Ordered line and skew-fields in the Desargues affine plane

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Abstract. This paper introduces ordered skew fields that result from the construction of a skew field over an ordered line in a Desargues affine plane. A special case of a finite ordered skew field in the construction of a skew field over an ordered line in a Desargues affine plane in Euclidean space, is also considered. Two main results are given in this paper: (1) every skew field constructed over a skew field over an ordered line in a Desargues affine plane is an ordered skew field and (2) every finite skew field constructed over a skew field over an ordered line in a Desargues affine plane in $\mathbb{R}^2$ is a finite ordered skew field.

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Key words: Affine Pappus Condition; ordered line; ordered skew-field; Desargues affine plane.

1 Introduction

The foundations for the study of the connections between axiomatic geometry and algebraic structures were set forth by D. Hilbert [10], recently elaborated and extended in terms of the algebra of affine planes in, for example, [13], [5, Sec.IX.3, p.574]. E.Artin in [3] shows that any ordering of a plane geometry is equivalent to a weak ordering of its skew field. He shows that that any ordering of a Desargues plane with more than four points is (canonically) equivalent to an ordering of its field. In his paper on ordered geometries [24], P. Scherk considers the equivalence of an ordering of a Desarguesian affine plane with an ordering of its coordinatizing division ring. Considerable work on ordered plane geometries has been done (see, *e.g.*, J. Lipman [15], V.H. Keiser [12], H. Tecklenburg [27], H. S. M. Coxeter [6] and L. A. Thomas [28, 29]).

In this paper, we utilize a method that is naive and direct, without requiring the concept of coordinates. Our results are straightforward and constructive. For this reason we begin by giving a suitable definition for our search for lines in Desargues affine planes, based on the meaning of betweenness given by Hilbert [10, §3].

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In addition, we introduce ordered skew files that result from the construction of a skew field over an ordered line in a Desargues affine plane. A special case of a finite ordered skew field in the construction of a skew field over an ordered line in a Desargues affine plane in Euclidean space, is also considered.

Based on the works of E.Artin [3] and J. Lipman [15] on ordered skew fields, we prove that the skew field that the constructed over an ordered line in a Desargues affine plane is an ordered skew field. To prove this, we consider ordered Desargues affine plane based on the definition given by E.Artin [3], but in this case the ordered line is given a suitable definition without the use of *coordinates*.

Two main results are given in this paper, namely, every skew field constructed over a skew field over an ordered line in a Desargues affine plane is an ordered skew field 4.5. and every finite skew field constructed over a skew field over an ordered line in a Desargues affine plane in $\mathbb{R}^2$ is a finite ordered skew field 4.7.

2 Preliminaries

Let $\mathcal{P}$ be a nonempty space, $\mathcal{L}$ a nonempty subset of $\mathcal{P}$. The elements p of $\mathcal{P}$ are points and an element ℓ of $\mathcal{L}$ is a line. Collinear points on a line $\mathcal{L}$ are denoted by $[A, E, B]$, where E is between A and B. Given *distinct* points A, B, there is a unique line ℓ^{AB} such that A, B lie on ℓ^{AB} and we write $\ell^{AB} = A + B$ [34, p. 52]. An **affine space** is a vector space with the origin removed [4, §4.1, p. 391]. The geometric structure $(\mathcal{P}, \mathcal{L})$ is an **affine plane**, a subspace of an affine space, provided

1^o For each $\{P, Q\} \in \mathcal{P}$, there is exactly one $\ell \in \mathcal{L}$ such that $\{P, Q\} \in \ell$.

2^o For each $P \in \mathcal{P}, \ell \in \mathcal{L}, P \notin \ell$, there is exactly one $\ell' \in \mathcal{L}$ such that $P \in \ell'$ and $\ell \cap \ell' = \varnothing$ (Playfair Parallel Axiom [22]). Put another way, if $P \notin \ell$, then there is a unique line ℓ' on P missing ℓ [23].

3^o There is a 3-subset $\{P, Q, R\} \in \mathcal{P}$, which is not a subset of any ℓ in the plane. Put another way, there exist three non-collinear points $\mathcal{P}$ [23].

An affine plane is a projective plane in which one line has been distinguished [12]. For simplicity, our affine geometry is on the Euclidean plane $\mathbb{R}^2$ and incident lines ℓ, ℓ' are represented by $\ell \cap \ell' \neq \varnothing$ (intersection). A **0-plane** is a point, a **1-plane** a line containing a minimum of 2 collinear points and a **2-plane** is an affine plane containing a minimum of 4 points, no 3 of which are collinear. An affine geometry is a geometry defined over vector spaces V, field $\mathbb{F}$ (vectors are points and subspaces of V) and subsets $\mathcal{P}$ (points), $\mathcal{L}$ (lines) and Π (planes) [7, §7.5].

Desargues' Axiom, circa 1630 [13, §3.9, pp. 60-61] [25]. Let $A, B, C, A', B', C' \in \mathcal{P}$ and let pairwise distinct lines $\ell_k, \ell_l, \ell_m, \ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ such that

$$\ell_k \parallel \ell_l \parallel \ell_m \text{ and } \ell^{A} \parallel \ell^{A'} \text{ and } \ell^{C} \parallel \ell^{C'}.$$
$$A, B \in \ell^{AB}, A'B' \in \ell^{A'B'}, \text{ and } B, C \in \ell^{BC}, B'C' \in \ell^{B'C'}.$$
$$A \neq C, A' \neq C', \text{ and } \ell^{AB} \neq \ell_l, \ell^{BC} \neq \ell_l.$$

Then $\boldsymbol{\ell^{AC} \parallel \ell^{A'C'}}$.

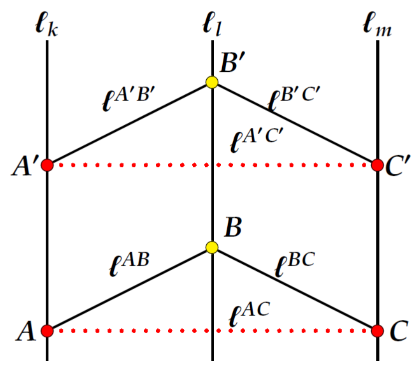

Figure 1: Desargues: $\boldsymbol{\ell^{AC}} \parallel \boldsymbol{\ell^{A'C'}}$

Example 2.1. The parallel lines $\ell^{AC}, \ell^{A'C'} \in \mathcal{L}$ in Desargues' Axiom are represented in Fig. 1. In other words, the base of $\triangle ABC$ is parallel with the base of $\triangle A'B'C'$, provided the restrictions on the points and lines in Desargues' Axiom are satisfied.

A **Desargues affine plane** is an affine plane that satisfies Desargues' Axiom.

Theorem 2.1. Pappus, circa 320 B.C. [5, §1.4, p. 18].
If $[ACE] \in \ell^{EA}$, $[BFD] \in \ell^{BD}$ *and* $\ell^{BD}, \ell^{CD}, \ell^{EF}$ *meet* $\ell^{DE}, \ell^{FA}, \ell^{BC}$, *then* $[NLM]$ *are collinear on* ℓ^{NM}.

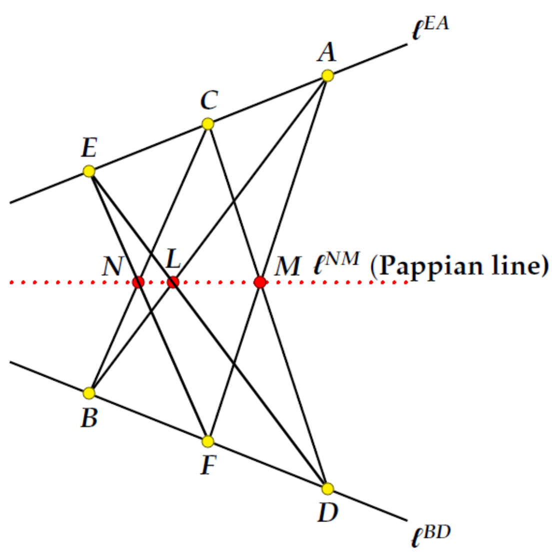

Figure 2: Pappian Line: $[\boldsymbol{NLM}] \in \boldsymbol{\ell^{NM}}$

Example 2.2. The lines $\ell^{BD}, \ell^{CD}, \ell^{EF}, \ell^{DE}, \ell^{FA}, \ell^{BC}$ in Pappus' Axiom are represented in Fig. 2. In that case, the points of intersection $[NLM]$ lie on the line ℓ^{NM}.

The affine Pappus condition in Theorem 2.1 has an effective formulation relative to $[NLM]$ on line ℓ^{NM} given by N.D. Lane [14], *i.e.*,

Affine Pappus Condition [14]. Let E, C, A, B, F, D be mutually distinct points as shown in Fig. 2 such that A, B, C lie on ℓ^{EA} and B, F, D lie on ℓ^{BD} and none of these points lie on $\ell^{EA} \cap \ell^{BD}, \ell^{BD} \cap \ell^{NM}$ or $\ell^{NM} \cap \ell^{EA}$. Then

$$\left.\begin{array}{l} \ell^{CB} \cap \ell^{EF} \text{ lies on } \ell^{NM} \\ \ell^{AF} \cap \ell^{CD} \text{ lies on } \ell^{NM} \end{array}\right\} \Rightarrow \ell^{AB} \cap \ell^{ED} \text{ lies on } \ell^{NM}.$$

This leads to the following result.

Theorem 2.2. Affine Pappus Condition [14]. *If the affine Pappus condition holds for all pairs of lines ℓ, ℓ' such that $\ell \not\parallel \ell'$, then the affine Pappus condition holds for all pairs ℓ, ℓ' with $\ell \parallel \ell'$.*

Every Desarguesian affine plane is isomorphic to a coordinate plane over a field [26] and every finite field is commutative [16, §3, p. 351]. From this, we obtain

Theorem 2.3. [**Tecklenburg**] [26].
Every finite Desarguesian affine plane is Pappian.

3 Ordered lines and ordered Desargues affine plane

An invariant way to describe an 'order' is by means of a ternary relation: the point B lies "between" A and C. Hilbert has axiomatized this ternary relation [10]. In this section, we begin by giving a suitable definition for our search for lines in Desargues affine plane, based on the meaning of *betweenness* given by D.Hilbert, *i.e.*, if B lies "between" A and C, we mark it with $[A, B, C]$.

Definition 3.1. An **ordered line** in a Desargues Affine plane (briefly, called the *line*) satisfies the following axioms.

Lo.1 For $A, B, C \in \ell$, $[A, B, C] \Longrightarrow [C, B, A]$.

Lo.2 For $A, B, C \in \ell$ are mutually distinct, then we have exactly one, $[A, B, C]$, $[B, C, A]$ or $[C, A, B]$.

Lo.3 For $A, B, C, D \in \ell$, then $[A, B, C]$ and $[B, C, D] \Longrightarrow [A, B, D]$ **and** $[A, C, D]$.

Lo.4 For $A, B, C, D \in \ell$, then $[A, B, C]$ and $[C, B, D] \Longrightarrow [D, A, B]$ **or** $[A, D, B]$.

Proposition 3.1. *For all $A, B, C, D \in \ell$, two, of $[B, A, C]$, $[C, A, D]$ and $[D, A, B]$ exclude the third.*

Proof. Let's get double combinations and see how third is excluded.

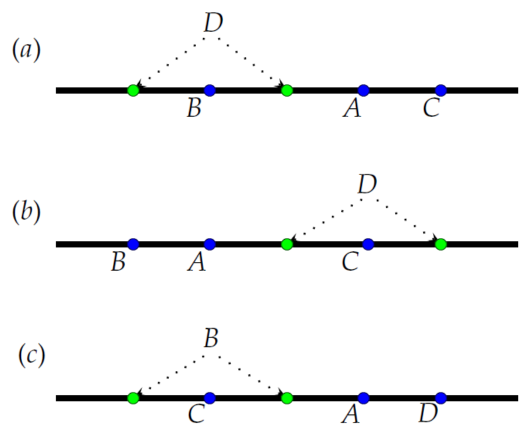

Figure 3: Possibilities of doubles combinations, of $[B, A, C]$, $[C, A, D]$ and $[D, A, B]$

(a) Suppose we are true $[B, A, C]$ and $[C, A, D]$, (see Fig. 3, (a)) then from the order axioms in Def. 3.1, we have:

$$\left.\begin{array}{l}[B, A, C]\\ [C, A, D]\end{array}\right\} \overset{\textbf{Lo.1}}{\Longrightarrow} \left.\begin{array}{l}[B, A, C]\\ [D, A, C]\end{array}\right\} \overset{\textbf{Lo.4}}{\Longrightarrow} [D, B, A] \vee [B, D, A].$$

From Axiom **Lo.2** loses the possibility that $[D, A, B]$ is true, so $[D, A, B]$ is false.

(b) Suppose we are true $[B, A, C]$ and $[D, A, B]$, (see Fig. 3, (b)) then from the axioms above we have:

$$\left.\begin{array}{l}[B, A, C]\\ [D, A, B]\end{array}\right\} \overset{\textbf{Lo.1}}{\Longrightarrow} \left.\begin{array}{l}[B, A, C]\\ [B, A, D]\end{array}\right\} \overset{\textbf{Lo.4}}{\Longrightarrow} [A, D, C] \vee [A, C, D].$$

From Axiom **Lo.2** loses the possibility that $[C, A, D]$ be true, so $[C, A, D]$ is false.

but each of them $[A, D, C] \vee [A, C, D]$ derives that $[C, A, D]$ is false.

(c) Suppose we are true $[B, A, C]$ and $[D, A, B]$, (see Fig. 3, (c)) then from the axioms above we have:

$$\left.\begin{array}{l}[C, A, D]\\ [D, A, B]\end{array}\right\} \overset{\textbf{Lo.4}}{\Longrightarrow} [B, C, A] \vee [C, A, B].$$

From Axiom **Lo.2** loses the possibility that $[B, A, C]$ be true, so $[B, A, C]$ is false.

□

Definition 3.2. For three different points $A, B, C \in \ell$, that, we say that points B and C lie on the same side of point A, if we have exactly one of $[A, B, C], [A, C, B]$.

Proposition 3.2. *For every four different points A, B, C, D in a line ℓ, in Desargues affine plane. Then*

$$\left.\begin{array}{l}[A,B,C]\\ [A,C,D]\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}[A,B,D]\\ [B,C,D]\end{array}\right.$$

Proof. By $[A, B, C]$, we have that the points A and B are in the same side of point C, by $[A, C, D]$, we have that the points A and D are on the opposite side of point C. Then, we have that, the point B and D are on the opposite side of point C (since otherwise we would have that points A, B and D would be on the same side of point C, this would exclude $[A, C, D]$), so we have that $[B, C, D]$.

By $[A, B, C]$, we have that the points A and C on the opposite side of point B. Since $[B, C, D]$, we have that the points C and D are in the same side of point B. Then, we have that, the point A and D are on opposite sides of the point B (since otherwise we would have that points A and D would be on the same side of point B, and point C this will give us $[D, B, C]$ this would exclude $[B, C, D]$), so we have that $[A, B, D]$. □

Definition 3.3. For three different points $A, B, C \in \ell$, that, we say that points B and C lie on the same side of point A, if we have exactly one of $[A, B, C], [A, C, B]$.

Proposition 3.3. *If we have, that, for four different points $A, B, C, D \in \ell$: points B and C lie on the same side of point A, and the points C and D lie on the same side of point A, then the points B and D lie on the same side of point A.*

Proof. Let's look at the possible cases of 'positioning' between the points,

If,

$$\left.\begin{array}{l}[A,D,C]\\ [B,C,A]\end{array}\right\} \overset{\textbf{Lo.1}}{\Longrightarrow} \left.\begin{array}{l}[A,D,C]\\ [A,C,B]\end{array}\right\} \overset{3.2}{\Longrightarrow} [A,D,B].$$

If,

$$\left.\begin{array}{l}[A,B,C]\\ [A,C,D]\end{array}\right\} \overset{3.2}{\Longrightarrow} [A,B,D].$$

If,

$$\left.\begin{array}{l}[A,D,C]\\ [A,B,C]\end{array}\right\} \overset{3.1}{\Longrightarrow} [A,B,D] \vee [A,D,B].$$

If,

$$\left.\begin{array}{l}[A,D,C]\\ [B,C,A]\end{array}\right\} \overset{\textbf{Lo.1}}{\Longrightarrow} \left.\begin{array}{l}[A,D,C]\\ [A,C,B]\end{array}\right\} \overset{\textbf{Lo.3}}{\Longrightarrow} [A,D,B].$$

So we see that any of the possible cases stays. The case when at least two of the four above points coincide the proof is clear. □

Definition 3.4. The parallel projection between the two lines in the Desargues affine plane, will be called, a function,

$$P_p : \ell_1 \to \ell_2, \forall A, B \in \ell_1, AB \parallel P_p(A)P_p(B).$$

It is clear that this function is a bijection between any two lines in Desargues affine planes.

Definition 3.5. [3] A Desargues Affine plane, is said to be **ordered**, provided

1. All lines in this plane are **ordered lines**.
2. Parallel projection of the points of one line onto the points of another line in this plane, either preserves or reverses the ordering.

Theorem 3.4. *Each translation (dilation which is different from $id_{\mathcal{P}}$ and has no fixed point) in a finite Desargues affine plane $\mathcal{A}_D = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ preserves order in a line ℓ of this plane.*

Proof. Let $A, B, C \in \ell$, and $[A, B, C]$. Let's have, too, a each translation φ on this plan. Here we will distinguish two cases regarding the direction of translation.

Case.1 The translation φ has not direction according to line ℓ, in this case $\varphi(\ell) \parallel \ell$, and $\varphi(\ell) \neq \ell$, (see Fig. 4). Mark $\varphi(\ell) = \ell'$.

From the translation properties, we have,

$$A, B, C \in \ell \Longrightarrow \varphi(A), \varphi(B), \varphi(C) \in \ell'.$$

From the translation properties described at [37], we have the following parallelisms,

$$AB \parallel \varphi(A)\varphi(B), AC \parallel \varphi(A)\varphi(C), BC \parallel \varphi(B)\varphi(C),$$

and

$$A\varphi(A) \parallel B\varphi(B) \parallel C\varphi(C).$$

So we have the following parallelograms (see [11]):

$$(A, B, \varphi(B), \varphi(A)), (A, C, \varphi(C), \varphi(A)), (B, C, \varphi(C), \varphi(B)),$$

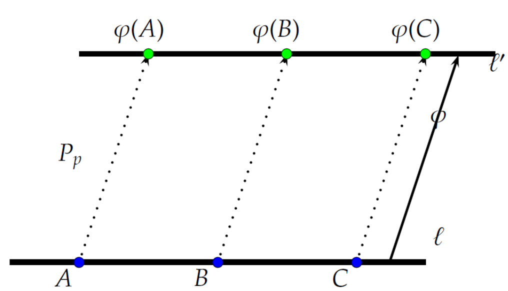

Figure 4: The translation φ as a parallel projection $P_p : \ell \to \ell'$.

Clearly, we see that the translation φ can be seen as parallel projection P_p, from line ℓ to line ℓ', with direction the line $\ell^{A\varphi(A)}$.

Thus by definition and so $[A, B, C]$, we have $[\varphi(A), \varphi(B), \varphi(C)]$.

Case.2 The translation φ has direction according to line ℓ, in this case $\varphi(\ell) \parallel \ell$, and $\varphi(\ell) = \ell$, (see Fig. 5). By translation properties have,

$$A, B, C \in \ell \Longrightarrow \varphi(A), \varphi(B), \varphi(C) \in \ell.$$

In this case, we choose a point $E \notin \ell$ of the plan. For this point, $\exists! \ell_\ell^E \in \mathcal{L}$, so it exists a translation φ_1, such that $\varphi_1(A) = E$, and exists a translation φ_2, such that $\varphi_2(E) = \varphi(A)$.

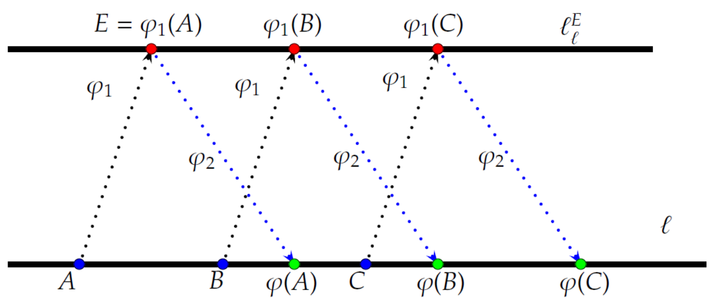

Figure 5: The translations φ as a composition of two translations φ_1 and φ_2. So $\varphi = \varphi_2 \circ \varphi_1$.

Well,

$$(\varphi_2 \circ \varphi_1)(A) = \varphi(A).$$

Two translations φ_1 and φ_2 have different directions from line ℓ. Now we repeat the first case twice, once for the translation φ_1 and once for the translation φ_2, and we have:

$$[A, B, C] \Longrightarrow [\varphi_1(A), \varphi_1(B), \varphi_1(C)] \Longrightarrow [\varphi_2(\varphi_1(A)), \varphi_2(\varphi_1(B)), \varphi_2(\varphi_1(C))]$$

thus,

$$[A, B, C] \Longrightarrow [(\varphi_2 \circ \varphi_1)(A), (\varphi_2 \circ \varphi_1)(B), (\varphi_2 \circ \varphi_1)(C)].$$

Hence

$$[A, B, C] \Longrightarrow [\varphi(A), \varphi(B), \varphi(C)]$$

□

Theorem 3.5. *Every finite ordered Desarguesian affine plane over $\mathbb{R}^2$ is Pappian.*

Proof. Let π be a finite Desarguesian affine plane over $\mathbb{R}^2$. From Theorem 2.3, π is Pappian. Since every line in π satisfies Axioms Lo.1-Lo.4 of definition 3.1, so every line in π is an ordered line, also, this plane meets the conditions of the definition 3.5. Consequently, π is a finite ordered Desarguesian affine plane. It has been observed that every Euclidean space is Pappian [27, §1.1, p. 195]. Hence, π is Pappian. □

4 The ordered skew-field in a line in ordered Desargues affine plane

In [35], [9], we have shown how to construct a skew-field over a line in Desargues affine plane. Let it be ℓ a line of Desargues affine plane $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$.

We mark $\mathbf{K} = (\ell, +, *)$ the skew-field constructed over the line ℓ, in Desargues affine plane $\mathcal{A}_{\mathcal{D}}$. In previous work [9], we have shown how we can transform a line in the Desargues affine plane into an additive Group of its points. We have also shown [35], [9] and [36] how to construct a skew-field with a set of points on a line in the Desargues affine plane. In addition, for a line of in any Desargues affine plane, we construct a skew-field with the points of this line, by appropriately defined addition and multiplication of points in a line (also, another very interesting construction of skew-fields in affine plane, is presented in the works [30] and [31]).

During the construction of the skew-field over a line of a Desargues affine plane, we choose two points (each affine plan has at least two points), which we write with O and I and call them zero and one, respectively. These points play the role of unitary elements regarding the two actions addition and multiplication, respectively.

Definition 4.1. [3],[24]A skew-field $\mathbf{K}$ is said to be ordered, if, satisfies the following conditions

1. $K = K_- \cup \{0_K\} \cup K_+$ and $K_- \cap \{0_K\} \cap K_+ = \varnothing$,
2. For all $k_1, k_2 \in K_+, k_1 + k_2 \in K_+$, ($K_+$ is closed under addition)
3. For all $k_1, k_2 \in K_+, k_1 * k_2 \in K_+$, (a product of positive elements is positive)

Consider a line ℓ in a Desargues affine plane, by defining the affine plane so that we have at least two points in this line, which we mark $\mathbf{O}$ and $\mathbf{I}$. For this line, we have shown in the previous works (see [9],[33]) that we can construct a skew-field $\mathbf{K} = (\ell, +, *)$. We have shown [9] the independence of choosing points $\mathbf{O}$ and $\mathbf{I}$ in a line, for the construction of a skew-field over this line.

To establish a separation of the set of points in line ℓ, we separate, firstly the points $\{\mathbf{O}\}$, which we mark with

$$\mathbf{K}_+ = \{X \in \ell \mid [O, X, I] \ or \ [O, I, X]\} \, (= \ell_+).$$

and

$$\mathbf{K}_- = \{X \in \ell \mid [X, O, I]\} \, (= \ell_-).$$

So, clearly, from the definition of the ordered line in a Desargues affine plane, we have

$$\mathbf{K} = \mathbf{K}_- \cup \{0_{\mathbf{K}}\} \cup \mathbf{K}_+,$$

where $0_{\mathbf{K}} = O$, and

$$\mathbf{K}_- \cap \{0_{\mathbf{K}}\} \cap \mathbf{K}_+ = \varnothing.$$

Lemma 4.1. *For all* $A, C \in \mathbf{K}_+ \Longrightarrow A + C \in \mathbf{K}_+$.

Proof. Let's have the points $A, C \in \ell$, such that $[O, A, C]$ or $[O, C, A]$. Suppose we have true $[O, A, C]$, from the construction of the $A + C$ point we have that $A + C \in \ell$, (see Fig. 6) which was built according to the algorithm

$$\forall A, C \in \ell, \left[\begin{array}{l} \textbf{Step.1}.\exists B \notin OI \\ \\ \textbf{Step.2}.\ell_{OI}^{B} \cap \ell_{OB}^{A} = D \\ \\ \textbf{Step.3}.\ell_{CB}^{D} \cap OI = E \end{array} \right] \Leftrightarrow A + C = E.$$

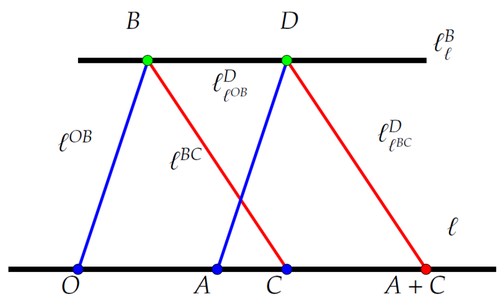

Figure 6: The addition of points in a line of Desargues affine plane, as a composition of tow parallel projections.

From the construction of the $A + C$ point, we have that:

$$\ell^{OB} \parallel \ell^{AD}; \ell^{BC} \parallel \ell^{D(A+C)}, \ell^{BD} \parallel OI.$$

From these parallelisms, there is the translation φ_1 with direction, according to line ℓ, such that, $\varphi_1(O) = A$.

$$[O, A, C] = [O, \varphi_1(O), C].$$

Since, by with Theorem 3.4, the translations preseve the line-order, we have

$$\begin{aligned} [O, A, C] = [O, \varphi_1(O), C] &\Longrightarrow [\varphi_1(O), \varphi_1(\varphi_1(O)), \varphi_1(C)] \\ &\Longrightarrow [\varphi_1(O), \varphi_1(A), \varphi_1(C)] = [A, \varphi_1(A), \varphi_1(C)]. \end{aligned}$$

Hence, we have defined a translation (see [34],[32]), such that we have true

$$[O, \varphi_1(O) = A, \varphi_1(A)] = [O, A, \varphi_1(A)]$$

From this translation, we also have,

$$\varphi_1(B) = D \text{ and } \varphi_1(C) = A + C,$$

By by with Theorem 3.4 we have

$$[O, A, C] \Longrightarrow [\varphi_1(O), \varphi_1(A), \varphi_1(C)] = [A, \varphi_1(A), A + C].$$

For four points $O, A, \varphi_1(A), A + C \in \ell$, we have

$$\left. \begin{array}{c} [O, A, \varphi_1(A)] \\ [A, \varphi_1(A), A + C] \end{array} \right\} \overset{\textbf{Lo.3}}{\Longrightarrow} [O, A, A + C] \wedge [O, \varphi_1(A), A + C].$$

By $[O, A, A + C]$, we have that

$$A + C \in K_+ (= \ell_+).$$

□

Corollary 4.2. *For all* $A, C \in \mathbf{K}_- \Longrightarrow A + C \in \mathbf{K}_-$.

Corollary 4.3. *For all* $A \in \mathbf{K}_+ \Longrightarrow -A \in \mathbf{K}_-$.

Proof. If $-A \in \mathbf{K}_+ \Longrightarrow A + (-A) = O(= 0_K) \in \mathbf{K}_+$, which is a contradiction. □

Lemma 4.4. *For all* $A, C \in \mathbf{K}_+ \Longrightarrow A * C \in \mathbf{K}_+$.

Proof. Let's have the points $A, C \in \ell$, such that $[O, A, C]$ or $[O, C, A]$. Suppose we have true $[O, A, C]$,and suppose also that we have $[O, I, A]$ from the construction of the $A * C$ point we have that $A * C \in \ell$, which was built according to the algorithm

$$\forall A, C \in \ell, \begin{bmatrix} \textbf{Step.1}. \exists B \notin OI \\ \\ \textbf{Step.2}. \ell^A_{IB} \cap OB = E \\ \\ \textbf{Step.3}. \ell^E_{BC} \cap OI = F \end{bmatrix} \Leftrightarrow A * C = F.$$

By construction of the point $A * C$, we have parallelisms

$$IB || AE, BC || E(A * C).$$

We take a parallel projection P_p with the direction of the line ℓ^{IB}, (see Fig. 7) such that

$$P_p : \ell^{OI} \to \ell^{OB}, P_p(O) = O; P_p(I) = B$$

and $P_p(A) = E$, since parallel projection preserves the order, we have

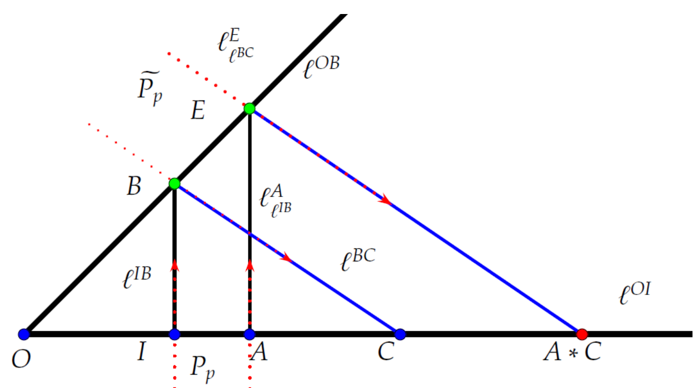

Figure 7: The multiplication of points in a line of Desargues affine plane, as a composition of tow parallel projections.

$$\left.\begin{array}{c} [O, I, A] \\ IB \parallel AE \end{array}\right\} \Longrightarrow [P_p(O), P_p(I), P_p(A)] = [O, B, E]$$

But by the multiplication algorithm, during the construction of the $A * C$ point, we have $BC || E(A * C)$.

We take a parallel projection $\widetilde{P_p}$ with the direction of the line ℓ^{BC}, (see Fig. 7) such that

$$\widetilde{P_p} : \ell^{OB} \to \ell^{OI}, \widetilde{P_p}(O) = O; \widetilde{P_p}(B) = C \text{ and } \widetilde{P_p}(E) = A * C.$$

Since the parallel projection preserves the order, we have

$$\left.\begin{array}{c}[O, B, E] \\ BC || E(A * C)\end{array}\right\} \Longrightarrow \left[\widetilde{P_p}(O), \widetilde{P_p}(B), \widetilde{P_p}(E)\right] = [O, C, A * C]$$

By $[O, C, A * C]$, we have that

$$A * C \in K_+(= \ell_+).$$

If we have, $[O, A, C]$ and $[A, I, C] \Longrightarrow [O, A, I]$, (see Fig. 8). By construction of the point $A * C$, we have parallelisms

$$IB || AE, BC || E(A * C).$$

We take a parallel projection P_p with the direction of the line ℓ^{IB}, such that

$$P_p : \ell^{OI} \to \ell^{OB}, P_p(O) = O; P_p(A) = E \text{ and } P_p(I) = B.$$

Since the parallel projection preserves the order, we have

$$[O, A, I] \Longrightarrow [P_p(O), P_p(A), P_p(I)] \Longrightarrow [O, E, B].$$

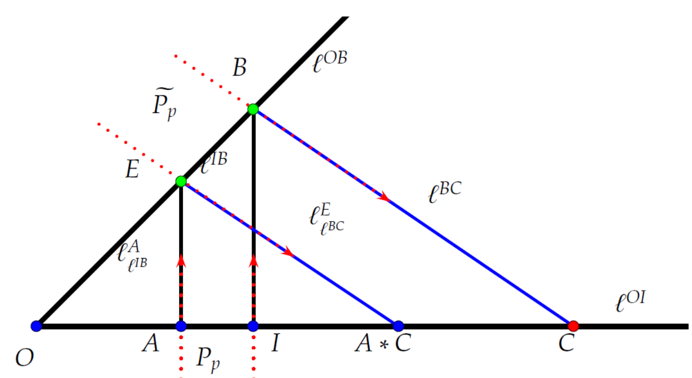

Figure 8: The case where, we have, $[O, A, C]$ and $[A, I, C]$.

But by the multiplication algorithm, during the construction of the $A * C$ point, we have $BC || E(A * C)$.We take a parallel projection $\widetilde{P_p}$ with the direction of the line ℓ^{BC}, such that

$$\widetilde{P_p} : \ell^{OB} \to \ell^{OI}, \widetilde{P_p}(O) = O; \widetilde{P_p}(E) = A * C \text{ and } \widetilde{P_p}(B) = C.$$

Since the parallel projection preserves the order, we have

$$[O,E,B] \Longrightarrow \left[\widetilde{P_p}(O), \widetilde{P_p}(E), \widetilde{P_p}(B)\right] = [O, A*C, C] \Longrightarrow A*C \in K_+(=\ell_+).$$

If we have, $[O,A,C]$ and $[A,C,I] \Longrightarrow [O,A,I] \wedge [O,C,I]$, (see Fig. 9). By construction of the point $A*C$, we have parallelisms

$$IB||AE, BC||E(A*C).$$

We take a parallel projection P_p with the direction of the line ℓ^{IB}, such that

$$P_p : \ell^{OI} \to \ell^{OB}, P_p(O) = O; P_p(A) = E \text{ and } P_p(I) = B.$$

Since the parallel projection preserves the order, we have

$$[O,A,I] \Longrightarrow [P_p(O), P_p(A), P_p(I)] \Longrightarrow [O,E,B].$$

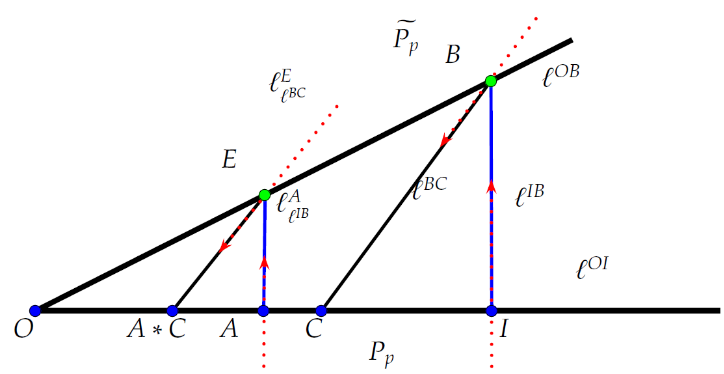

Figure 9: The case where we have $[O,A,C]$ and $[A,C,I]$.

But by the multiplication algorithm, during the construction of the $A*C$ point, we have $BC||E(A*C)$.We take a parallel projection $\widetilde{P_p}$ with the direction of the line ℓ^{BC}, such that

$$\widetilde{P_p} : \ell^{OB} \to \ell^{OI}, \widetilde{P_p}(O) = O; \widetilde{P_p}(E) = A*C; \widetilde{P_p}(B) = C,$$

and $\widetilde{P_p}$, since parallel projection preserves the order, we have

$$[O,E,B] \Longrightarrow \left[\widetilde{P_p}(O), \widetilde{P_p}(E), \widetilde{P_p}(B)\right] = [O, A*C, C] \Longrightarrow A*C \in K_+(=\ell_+).$$

□

We have indirectly from Lemma 4.4 the case when $[O,C,A]$, since $A*C \neq C*A$. From Lemmas 4.1 and 4.4, we have the following result.

Theorem 4.5. *Every skew Field that is constructed over an ordered-line of a Desargues affine plane is an ordered skewField.*

If we consider the special case where ordered-line of a Desargues affine plane in $\mathbb{R}^2$, we obtain the following result.

Corollary 4.6. *Every finite skew Field that is constructed over an ordered-line of a Desargues affine plane in $\mathbb{R}^2$ is a finite ordered skew-field.*

Putting together the result from Corollary 4.6 and Theorem 3.5, we obtain the following result for an ordered lines o a finite Desargues affine plane in $\mathbb{R}^2$.

Theorem 4.7. *A finite skew Field constructed over an ordered-line on a finite Desargues affine plane in $\mathbb{R}^2$ is Pappian.*

Proof. Let ℓ be an ordered-line in a finite Desargues affine plane in $\mathbb{R}^2$. By definition, ℓ is a Desarguesian affine 1-plane. From Corollary 4.6, ℓ is a finite ordered skew-field. Hence, from Theorem 3.5, ℓ is Pappian. $\square$

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