k_1 b^2 > 0, \quad 1 + (k_1 + k_3)b^2 + k_2 b^4 > 0. \tag{3.2}$$

Proof. By the ODE (1.1), we have $\phi - s\phi' = exp\big(-\frac{1}{2}\int_0^{s^2} \frac{k_1 + k_2\theta}{1 + (k_1 + k_3)\theta + k_2\theta^2}\big) > 0$, and

$$\phi - s\phi' + (b^2 - s^2)\phi'' = (\phi - s\phi') \cdot \frac{1 + k_1 b^2 + (k_3 + k_2 b^2)s^2}{1 + (k_1 + k_3)s^2 + k_2 s^4}.$$

Since the above expression is positive for $|s| \leq b$, we easily obtain (3.2). □

Using the condition that β is closed ($s_{ij} = 0$) and multiplying ${W^i}_k = 0$ by

$$4(n^2 - 1)\big[\phi - s\phi' + (b^2 - s^2)\phi''\big]^5 \alpha^4$$

and we get an equation denoted by $Eq_0 = 0$. By the ODE (1.1) we can get $\phi^{(i)}$ $(2 \leq i \leq 5)$ expressed by ϕ, ϕ'. Plug them into $Eq_0 = 0$ and then multiply $Eq_0 = 0$ by $\big[1 + (k_1 + k_3)s^2 + k_2 s^4\big]^5$. By this way, we have

$$4(\phi - s\phi')^5 Eq_1 = 0.$$

It is surprising that Eq_1 is independent of ϕ and by $Eq_1 = 0$ we have

$$\begin{aligned} 0 &= 24(n-2)(k_2 - k_1k_3)^3(\alpha^2 b_k - \beta y_k)y^i\beta^3\big[\alpha^4 + (k_1+k_3)\alpha^2\beta^2 + k_2\beta^4\big]^2 r_{00}^2 \\ &\quad + C^i_k\big[(1+k_1b^2)\alpha^2 + (k_2b^2+k_3)\beta^2\big], \end{aligned} \tag{3.3}$$

where C^i_k are homogeneous polynomials in (y^i).

Lemma 3.3. *For some k, $\alpha^2 b_k - \beta y_k$ cannot be divisible by $(1+k_1b^2)\alpha^2 + (k_2b^2+k_3)\beta^2$.*

Proof. Otherwise, for some scalar functions $f_k = f_k(x)$ we have

$$\alpha^2 b_k - \beta y_k = f_k\big[(1+k_1b^2)\alpha^2 + (k_2b^2+k_3)\beta^2\big].$$

Then we have

$$b^2\alpha^2 - \beta^2 = f\big[(1+k_1b^2)\alpha^2 + (k_2b^2+k_3)\beta^2\big], \quad f := b^k f_k,$$

which implies $1 + k_1b^2 + (k_2b^2+k_3)b^2 = 0$. This is a contradiction by Lemma 3.2. □

Lemma 3.4. *$\alpha^4 + (k_1+k_3)\alpha^2\beta^2 + k_2\beta^4$ cannot be divisible by $(1+k_1b^2)\alpha^2 + (k_2b^2 + k_3)\beta^2$, provided that $k_2 \neq k_1k_3$.*

Proof. We can prove it in two cases: $k_2b^2 + k_3 = 0$ and $k_2b^2 + k_3 \neq 0$. We need Lemma 3.2 and the fact $k_2 \neq k_1k_3$. The details are omitted. □

By (3.3), Lemma 3.3 and Lemma 3.4 we have (3.1) for some scalar function $\tau = \tau(x)$. Thus F in Theorem 1.1 is a Douglas metric by Lemma 3.1.

3.2 The case of F being metric of type (1.2)

In this section, we will prove Theorem 1.1 when F is of the metric type (1.2). In the following discussion, we put $F = \alpha + \epsilon\beta + \beta^2/\alpha$. The proof for the case $F = \alpha + \epsilon\beta - \beta^2/\alpha$ is similar.

To complete the proof of Theorem 1.1, we only need to show that β is closed when F is of scalar flag curvature in $n \geq 3$. Then Theorem 1.1 follows from the result in Subsection 3.1.

Lemma 3.5. *β is closed $\Longleftrightarrow t_{ij} = 0 \Longleftrightarrow t^k{}_k = 0$.*

Now we begin our discussion. We will prove our results in two cases: $\epsilon \neq 0$ and $\epsilon = 0$. The method used in the following proof is similar to the idea in [9] for the consideration of square metrics. Multiplying $W^i{}_k = 0$ by $(n^2-1)\alpha^{18}(\alpha^2-\beta^2)^4\big[(1 + 2b^2)\alpha^2 - 3\beta^2\big]^5$ gives an equation in the following form

$$H + \alpha P = 0,$$

where H, P are homogeneous polynomials in (y^i). This is equivalent to

$$H = 0, \quad P = 0, \quad \Big(H = \sum_{i=0}^{10} A_i \alpha^{2i}, \quad P = \sum_{i=0}^{9} B_i \alpha^{2i} \Big), \tag{3.4}$$

where A_i, B_i are homogeneous polynomials in (y^i).

Case I: Assume $\epsilon \neq 0$. We shall first show the following

Lemma 3.6. *If $H = 0, P = 0$, then $s_0 = 0$.*

Proof. The equation $P = 0$ can be written as

$$(\cdots)\big[(1+2b^2)\alpha^2 - 3\beta^2\big] + 2592(n-2)\epsilon\beta^3(\alpha^2-\beta^2)^4(\alpha^2 b_k - \beta y_k)y^i s_0 \widetilde{P} = 0, \tag{3.5}$$

where

$$\widetilde{P} := (\beta^2 - \alpha^2)r_{00} + 4\alpha^2\beta s_0, \tag{3.6}$$

and the equation $\alpha^2 H = 0$ can be written as

$$(\cdots)\big[(1+2b^2)\alpha^2 - 3\beta^2\big] + 648(n-2)\beta^3(\alpha^2-\beta^2)^4(\alpha^2 b_k - \beta y_k)y^i \widetilde{H} = 0, \tag{3.7}$$

where

$$\widetilde{H} := 4\epsilon^2 s_0^2 \alpha^6 + (r_{00} - 4\beta s_0)^2\alpha^4 - 2\beta^2 r_{00}(r_{00} - 4\beta s_0)\alpha^2 + r_{00}^2\beta^4. \tag{3.8}$$

The omitted terms in the parentheses in (3.5) and (3.7) are homogeneous polynomials in (y^i). Since $(1+2b^2)\alpha^2 - 3\beta^2$ is irreducible ($b^2 < 1$), it is easy to see from (3.5) and (3.7) that both $\widetilde{P}$ (if $s_0 \neq 0$) and $\widetilde{H}$ are divisible by $(1+2b^2)\alpha^2 - 3\beta^2$.

Suppose $s_0 \neq 0$. Then it follows from (3.5) that $\widetilde{P}$ is divisible by $(1+2b^2)\alpha^2 - 3\beta^2$. Thus from (3.6), there is a homogeneous polynomial f in (y^i) of degree two satisfying

$$(\beta^2 - \alpha^2)r_{00} + 4\alpha^2\beta s_0 = f\big[(1+2b^2)\alpha^2 - 3\beta^2\big]. \tag{3.9}$$

It is clear that (3.9) can be rewritten as

$$(4\beta s_0 - r_{00} - f - 2b^2 f)\alpha^2 + (r_{00} + 3f)\beta^2 = 0. \tag{3.10}$$

It follows from (3.10) that $r_{00} + 3f = 2(b^2-1)\tau\alpha^2$ for some scalar function $\tau = \tau(x)$. Solving f from this and plugging it into (3.10) again yields

$$r_{00} = \tau\big[(1+2b^2)\alpha^2 - 3\beta^2\big] + \frac{6\beta s_0}{1-b^2}. \tag{3.11}$$

From (3.7), $\widetilde{H}$ is divisible by $(1+2b^2)\alpha^2 - 3\beta^2$. Then it follows from (3.8) that there is a homogeneous polynomial h in (y^i) of degree six such that

$$4\epsilon^2 s_0^2\alpha^6 + (r_{00} - 4\beta s_0)^2\alpha^4 - 2\beta^2 r_{00}(r_{00} - 4\beta s_0)\alpha^2 + r_{00}^2\beta^4 = h\big[(1+2b^2)\alpha^2 - 3\beta^2\big]. \tag{3.12}$$

Plugging (3.11) into (3.12) yields

$$(\cdots)\big[(1+2b^2)\alpha^2 - 3\beta^2\big] + 36\epsilon^2\alpha^6(\alpha^2-\beta^2)^2 s_0^2 = 0, \tag{3.13}$$

where the omitted term in the parenthesis above is a homogeneous polynomial in (y^i). It is easy to get a contradiction from (3.13) since $s_0 \neq 0$ by assumption. □

Lemma 3.7. *If $H = 0, P = 0$, then*

$$t_{00} = \gamma(\alpha^2 - \beta^2), \tag{3.14}$$

where $\gamma = \gamma(x)$ is a scalar function on M.

Proof. By Lemma 3.6, we have $s_0 = 0$. Now plug $s_0 = 0$ into $H = 0$, and then $H = 0$ can be written as

$$(\cdots)(\alpha^2 - \beta^2) + 384(n+1)(4+\epsilon^2)(1-b^2)^5\beta^{16}y^i(\beta b_k - y_k)t_{00} = 0,$$

where the omitted term in the parenthesis above is a homogeneous polynomial in (y^i). Now it is clear from the above equation that (3.14) holds for some scalar function $\gamma = \gamma(x)$. □

Lemma 3.8. *If $H = 0, P = 0$, then β is closed.*

Proof. By (3.14) we have

$$t_{i0} = \gamma(y_i - \beta b_i), \quad t_{k0} = \gamma(y_i - \beta b_i), \quad t_0 = 0, \quad t^m_{\ m} = \gamma(n - b^2). \tag{3.15}$$

Plugging (3.14), (3.15) and $s_0 = 0$ into $H/(\alpha^2 - \beta^2) = 0$ yields

$$\begin{aligned} 0 \;=\; & (\cdots)(\alpha^2 - \beta^2) + 64(n+1)(4+\epsilon^2)(1-b^2)^5\beta^{16} \times \\ & \big[\gamma(y_k - \beta b_k)(n\beta b_i - \beta b_i - 3y_i - b^2 y_i) - 3(n-1)s_{i0}s_{k0}\big], \end{aligned} \tag{3.16}$$

where the omitted term in the parenthesis above is a homogeneous polynomial in (y^i). Then it follows from (3.16) that there are scalar functions $\sigma_{ik} = \sigma_{ik}(x)$ such that

$$\gamma(y_k - \beta b_k)(n\beta b_i - \beta b_i - 3y_i - b^2 y_i) - 3(n-1)s_{i0}s_{k0} = \sigma_{ik}(\alpha^2 - \beta^2). \tag{3.17}$$

It has been prove in [9] that β is closed by (3.17). Here we also show it. Exchanging the indices i and k in (3.17), we have

$$\gamma(y_i - \beta b_i)(n\beta b_k - \beta b_k - 3y_k - b^2 y_k) - 3(n-1)s_{i0}s_{k0} = \sigma_{ki}(\alpha^2 - \beta^2). \tag{3.18}$$

Then (3.17) – (3.18) gives

$$\gamma(n - 4 - b^2)\beta(b_k y_i - b_i y_k) = (\sigma_{ik} - \sigma_{ki})(\alpha^2 - \beta^2),$$

which implies that $\gamma = 0$ since $n - 4 - b^2 \neq 0$ ($b^2 < 1$). Now since $\gamma = 0$, (3.17) reduces to

$$-3(n-1)s_{i0}s_{k0} = \sigma_{ik}(\alpha^2 - \beta^2). \tag{3.19}$$

It is clear from (3.19) that $s_{i0} = 0$ and thus β is closed. □

Case II: Assume $\epsilon = 0$. In this case, there are some different steps from that in Case I.

In (3.4), we have $P = 0$ and $A_{10} = 0$ for H. First we have the following lemma.

Lemma 3.9. *If $H = 0$, then*

$$t_{00} = \gamma(\alpha^2 - \beta^2) + \frac{s_0^2}{1 - b^2}, \tag{3.20}$$

where $\gamma = \gamma(x)$ is a scalar function.

Proof. Rewrite $H = 0$ in the following form

$$(\cdots)(\alpha^2 - \beta^2) + 1536(n+1)(1-b^2)^4\beta^{16}\big[(1-b^2)t_{00} - s_0^2\big]y^i(b_k\beta - y_k) = 0,$$

where the omitted term in the parenthesis above is a homogeneous polynomial in (y^i). It is clear that the above equation shows that $\alpha^2 - \beta^2$ is divisible by $(1-b^2)t_{00} - s_0^2$. This fact implies that (3.20) holds for some scalar function $\gamma = \gamma(x)$. □

Lemma 3.10. *If $H = 0$, then β is closed.*

Proof. By Lemma 3.9 that (3.20) holds. Then it follows from (3.20) that

$$t^i{}_k = \gamma(\delta^i_k - b^i b_k) + \frac{6s^i s_k}{1-b^2}, \quad t_{k0} = \gamma(y_k - \beta b_k) + \frac{s_k s_0}{1-b^2}, \quad t^i{}_0 = \gamma(y^i - \beta b^i) + \frac{s^i s_0}{1-b^2},$$
$$t^m{}_m = \gamma(n - 2b^2), \quad t_k = (1-b^2)\gamma b_k, \quad t_0 = (1-b^2)\gamma\beta \quad s_m s^m = -b^2(1-b^2)\gamma.$$

Plugging the above formula and (3.20) into $H \cdot (1-b^2)/(\alpha^2 - \beta^2) = 0$ gives

$$\widetilde{A}^i_k(\alpha^2 - \beta^2) + 24(1-b^2)\beta^4 B^i_k = 0, \tag{3.21}$$

where $\widetilde{A}^i_k$ and B^i_k are homogeneous polynomials in (y^i), and B^i_k are in the following form

$$B^i_k = (n^2-1)(1-b^2)\beta\big[(1-b^2)\beta s_{k0} - 3s_0(y_k - \beta b_k)\big]s^i{}_0 + (n-1)(1-b^2)\beta s_0\big[(n+1)\beta b^i$$
$$-3y^i\big]s_{k0} + 3(y_k - \beta b_k)\Big\{\big[2(1-b^2)^2\gamma\beta^2 + (3n-5)s_0^2\big]y^i - (n^2-1)\beta s_0^2 b^i\Big\}.$$

By (3.21), there are polynomials A^i_k such that

$$B^i_k = A^i_k(\alpha^2 - \beta^2). \tag{3.22}$$

Contracting (3.22) by $b_i b^k$ we obtain

$$X\alpha^2 - \beta^2\big[6(1-b^2)^3\gamma\beta^2 - 2(n^2-3n+5)(1-b^2)s_0^2 + X\big] = 0, \quad (X := A^i_k b_i b^k), \tag{3.23}$$

By (3.23), there is a scalar function $\xi = \xi(x)$ such that $X = \xi\beta^2$, and then plugging it into (3.23) yields

$$\xi(\alpha^2 - \beta^2) - 6(1-b^2)^3\gamma\beta^2 + 2(n^2-3n+5)(1-b^2)s_0^2 = 0, \tag{3.24}$$
$$\xi(a_{ij} - b_i b_j) - 6(1-b^2)^3\gamma b_i b_j + 2(n^2-3n+5)(1-b^2)s_i s_j = 0. \tag{3.25}$$

Contracting (3.25) by a^{ij} and using the expression of $s_m s^m$ implied by (3.20) we obtain

$$\xi(n - b^2) - 2b^2(1-b^2)^2(n^2 - 3n + 8 - 3b^2)\gamma = 0. \tag{3.26}$$

Contracting (3.25) by b^ib^j gives

$$\xi = 6b^2(1-b^2)^2\gamma. \tag{3.27}$$

Substitute (3.27) into (3.26) and we have

$$2(n-2)(n-4)b^2(1-b^2)^2\gamma = 0. \tag{3.28}$$

(1). If $n \neq 4$, then by (3.28) we have $\gamma = 0$. In this case, by $\gamma = 0$ and (3.24) we have $s_0 = 0$ and then by (3.20) we get $t_{00} = 0$. Therefore, it shows β is closed by Lemma 3.5.

(2). If $n = 4$, then plugging $n = 4$ and (3.27) into (3.24) shows

$$(1-b^2)\gamma(b^2\alpha^2 - \beta^2) + 3s_0^2 = 0. \tag{3.29}$$

Since $n \geq 3$, clearly by (3.29) we have $s_0 = 0, \gamma = 0$, and then again by (3.20) we get $t_{00} = 0$. Therefore, β is closed by Lemma 3.5. □

4 Characterizations for locally projective flatness

Let $\phi(s)$ satisfies (1.1). It is shown in [8] that an $n(\geq 3)$-dimensional (α,β)-metrics $F = \alpha\phi(\beta/\alpha)$ is locally projectively flat iff.

$$\begin{aligned} b_{i|j} &= \tau\big\{(1+k_1b^2)a_{ij} + (k_2b^2+k_3)b_ib_j\big\}, \\ G^i_\alpha &= \theta y^i - \tau(k_1\alpha^2 + k_2\beta^2)b^i. \end{aligned}$$

where G^i_α are the spray coefficients of α, θ is a 1-form and $\tau = \tau(x)$ is a scalar function.

In the above characterization, G^i_α hold in a special coordinate system. On the other hand, locally projectively flat Finsler metrics can be also characterized by projective quantities $W^i{}_k = 0$ and $D_j{}^i{}_{kl} = 0$. Basing on this, we have the following different characterization theorem.

Theorem 4.1. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric on an $n(\geq 3)$-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is Riemannian and $\beta = b_i(x)y^i$ is a 1-form, and $\phi(s)$ satisfies (1.1). Then F is locally projectively flat iff. the Riemann curvature $\bar{R}^i{}_k$ of α and the covariant derivatives $b_{i|j}$ of β with respect to α satisfy the following equations*

$$b_{i|j} = \tau\big\{(1+k_1b^2)a_{ij} + (k_2b^2+k_3)b_ib_j\big\}, \tag{4.1}$$

$$\bar{R}^i{}_k = \lambda(\alpha^2\delta^i_k - y^iy_k) + \eta\big(\beta^2\delta^i_k + \alpha^2b^ib_k - \beta b^iy_k - \beta b_ky^i\big), \tag{4.2}$$

$$\tau_{x^i} = qb_i, \tag{4.3}$$

where $\lambda = \lambda(x), \tau = \tau(x)$ are scalar functions on M and η, q are defined by

$$\eta := \big\{k_1^2 + k_2 - 2k_1k_3 - k_1(k_2 - k_1^2)b^2\big\}\tau^2 + k_1\lambda, \quad q := (k_3 - 2k_1 - k_1^2b^2)\tau^2 - \lambda. \tag{4.4}$$

In this case, the flag curvature K is given by

$$\begin{aligned} 32\phi^2K &= f\phi'\phi^{-1}\Big\{\big[24f\phi^{-1}\phi' + (3-3f+hs^2)^2s^{-3}b^2 - 16hs\big]\tau^2 + 16\lambda s\Big\} \\ &\quad + \big[8(2ghs^2 + 12f - 3g^2) - g(3+hs^2-3f)^2s^{-2}b^2\big]s^{-2}\tau^2 - 16\lambda g, \end{aligned} \tag{4.5}$$

where f, g, h are defined by

$$f := 1 + (k_1 + k_3)s^2 + k_2 s^4, \quad g := k_2 s^4 - k_1 s^2 - 2, \quad h := 3k_2 s^2 - k_1 + 3k_3. \tag{4.6}$$

Proof. Let $F = \alpha\phi(\beta/\alpha)$ be locally projectively flat in dimension $n \geq 3$, where $\phi(s)$ satisfies (1.1). Then the Weyl curvature $W^i{}_k$ vanish. It is shown in [8] that (4.1) holds. Now by (4.1) we can obtain the expressions of these quantities:

$$r_{00},\ r_i,\ r^m_m,\ r,\ r_{00|0},\ r_{0|0},\ r_{00|k},\ r_{k0|m},\ r_{k|0},\ etc.$$

For example, we have

$$r_{0|0} = \big[1 + (k_1 + k_3)b^2 + k_2 b^4\big]\Big\{\big[(1 + k_1 b^2)\alpha^2 + (2k_1 + 3k_3 + 5k_2 b^2)\beta^2\big]\tau^2 + \tau_0\beta\Big\}.$$

Define $\bar{W}_{ik} := a_{im}\bar{W}^m_k$, where $\bar{W}^m_k$ are the Weyl curvature of α. Now plug all the above quantities into (3.3) and then we can get $\bar{W}_{ik}$. We will discuss it under two cases.

Case I: Assume $k_1 \neq 0$. We have

$$\begin{aligned} \bar{W}_{ik} \ &= \ \frac{k_1}{n-1} b^m\omega_m(\alpha^2 a_{ik} - y_i y_k) - \frac{(k_1 + k_2 b^2)\omega_0 - k_2 b^m\omega_m\beta}{n-1}\beta a_{ik} \\ &\quad + \frac{1}{n-1}\Big\{\frac{k_1 + k_2 b^2}{n+1}\big[(2n-1)\beta\omega_k - (n-2)\omega_0 b_k\big] - k_2 b^m\omega_m\beta b_k\Big\} y_i \\ &\quad + \omega_0 b_i(k_1 y_k + k_2\beta b_k) - (k_1\alpha^2 + k_2\beta^2) b_i\omega_k, \end{aligned} \tag{4.7}$$

where $\tau_i := \tau_{x^i}$ and

$$\omega_i := \tau_i + \Big(k_1 + k_3 + k_2 b^2 - \frac{k_2}{k_1}\Big)\tau^2 b_i. \tag{4.8}$$

Lemma 4.2. *(4.7) $\Longleftrightarrow$ (4.2) and (4.3), where q is defined by*

$$q := -\frac{\eta}{k_1} - \big(k_1 + k_3 + k_2 b^2 - \frac{k_2}{k_1}\big)\tau^2. \tag{4.9}$$

Proof. $\Longrightarrow$ **:** By the definition of the Weyl curvature $\bar{W}_{ik}$ of α we have

$$\bar{W}_{ik} = \bar{R}_{ik} - \frac{1}{n-1}\bar{R}ic_{00} a_{ik} + \frac{1}{n-1}\bar{R}ic_{k0} y_i, \tag{4.10}$$

where $\bar{R}_{ik} := a_{im}\bar{R}^m_k$ and $\bar{R}ic_{ik}$ denote the Ricci tensor of α. Using the fact $\bar{R}_{ik} = \bar{R}_{ki}$ we get from (4.10)

$$\bar{W}_{ik} - \bar{W}_{ki} = \frac{1}{n-1}\big(\bar{R}ic_{k0} y_i - \bar{R}ic_{i0} y_k\big). \tag{4.11}$$

By (4.7) we can get another expression of $\bar{W}_{ik} - \bar{W}_{ki}$. Thus by (4.7) and (4.11) we have

$$T_i y_k - T_k y_i + (n^2 - 1)(k_1\alpha^2 + k_2\beta^2)(\omega_i b_k - \omega_k b_i) = 0, \tag{4.12}$$

where we define

$$T_i: = (n+1)\bar{R}ic_{i0} - (2n-1)(k_1+k_2b^2)\beta\omega_i + \big\{[(n^2+n-3)k_1 + (n-2)k_2b^2]\omega_0 + (n+1)k_2b^m\omega_m\beta\big\}b_i.$$

Contracting (4.12) by y^k we get

$$\big[T_i + (n^2-1)k_1(\omega_i\beta - \omega_0 b_i)\big]\alpha^2 - T_0 y_i + (n^2-1)k_2\beta^2(\omega_i\beta - \omega_0 b_i) = 0. \tag{4.13}$$

Contracting (4.13) by b^i we obtain

$$\big[b^m T_m + (n^2-1)k_1(b^m\omega_m\beta - b^2\omega_0)\big]\alpha^2 + \big[(n^2-1)k_2\beta(b^m\omega_m\beta - b^2\omega_0) - T_0\big]\beta = 0. \tag{4.14}$$

So by (4.14) there is some scalar function $\bar{\eta} = \bar{\eta}(x)$ such that

$$T_0 = (n^2-1)k_2\beta(b^m\omega_m\beta - b^2\omega_0) + (n+1)\bar{\eta}\alpha^2. \tag{4.15}$$

Then by the definition of T_i and (4.15) we have

$$\bar{R}ic_{00} = \bar{\eta}\alpha^2 - (n-2)\big[(k_1+k_2b^2)\omega_0 - k_2b^m\omega_m\beta\big]\beta, \tag{4.16}$$

$$\bar{R}ic_{i0} = \bar{\eta}y_i - (n-2)\Big\{\frac{k_1+k_2b^2}{2}(\beta\omega_i + b_i\omega_0) - k_2b^m\omega_m\beta b_i\Big\}. \tag{4.17}$$

Now plugging (4.16) and (4.17) into (4.13) yields

$$2(n+1)k_2A_i\beta + B_i\alpha^2 = 0, \tag{4.18}$$

where A_i and B_i are defined by

$$A_i: = (b^2\omega_0 - b^m\omega_m\beta)y_i + \beta^2\omega_i - \beta\omega_0 b_i,$$

$$B_i: = k_2\Big\{\big[2(n+1)b^m\omega_m\beta - (n-2)b^2\omega_0\big]b_i - (n+4)b^2\beta\omega_i\Big\} + (n-2)k_1(\beta\omega_i - \omega_0 b_i).$$

If $k_2 = 0$, then by (4.18) we have

$$\beta\omega_i - \omega_0 b_i = 0. \tag{4.19}$$

If $k_2 \neq 0$, then by (4.18) we have

$$A_i = f_i\alpha^2, \tag{4.20}$$

where $f_i = f_i(x)$ are scalar functions. Contracting (4.20) by y^i we get

$$f_i = b^2\omega_i - b^m\omega_m b_i \tag{4.21}$$

Plugging (4.20) and (4.21) into (4.18) gives

$$(n-2)(k_1+k_2b^2)(\beta\omega_i - \omega_0 b_i)\alpha^2 = 0. \tag{4.22}$$

If $\tau = 0$, we can naturally find λ, η, q such that (4.1)–(4.3) hold, since in this case, $\beta(\neq 0)$ is parallel and α is flat. So we may assume $\tau \neq 0$, and then by (4.1) and

Lemma 3.2, we have $b^2 \neq constant$. So by (4.22) we also get (4.19). Thus it follows from (4.19) that

$$\omega_i = eb_i, \tag{4.23}$$

for some scalar function $e = e(x)$. Now plugging (4.16), (4.17) and (4.23) into (4.7) and (4.10) we obtain

$$\bar{R}_{ik} = \lambda(\alpha^2 a_{ik} - y^i y_k) + \eta\big(\beta^2 a_{ik} + \alpha^2 b_i b_k - \beta b_i y_k - \beta b_k y_i\big), \tag{4.24}$$

where we define

$$\lambda := \frac{k_1 e b^2 + \bar{\eta}}{n-1}, \quad \eta := -k_1 e. \tag{4.25}$$

Clearly, (4.24) is just (4.2). It follows from (4.8), (4.23) and (4.25) that

$$\tau_i = q b_i = \Big\{ -\frac{\eta}{k_1} - \big(k_1 + k_3 + k_2 b^2 - \frac{k_2}{k_1}\big)\tau^2 \Big\} b_i,$$

which implies (4.3) with q given by (4.9).

$\Longleftarrow$ **:** We verify that both sides of (4.7) are equal. By (4.2) we have (4.16) and (4.17). Since (4.10) naturally holds, we plug (4.16), (4.17) and (4.2) into (4.10) and then we obtain the left side of (4.7). By (4.3) and (4.9) we get

$$\omega_i = eb_i = -\frac{\eta}{k_1} \tag{4.26}$$

from (4.8). Then plugging (4.26) into the right side of (4.7) we obtain the result equal to the left side of (4.7). □

In the final we compute the flag curvature and prove that η, q are given by (4.4). As shown above, we have (4.1)–(4.3) with q being given by (4.9) since F is locally projectively flat. Plug (1.1) and (4.1)–(4.3) and (4.9) into the Riemann curvature $R^i{}_k$ of F, and then a direct computation gives

$$\begin{aligned} R^i{}_k &= KF^2(\delta^i_k - F^{-1} y^i F_{y^k}) + \frac{\phi'(sF_{y^k} - \phi b_k)}{k_1 \phi^2(\phi - s\phi')} \times \\ & \quad \big[\eta - \big\{k_1^2 + k_2 - 2k_1 k_3 - k_1(k_2 - k_1^2) b^2\big\}\tau^2 - k_1\lambda\big] F y^i, \end{aligned} \tag{4.27}$$

where the expression of $K = K(x, y)$ is omitted. Since F is of scalar flag curvature and $n \geq 3$, by (4.27) we must have

$$\eta - \big\{k_1^2 + k_2 - 2k_1 k_3 - k_1(k_2 - k_1^2) b^2\big\}\tau^2 - k_1\lambda = 0. \tag{4.28}$$

Thus we get η given by (4.4). Plug η given by (4.4) into K, and then we obtain the flag curvature K given by (4.5). By (4.9) and η in (4.4), we get q given by (4.4).

Case II: Assume $k_1 = 0$. Then $k_2 \neq 0$ since $k_2 \neq k_1 k_3$. We have

$$\begin{aligned} \frac{\bar{W}_{ik}}{k_2} &= \frac{(\tau^2 + b^m \tau_m)\beta^2 - b^2(\beta\tau_0 + \tau^2\alpha^2)}{n-1}\delta_{ik} + (\tau_0 b_k - \beta\tau_k)\beta b_i + \tau^2(\alpha^2 b_k - \beta y_k) b_i \\ & + \frac{\big[(2n-1)\beta\tau_k - (n-2)\tau_0 b_k\big] b^2 + (n+1)\big[b^2\tau^2 y_k - (\tau^2 + b^m\tau_m)\beta b_k\big]}{n^2-1} y_i. \end{aligned} \tag{4.29}$$

Lemma 4.3. *(4.29) $\Longleftrightarrow$ (4.2) and (4.3), where $q = q(x)$ is some scalar function and $\eta = k_2\tau^2$.*

Proof. We only show the final results of the necessity. Assume that (4.29) holds. By similar steps as that in Lemma 4.2, we have

$$\tau_i = qb_i, \tag{4.30}$$

$$\bar{R}_{ik} = \lambda(\alpha^2 a_{ik} - y^i y_k) + k_2\tau^2\big(\beta^2 a_{ik} + \alpha^2 b_i b_k - \beta b_i y_k - \beta b_k y_i\big), \tag{4.31}$$

where $q = q(x), \lambda = \lambda(x)$ are scalar functions. By (4.31), η in (4.2) is given by $\eta = k_2\tau^2$, which is equal to the η in (4.4) with $k_1 = 0$. □

Now we compute the Riemann curvature $R^i{}_k$ of F in this case and prove that q are given by (4.4) with $k_1 = 0$. By (1.1) and (4.1)–(4.3) with $\eta = k_2\tau^2$, a direct computation gives

$$R^i{}_k = KF^2(\delta^i_k - F^{-1}y^i F_{y^k}) - \frac{\phi'(sF_{y^k} - \phi b_k)}{\phi^2(\phi - s\phi')}(\lambda + q - k_3\tau^2)Fy^i, \tag{4.32}$$

where the expression of $K = K(x, y)$ is omitted. Since F is of scalar flag curvature and $n \geq 3$, by (4.32) we must have

$$\lambda + q - k_3\tau^2 = 0. \tag{4.33}$$

Thus we get q given by (4.4) with $k_1 = 0$.

5 Appendix

In this appendix, as an application of Theorem 4.1, we would like to use the characterization (4.1)–(4.4) and the scalar flag curvature (4.5) to verify two known results in [15] and [5] respectively. One is about the local structure of locally projectively flat (α, β)-metrics and the other is about the classification on locally projectively flat (α, β)-metrics of constant flag curvature.

5.1 A deformation on (α, β)-metrics

In Theorem 4.1, (4.1)–(4.4) are necessary and sufficient conditions for an $n(\geq 3)$-dimensional (α, β)-metric satisfying (1.1) to be locally projectively flat. In [8], Shen gives another characterization, and then in [15], Yu finds a deformation to obtain the local structure based on Shen's result. In this section, we will give the local structure using (4.1)–(4.4) by a similar deformation.

Let $u = u(t), v = v(t), w = w(t)$ satisfy the following ODEs:

$$u' = \frac{v - k_1 u}{1 + (k_1 + k_3)t + k_2 t^2}, \tag{5.1}$$

$$v' = \frac{u(k_2 u - k_3 v - 2k_1 v) + 2v^2}{u[1 + (k_1 + k_3)t + k_2 t^2]}, \tag{5.2}$$

$$w' = \frac{w(3v - k_3 u - 2k_1 u)}{2u\big[1 + (k_1 + k_3)t + k_2 t^2\big]}. \tag{5.3}$$

Let α and β satisfy (4.1)–(4.4), and define a new Riemann metric $h = \sqrt{h_{ij}(x)y^iy^j}$ and a new 1-form $\rho = p_i(x)y^i$ by

$$h := \sqrt{u\alpha^2 + v\beta^2}, \quad \rho := w\beta, \tag{5.4}$$

where $u = u(b^2) \neq 0, v = v(b^2), w = w(b^2) \neq 0$ are determined by (5.1)–(5.3). We will show in the following that h is of constant sectional curvature and ρ is a closed 1-form which is conformal with respect to h.

By (4.1), a direct computation shows that the sprays G^i_h of h and G^i_α of α satisfy

$$G^i_h = G^i_\alpha + \tau\Big\{\frac{1}{2}(k_1\alpha^2 + k_2\beta^2)b^i - \frac{(k_1u - v)\beta}{u}y^i\Big\}, \tag{5.5}$$

By (4.1) and (5.5) we can directly get

$$p_{i|j} = \frac{w\tau}{u}h_{ij} \ (= -2ch_{ij}), \tag{5.6}$$

where the covariant derivatives are taken with respect to h. Now (5.6) implies that ρ is a closed conformal 1-form with respect to h.

By (5.5) and using (4.1)–(4.4), we obtain

$$\widetilde{R}^i{}_k = \frac{\lambda u + (k_1^2ub^2 + 2k_1u - v)\tau^2}{u^2}(h^2\delta^i_k - y^i\widetilde{y}_k), \tag{5.7}$$

where $\widetilde{R}^i{}_k$ are the Riemann curvatures of h and $\widetilde{y}_k := h_{km}y^m$. It follows from (5.7) that h is of constant sectional curvature. We put it as μ, and then we obtain

$$\lambda = \mu u - \frac{k_1u(2 + k_1b^2) - v}{u}\ \tau^2. \tag{5.8}$$

It is already known that the local solution can be determined for a conformal vector field on a Riemannian space of constant sectional curvature. In some local coordinate system we may put $h = h_\mu$ in the form

$$h_\mu = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1 + \mu|x|^2}, \tag{5.9}$$

and then by (5.6) and (5.9) we obtain the 1-form $\rho = p_iy^i$ given by (cf. [15])

$$p_i = \frac{(k - \mu\langle a, x\rangle)x^i + (1 + \mu|x|^2)a^i}{(1 + \mu|x|^2)^{\frac{3}{2}}}, \quad p^i = \sqrt{1 + \mu|x|^2}(kx^i + a^i). \tag{5.10}$$

where k is a constant and $a = (a^i)$ is a constant vector, and $p_i = h_{im}p^m$. By (5.10) we have

$$p^2 = ||\rho||^2_h = |a|^2 + \frac{k^2|x|^2 + 2k\langle a, x\rangle - \mu\langle a, x\rangle^2}{1 + \mu|x|^2}. \tag{5.11}$$

By (5.4) we have

$$p^2 = \frac{w^2b^2}{u + vb^2}, \quad (p := ||\rho||_h). \tag{5.12}$$

Thus we can get the local expression of b^2 from (5.11) and (5.12) for a given triple (u, v, w). Additionally, c and τ in (5.6) are given by

$$c = \frac{-k + \mu\langle a, x\rangle}{2\sqrt{1+\mu|x|^2}}, \quad \tau = -2c\,\frac{u}{w}. \tag{5.13}$$

If we choose a triple (u, v, w) determined by (5.1)–(5.3), then by the above discussion, we can obtain the local expressions of α and β by (5.4).

Remark 5.1. We can have different suitable choices of u, v, w satisfying (5.1)–(5.3). For a square metric $F = (\alpha+\beta)^2/\alpha$, the triple (u, v, w) can be chosen as ([9])

$$u = (1-b^2)^2, \quad v = 0, \quad w = \sqrt{1-b^2}.$$

For the general case in Theorem 4.1, we may choose the triple (u, v, w) as ([15])

$$u = e^{2\sigma}, \quad v = (k_1 + k_3 + k_2 b^2)u, \quad w = \sqrt{1 + (k_1+k_3)b^2 + k_2 b^4}\; e^{\sigma}, \tag{5.14}$$

where σ is defined by

$$2\sigma := \int_0^{b^2} \frac{k_2 t + k_3}{1 + (k_1+k_3)t + k_2 t^2} dt.$$

5.2 Constant flag curvature

In this section, we consider the classification in Theorem 4.1 when F is of constant flag curvature. The following corollary has been proved in [5] [14] in a different way.

Corollary 5.1. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an $n(\geq 2)$-dimensional manifold M, where $\phi(s)$ satisfies (1.1). Suppose F is locally projectively flat with constant flag curvature K and β is not parallel with respect to α. Then F must be in the following form*

$$F = \frac{(\sqrt{\alpha^2 + k\beta^2} + \epsilon\beta)^2}{\sqrt{\alpha^2 + k\beta^2}}, \tag{5.15}$$

where k and $\epsilon \neq 0$ are constant. In this case, we have $K = 0$ and $k = k_1 - \phi'(0)^2/2$.

Proof. Note that β is closed if F is locally projectively flat in $n \geq 3$ ([8]). We can use Theorem 4.1 to prove this corollary in $n \geq 3$, and for the case $n = 2$, see [14]. Since F is of constant flag curvature, K given by (4.5) is a constant. Put $\phi(s) = 1 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots$. Then by (1.1), we can express all a_i's ($i \geq 2$) in terms of k_1, k_2, k_3. Multiply (4.5) by ϕ^2 and we get an equation. Let p_i be the coefficients of s^i. Firstly, by $p_0 = 0$, we get

$$K = \lambda + \big(k_1^2 b^2 + k_1 + \frac{3}{4}a_1^2\big)\tau^2. \tag{5.16}$$

We show $a_1 \neq 0$. If $a_1 = 0$, then plugging $a_1 = 0$ and (5.16) into $p_2 = 0$ yields

$$12\tau^2(k_2 - k_1 k_3) = 0.$$

Since $k_2 \neq k_1 k_3$, we get $\tau = 0$ on the whole M. Thus by (4.1), β is parallel with respect to α. So $a_1 \neq 0$. Now substitute (5.16) into $p_1 = 0$ and then using $a_1 \neq 0$ we obtain

$$\lambda = -(k_1^2 b^2 + k_3 + 2a_1^2)\tau^2. \tag{5.17}$$

Next plugging (5.16) and (5.17) into $p_3 = 0$ and using $a_1 \neq 0$ and $\tau \neq 0$ we get

$$k_2 = -a_1^4 + \frac{3}{5}(k_1 - k_3)a_1^2 + \frac{1}{5}(k_1 k_3 + 2k_1^2 + 2k_3^2). \tag{5.18}$$

Then similarly, by (5.16)–(5.18) and $p_4 = 0$ we have

$$k_3 = k_1 - a_1^2,\ k_1 - \frac{5}{4}a_1^2,\ -k_1 + \frac{10}{3}a_1^2.$$

If $k_3 = -k_1 + 10a_1^2/3$, then plugging it and (5.16)–(5.18) into $p_5 = 0$ yields

$$k_1 = \frac{5}{12}a_1^2,\ \frac{13}{6}a_1^2,\ \frac{55}{24}a_1^2.$$

It can be easily verified that if

$$k_3 = k_1 - a_1^2, \quad \text{or} \quad k_3 = -k_1 + \frac{10}{3}a_1^2 \quad \text{and} \quad k_1 = \frac{5}{12}a_1^2,\ \frac{13}{6}a_1^2,$$

then we have $k_2 = k_1 k_3$. Therefore, we have

$$k_3 = k_1 - \frac{5}{4}a_1^2, \quad \text{or} \quad k_3 = -k_1 + \frac{10}{3}a_1^2 \quad \text{and} \quad k_1 = \frac{55}{24}a_1^2. \tag{5.19}$$

The second case in (5.19) is a special case of the first case in (5.19). So by (5.18) and (5.19), we have

$$k_2 = \frac{3}{8}a_1^4 - \frac{5}{4}k_1 a_1^2 + k_1^2, \quad k_3 = k_1 - \frac{5}{4}a_1^2. \tag{5.20}$$

Now by (5.16), (5.17) and (5.20) we get $K = 0$. Plug (5.20) into (1.1) and solving the ODE we obtain (5.15) with $k := k_1 - a_1^2/2,\ \epsilon := a_1/2$. □

Corollary 5.2. *Let $F = (\alpha+\beta)^2/\alpha$ be a non-Riemannian square metric on an $n(\geq 2)$-dimensional manifold M. Then F is of constant flag curvature iff. either α is flat and β is parallel with respect to α, or up to a scaling on F, α and β can be locally expressed as*

$$\alpha = \frac{(1+\langle a,x\rangle)^2}{1-|x|^2}\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{1-|x|^2}, \tag{5.21}$$

$$\beta = \pm\frac{(1+\langle a,x\rangle)^2}{1-|x|^2}\Big\{\frac{\langle a,y\rangle}{1+\langle a,x\rangle} + \frac{\langle x,y\rangle}{1-|x|^2}\Big\}, \tag{5.22}$$

where $a = (a^i) \in R^n$ is a constant vector. In this case, the constant flag curvature $K = 0$.

Proof. This corollary has been verified in [10] [16]. Based on Theorem 4.1, we can prove Corollary 5.1 in dimension $n \geq 3$ in a different way. Let F be of constant flag curvature. Then F is locally projectively flat by Theorem 1.1 (cf. [16]). For $F = \alpha(1+s)^2$, we can put $k_1 = 2, k_2 = 0, k_3 = -3$ in Section 5.1 and in the proof of Corollary 5.1. Now put $u = (1-b^2)^2,\ v = 0,\ w = \sqrt{1-b^2}$ as shown in Remark 5.1, and define h and ρ by (5.4). Now it follows from (5.8) and (5.17) that

$$\mu(1-b^2)^2 - 4(1+b^2)\tau^2 + (5+4b^2)\tau^2 = 0. \tag{5.23}$$

Plug (5.12), (5.11) and (5.13) into (5.23) and then we get

$$(1+\mu|x|^2)^3(k^2+\mu+\mu|a|^2) = 0. \tag{5.24}$$

Therefore by (5.24) we get

$$\mu = -\frac{k^2}{1+|a|^2}. \tag{5.25}$$

If $k = 0$, then $\mu = 0$ by (5.25). In this case, we easily see that α is flat and β is parallel. If $k \neq 0$, using (5.25), we plug (5.12), (5.9)–(5.11) into (5.4) and then put

$$k = \delta d, \quad a = \frac{\bar{a}}{d}, \quad 1+|a|^2 = \delta^2,$$

and next put

$$\delta = k, \quad d^2 = -\mu, \quad \bar{a} = a,$$

and finally we get

$$\begin{aligned} \alpha &= \frac{(k+\langle a,x\rangle)^2}{1+\mu|x|^2} h_\mu, \quad \mu < 0, \\ \beta &= \pm\frac{1}{\sqrt{-\mu}}\frac{(k+\langle a,x\rangle)^2}{1+\mu|x|^2}\Big\{\frac{\langle a,y\rangle}{k+\langle a,x\rangle} - \frac{\mu\langle x,y\rangle}{1+\mu|x|^2}\Big\}. \end{aligned}$$

Then by choosing $\bar{x}^i = \sqrt{-\mu}x^i$ and a scaling on F we obtain (5.21) and (5.22). $\square$

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