η-Ricci solitons on Lorentzian para-Kenmotsu manifolds

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Abstract. In the present paper, we initiate the study of η-Ricci Soliton on a Lorentzian Para-Kenmotsu manifold. At first we give the historical background of the η-Ricci solitons and put down some curvature conditions of Riemannian manifolds. Next, in section 2, we introduce the basic formulas and rudimentary facts used in research work. The next section 3, deals with η-Ricci solitons on Lorentzian para-Kenmotsu manifold and deduces some results. Again in section 4, we developed and proof the Results on harmonic and Weyl harmonic curvature tensor. Section 5, about Ricci and η-parallel Ricci tensor on Ricci solitons. In the end section, we emphasized results on Lorentzian para-Kenmotsu manifold with η-Ricci soliton satisfying the curvature condition $P.\varphi = 0$.

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1 Introduction

The concept of Ricci flow, the notion of Ricci soliton and its existence introduced by Hamilton [12]. We need this concept to answer Thurston's geometric conjecture that is a 3-dimensional manifold admits a geometric decomposition if it is closed. All compact manifolds of dimensional –four with positive curvature also classified by Hamilton. The equation of Ricci flow is as follows

$$\frac{\partial g}{\partial t} = -2S.$$

Ricci soliton emerges as the limit of the soliton of Ricci flow. A soliton to the Ricci flow to be a Ricci soliton ([19], [12], [13]) if it has only one moving parameter group of diffeomorphism and scaling. The Ricci soliton equation is expressed as

$$\ell_X g + 2S - 2\lambda g = 0,$$

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where S denotes the Ricci tensor, g is the Riemannian metric, ℓ_X is the Lie derivative, X is a vector field, and λ is a scalar.

In the above equation if

(a) $\lambda > 0$ (positive) then it defines as shrinking Ricci soliton.
(b) $\lambda < 0$ (negative) then it is defined as expanding Ricci soliton.
(c) $\lambda = 0$ then it is defined as steady Ricci soliton.

Next, A. E. Fischler [10] invented a new concept of conformal Ricci flow, known as modification of the classical Ricci flow equation, which transforms the unit volume constraint of that equation to a scalar curvature constraint. The new equations defines the conformal Ricci flow on M, where M is regarded as a smooth connected n-manifold and the equation is as follows [10]:

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg,$$

where $r(g)$ is the scalar curvature of the manifold defined by $r(g) = -1$, p is a scalar non-dynamical field and the dimension of the manifold is n.

Similarly, N. Basu and A. Bhattacharya [1] proposes as follows the definition of the conformal Ricci soliton equation

$$\ell_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g.$$

The concept of Ricci almost soliton was firstly introduced by S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti [15]. R. Sharma [17] also started the study of Ricci soliton and done excellent work.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura [6], for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple $(g, \Upsilon, \lambda, \mu)$, where Υ is a vector field on M, λ and μ are real scalars and g is Riemannian metric satisfying the equation

$$\ell_\Upsilon g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where S is the Ricci tensor associated with g. In this connection, we mention the works of Blaga ([2], [3], [4]), Prakasha et al. [16], De and De [8], Kar, D. et al. [14], Eyasmin et al. [9] and many others on η-Ricci solitons.

In general

(i) If $\mu = 0$ then η-Ricci soliton reduces to Ricci soliton.

(ii) If $\mu \neq 0$ then the η-Ricci solitons is named proper η-Ricci solitons.

Gray [11] introduces the notion of cyclic parallel Ricci tensor and Codadazi type of Ricci tensor. A Riemannian manifold or semi-Riemannian manifold is said to have cyclic parallel Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$(\nabla_X S)(\Upsilon, Z) + (\nabla_\Upsilon)(Z, X) + (\nabla_Z S)(X, \Upsilon) = 0. \tag{1.1}$$

Assume that the curvature tensor is harmonic that is $divR = 0$, this implies

$$(\nabla_S)(\Upsilon, Z) = (\nabla_\Upsilon S)(X, Z), \tag{1.2}$$

where *div* denotes divergence. It means that the Levi-Civita connection ∇ of such a metric is Yang-Mills connection while keeping the metric on the manifold fixed. Equation (1.2) implies that the Ricci tensor S is of Coddazi type.

Also, Weyl tensor is harmonic, then we get

$$(\nabla_X S)(\Upsilon, Z) - (\nabla_\Upsilon S)(X, Z) = \frac{1}{2(n-1)}[(Xr)g(\Upsilon, Z) - (\Upsilon r)g(X, Z)], \tag{1.3}$$

where r is the scalar curvature.
The projective curvature tensor P [20] in a manifold (M, g) is defined by

$$P(X, \Upsilon)Z = R(X, \Upsilon)Z - \frac{1}{n-1}\left[g(\Upsilon, Z)QX - g(X, Z)Q\Upsilon\right], \tag{1.4}$$

where Q is the Ricci tensor operator defined by $S(X, \Upsilon) = g(QX, \Upsilon)$ and $X, Y, Z \epsilon \chi(M)$, being the Lie algebra of vector fields of M.

2 Preliminaries

Let M^n be Lorentzian metric manifold, with a $(1, 1)$ tensor field φ. We consider a vector field ξ, a Lorentzian metric g, and a 1 form η on M and assume that the structure tensor (φ, ξ, η, g) satisfies

$$\begin{cases} \varphi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ \eta(\varphi X) = 0, \\ g(\varphi X, \varphi \Upsilon) = g(X, \Upsilon) + \eta(X)\eta(\Upsilon), \end{cases}$$

provides a Lorentzian almost para-contact manifold for all X, Υ on M. In a Lorentzian almost para-contact manifolds, we have

$$\varphi\xi = 0,\ g(X, \xi) = \eta(X),\ g(\xi, \xi) = -1$$
$$\varphi(X, \Upsilon) = \varphi(\Upsilon, X), \quad \text{where} \quad \varphi(X, \Upsilon) = g(X, \varphi\Upsilon).$$

Definition 2.1. A Lorentzian almost para-contact manifold M, is a Lorentzian para-Kenmotsu manifold, if for any vector fields X, Υ on M, we have

$$(\nabla_X \varphi)\Upsilon = -g(\varphi X, \Upsilon)\xi - \eta(\Upsilon)\varphi X,$$

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.1}$$

$$(\nabla_X \eta)\Upsilon = -g(X, \Upsilon) - \eta(X)\eta(\Upsilon), \tag{2.2}$$

where ∇ represents the covariant differentiation.

Remark 2.2. In a Lorentzian para-Kenmotsu manifold M, the following relations hold:

$$g(R(X,\Upsilon),Z,\xi) = \eta(R(X,\Upsilon)Z) = g(\Upsilon,Z)\eta(X) - g(X,Z)\eta(\Upsilon),$$
$$R(\xi,X)\Upsilon = g(X,\Upsilon)\xi - \eta(\Upsilon)X,$$

$$R(X,\Upsilon)\xi = \eta(\Upsilon)X - \eta(X)\Upsilon, \tag{2.3}$$

$$R(\xi,X)\xi = X + \eta(X)\xi,$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.4}$$

$$Q\xi = (n-1)\xi,$$
$$S(\varphi X,\varphi\Upsilon) = S(X,\Upsilon) + (n-1)\eta(X)\eta(\Upsilon),$$

where R is the Riemannian curvature tensor and S is the Ricci tensor.
pasn In a 3-dimensional Riemannian manifold M, we have

$$\begin{aligned} R(X,\Upsilon)Z &= g(\Upsilon,Z)QX - g(X,\Upsilon)Q\Upsilon + S(\Upsilon,Z)X - S(X,Z)\Upsilon \\ &\quad -\frac{r}{2}[g(\Upsilon,Z)X - g(X,Z)\Upsilon],, \end{aligned} \tag{2.5}$$

where $Q(g(QX,\Upsilon)) = S(X,\Upsilon)$ and r are the Ricci operator and the scalar curvature, respectively. Also

$$QX = \frac{1}{2}[(r-2)X + (r-6)\eta(X)\eta(\Upsilon)].$$

3 η-Ricci soliton on Lorentzian para-Kenmotsu manifold

Consider the equation

$$\ell_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta = 0,$$

where ℓ_ξ is the Lie derivative operator, ξ is a vector field, S is the Ricci curvature tensor field, λ and μ are real constants. Expressing $\ell_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,\Upsilon) = -g(\nabla_X\xi,\Upsilon) - g(X,\nabla_\Upsilon\xi) - 2\lambda g(X,\Upsilon) - 2\mu\eta(X)\eta(\Upsilon), \tag{3.1}$$

for any $X,\Upsilon\epsilon\chi(M)$.
The expression (g,ξ,λ,μ) which satisfies the relation (3.1) is said to be an η-Ricci soliton over M. If $\mu = 0$, (g,ξ,λ) forms a Ricci soliton [12] and differentiate shrinking, steady, or expanding according to $\lambda < 0,\ \lambda = 0,$ or $\lambda > 0$ respectively [7].

Now we know that in a Lorentzian para-Kenmotsu manifold $\nabla_X\xi = -X - \eta(X)\xi$, so equation (3.1) becomes:

$$2S(X,\Upsilon) = -g(-X-\eta(X)\xi,\Upsilon) - g(X,-\Upsilon-\eta(\Upsilon)\xi) - 2\lambda g(X,\Upsilon) - 2\mu\eta(X)\eta(\Upsilon).$$

Solving this, we get

$$S(X,\Upsilon) = (1-\lambda)g(X,\Upsilon) - (1+\mu)\eta(X)\eta(\Upsilon). \tag{3.2}$$

Putting $\Upsilon = \xi$ in the above equation, we have

$$S(X, \xi) = (2 - \lambda + \mu)\eta(X). \tag{3.3}$$

By comparing equation (2.4) and (3.3), we have

$$\mu - \lambda = n - 3. \tag{3.4}$$

In this scenario, the description of the Ricci operator Q is define as

$$QX = (1 - \lambda)X - (1 + \mu)\eta(X)\xi. \tag{3.5}$$

Taking into account α such a symmetrical $(0, 2)$ type tensor field, parallel to the Levi-Civita connection $\nabla\alpha = 0$. From the Ricci identity $\nabla^2\alpha(X, \Upsilon; Z, W) - \nabla^2\alpha(X, \Upsilon; W, Z) = 0$, for any $X, \Upsilon, W, Z \epsilon \chi(M)$ [18]. Then

$$\alpha(R(X, \Upsilon)Z, W) + \alpha(Z, R(X, \Upsilon)W) = 0.$$

If we replace $Z = W = \xi$, then from the symmetry of α follows $\alpha(R(X, \Upsilon), \xi, \xi) = 0$, for any $X, \Upsilon \epsilon \chi(M)$.
If (φ, ξ, η, g) will be a Lorentzian para-Kenmotsu structure on M, from remark (2.1) we see that $R(X, \Upsilon)\xi = \eta(X)\Upsilon - \eta(\Upsilon)X$ and replacing this expression in α, we get:

$$\alpha(\Upsilon, \xi) + \eta(\Upsilon)\alpha(\xi, \xi) = 0,$$

for any $\Upsilon \epsilon \chi(M)$, equivalent to

$$\alpha(\Upsilon, \xi) + \alpha(\xi, \xi)g(\Upsilon, \xi) = 0.$$

Covariantly differentiating the later equation w.r.t. the vector field X, we obtain

$$\alpha(\nabla_X \Upsilon, \xi) + \alpha(\Upsilon, \nabla_X \xi) + \alpha(\xi, \xi)\left[g(\nabla_X \Upsilon, \xi) + g(\Upsilon, \nabla_X \xi)\right] = 0.$$

Substituting from equation (2.1), we get

$$\alpha(\Upsilon, X) = -\alpha(\xi, \xi)g(\Upsilon, X),$$

for any $X, \Upsilon \epsilon \chi(M)$; as α is ∇-parallel, it follows that $\alpha(\xi, \xi)$ is constant, and we have:

Lemma 3.1. Under the hypothesis mentioned above, every symmetric $(0, 2)$-tensor parallel field is a constant (negative) multiple of the metric.

4 Harmonic and Weyl harmonic curvature tensors of LP-Kenmotsu manifold with η -Ricci soliton

Theorem 4.1. *A Lorentzian para-Kenmotsu manifold admitting an η-Ricci solitons is of harmonic curvature if and only if the manifold is an Einstien manifold. Also $\mu = -1$ and $\lambda = 2 - n$.*

Proof. By covariantly differentiating (3.2) with respect to W, we obtain

$$(\nabla_Z S)(X, \Upsilon) = -(1+\mu)\left[(\nabla_Z \eta)X\eta(\Upsilon) + (\nabla_Z \eta)\Upsilon\eta(X)\right].$$

Using (2.2) in the above equation, we get

$$(\nabla_Z S)(X, \Upsilon) = (1+\mu)\left[g(Z, X)\eta(\Upsilon) + g(Z, \Upsilon)\eta(X) + 2\eta(X)\eta(\Upsilon)\eta(Z)\right]. \tag{4.1}$$

Given (4.1), it follows that

$$(\nabla_Z S)(X, \Upsilon) - (\nabla_X S)(\Upsilon, Z) = (1+\mu)\left[g(Z, \Upsilon)\eta(X) - g(X, \Upsilon)\eta(Z)\right]. \tag{4.2}$$

By hypothesis, the manifold is of harmonic curvature, that is

$$(\nabla_Z S)(X, \Upsilon) = (\nabla_X S)(\Upsilon, Z).$$

Hence from (4.2), we get

$$(1+\mu)g(\varphi X, \varphi\Upsilon) = 0.$$

It follows that $\mu = -1$, therefore (3.4) implies $\lambda = 2 - n$. Thus from (3.2), we have

$$S(X, \Upsilon) = (n-1)g(X, \Upsilon).$$

Conversely, suppose that the manifold is an Einstien manifold. Then obviously, we have Codazzi type Ricci tensor. Therefore the manifold is of harmonic curvature, which proves our statement. □

Theorem 4.2. *Let M be a Lorentzian para-Kenmotsu manifold admitting an η-Ricci soliton. The manifold M is of harmonic Weyl tensor if and only if the manifold is an Einstein manifold, provided the scalar curvature r is invariant under the characteristic vector field ξ. Also $\mu = -1$ and $\lambda = 2 - n$.*

Proof. Let the Lorentzian para-Kenmotsu manifold M is of harmonic Weyl tensor. Then (1.3) gives

$$(\nabla_X S)(\Upsilon, Z) - (\nabla_\Upsilon S)(X, Z) = \frac{1}{2(n-1)}\left[(Xr)g(\Upsilon, Z) - (\Upsilon r)g(X, Z)\right]. \tag{4.3}$$

Making use of (4.1) in (4.3), we have

$$(1+\mu)\left[g(X, Z)\eta(\Upsilon) - g(\Upsilon, Z)\eta(X)\right] = \frac{1}{2(n-1)}\left[(Xr)g(\Upsilon, Z) - (\Upsilon r)g(X, Z)\right].$$

Putting $\Upsilon = \xi$ in the above equation, gives

$$(1+\mu)\left[-g(X, Z) - \eta(Z)\eta(X)\right] = \frac{1}{2(n-1)}\left[(Xr)\eta(Z) - (\xi r)g(X, Z)\right].$$

Replacing $Z = \varphi Z$ in the above equation provides

$$(1+\mu)\left[-g(X, \varphi Z)\right] = -\frac{1}{2(n-1)}\left[(\xi r)g(X, \varphi Z)\right. \tag{4.4}$$

Taking $\xi r = 0$, then the above equation implies

$$(1+\mu)g(X,\varphi Z) = 0.$$

It follows that $\mu = -1$. Then from (3.2), we have $\lambda = 2-n$, and hence (3.2) implies

$$S(X,\Upsilon) = (n-1)g(X,\Upsilon).$$

In contrast, if the manifold is an Einstein manifold, then it is Ricci symmetric $\nabla S = 0$ and the scalar curvature is constant. Hence the Weyl tensor is harmonic. This completes the proof. □

5 Ricci tensors on Lorentzian para-Kenmotsu manifold with η-Ricci soliton

Theorem 5.1. *A Lorentzian para-Kenmotsu manifold satisfies cyclic parallel Ricci tensor admitting an η-Ricci solitons if and only if the manifold is an Einstien manifold. Moreover $\mu = -1$ and $\lambda = 2-n$.*

Proof. Using (4.1) in (1.1), we get

$$\begin{aligned}(1+\mu)[g(X,\Upsilon)\eta(Z) + g(X,Z)\eta(\Upsilon) + 2\eta(X)\eta(\Upsilon)\eta(Z)\\ +g(X,\Upsilon)\eta(Z) + g(\Upsilon,Z)\eta(X) + 2\eta(X)\eta(\Upsilon)\eta(Z)\\ +g(X,Z)\eta(\Upsilon) + g(\Upsilon,Z)\eta(X) + 2\eta(X)\eta(\Upsilon)\eta(Z)] = 0.\end{aligned}$$

Putting $Z = \xi$ in the above equation, we obtain

$$(1+\mu)\left[-2g(X,\Upsilon)\right] = 0.$$

Then it follows that $1+\mu = 0 \Rightarrow \mu = -1$. Hence from (3.4), we get $\lambda = 2-n$. Then (3.2) implies

$$S(X,\Upsilon) = (n-1)g(X,\Upsilon).$$

Conversely, if the manifold is an Einstien manifold, then the Ricci tensor can clearly be shown as being cyclic parallel. This completes the proof. □

Lemma 5.1. In a 3-dimensional Lorentzian para-Kenmotsu manifold, we have

$$\xi(r) = -2(r-6).$$

Proof. If we replace $\Upsilon = Z = \xi$ in equation (2.5) and using equation (2.3) and (2.4), we get

$$QX = \frac{1}{2}\left[(r-2)X + (r-6)\eta(X)\xi\right], \tag{5.1}$$

for any vector field $X \epsilon \chi(M)$. Taking into account (5.1), (2.1), and (2.4) in the following formula for semi-Riemannian manifold, we get

$$trace\Upsilon \longrightarrow (\nabla_\Upsilon Q)X = \frac{1}{2}\nabla_X r.$$

and we obtain the required equation. □

We state the following consequence of this Lemma,

Corollary 5.2. *A 3-dimensional Lorentzian para-Kenmotsu manifold admitting an η-Ricci soliton is of harmonic Weyl tensor if and only if the manifold is of constant positive sectional curvature 1.*

Proof. Let us consider a 3-dimensional Lorentzian para-Kenmotsu manifold; it is known that [5]

$$R(X,\Upsilon)Z = g(\Upsilon,Z)QX - g(X,Z)Q\Upsilon + S(\Upsilon,Z)X - S(X,Z)\Upsilon - \frac{r}{2}\left[g(\Upsilon,Z)X - g(X,Z)\Upsilon\right] \tag{5.2}$$

and

$$QX = \frac{1}{2}\left[(r-2)X - (r-6)\eta(X)\xi\right], \tag{5.3}$$

for any vector fields X, Υ, Z, where r is the scalar curvature, Q is the Ricci operator, and S is the Ricci tensor. Also from the lemma (4.1) in a 3-dimensional Lorentzian para-Kenmotsu manifold, we have

$$\xi r = -2(r-6). \tag{5.4}$$

Using the above result in (4.4), we get

$$r = 2(\mu+2) = \text{constant}, \tag{5.5}$$

which implies

$$\xi r = 0. \tag{5.6}$$

Using this in (5.4), we obtain $r = 6$, and hence from (5.3), we get

$$QX = 2X.$$

Therefore from (5.2) the manifold has a constant sectional curvature 1.

Hence the manifold is an Einstein manifold, and the scalar curvature (positive) is constant. Therefore from (1.3), it follows that the manifold is of harmonic Weyl tensor, which concludes the proof. □

Theorem 5.3. *Let M be a Lorentzian para-Kenmotsu manifold admitting an η-Ricci soliton with η-parallel Ricci tensor. Then $\mu = 1, \lambda = 2 - n$ and the manifold is an Einstein manifold.*

Proof. Let the Ricci tensor of a Lorentzian para-Kenmotsu manifold be η-parallel [5]. Then

$$g\left((\nabla_\Upsilon Q)X, Z\right) = 0, \tag{5.7}$$

for arbitrary vector fields X, Υ, Z. Taking the covariant derivative of (3.5) concerning an arbitrary vector field Υ, we get

$$(\nabla_\Upsilon Q)X = \nabla_\Upsilon QX - Q(\nabla_\Upsilon X) = -(1+\mu)\left[((\nabla_\Upsilon \eta)X)\xi - \eta(X)\nabla_\Upsilon \xi\right].$$

Using (2.1) and (2.2) in the above equation, we get

$$(\nabla_{\Upsilon} Q)X = -(1+\mu)\left[-g(X,\Upsilon)\xi + \eta(X)\Upsilon + 2\eta(X)\eta(\Upsilon)\xi\right].$$

Using the above equation in (5.7), we infer

$$(1+\mu)\left[g(X,\Upsilon)\eta(Z) - \eta(X)\eta(\Upsilon) + 2\eta(X)\eta(\Upsilon)\eta(Z)\right] = 0. \tag{5.8}$$

Putting $Z = \xi$ in (5.8) yields

$$(1+\mu)\left[-g(X,\Upsilon) - 3\eta(X)\eta(\Upsilon)\right] = 0.$$

It follows that $\mu = -1$, and hence from (3.4), we get $\lambda = 2 - n$. Therefore (3.2) gives

$$S(X,\Upsilon) = (n-1)g(X,\Upsilon),$$

and consequently, the theorem is proved. □

6 η-Ricci solitons on Lorentzian para-Kenmotsu manifold satisfying the condition $P.\varphi = 0$

We assume that the LP-Kenmotsu manifold admitting an η-Ricci soliton satisfies the curvature condition

$$P.\varphi = 0.$$

This implies that

$$P(X,\Upsilon)\varphi Z - \varphi(P(X,\Upsilon)Z) = 0.$$

Putting $Z = \xi$, we get

$$\varphi(P(X,\Upsilon)\xi) = 0. \tag{6.1}$$

Now putting $Z = \xi$ in (1.4), we get

$$P(X,\Upsilon)\xi = R(X,\Upsilon)\xi - \frac{1}{n-1}\left[\eta(\Upsilon)QX - \eta(X)Q\Upsilon\right].$$

Now using (2.3) and (3.5) in the above equation, we have

$$\begin{aligned} P(X,\Upsilon)\xi = \eta(\Upsilon)X - \eta(X)\Upsilon - \frac{1}{n-1}[&\eta(\Upsilon)((1-\lambda)X - (1+\mu)\eta(X)\xi) \\ &-\eta(X)((1-\lambda)\Upsilon - (1+\mu)\eta(\Upsilon)\xi)], \end{aligned}$$

which implies

$$P(X,\Upsilon)\xi = \left[1 - \frac{(1-\lambda)}{(n-1)}\right](\eta(\Upsilon)X - \eta(X)\Upsilon). \tag{6.2}$$

Using (6.2) in (6.1), we get

$$\left[1 - \frac{(1-\lambda)}{(n-1)}\right][\eta(\Upsilon)\varphi X - \eta(X)\varphi\Upsilon] = 0.$$

Replacing $X = \varphi X$ in the above expression yields

$$\left(1 - \frac{(1-\lambda)}{(n-1)}\right)\eta(\Upsilon)\varphi^2 X = 0.$$

Putting $\Upsilon = \xi$, we get

$$-\left(1 - \frac{(1-\lambda)}{(n-1)}\right)[X + \eta(X)\xi] = 0.$$

Again replacing X by φX in the following equation, we infer

$$\left(1 - \frac{(1-\lambda)}{(n-1)}\right)\varphi X = 0. \tag{6.3}$$

Taking the inner product of (6.3) with respect to W, we get

$$\left(1 - \frac{(1-\lambda)}{(n-1)}\right)g(\varphi X, W) = 0.$$

It follows that $\left(1 - \frac{(1-\lambda)}{(n-1)}\right) = 0$, which implies $\lambda = 2 - n$. Therefore from (3.4), we get $\mu = -1$. Hence (3.2) infers

$$S(X, \Upsilon) = (n-1)g(X, \Upsilon).$$

We conclude that

Theorem 6.1. *If a Lorentzian para-Kenmotsu manifold admits an η-Ricci soliton and satisfies the curvature condition $P.\varphi = 0$, then $\mu = -1, \lambda = 2 - n$ and the manifold is an Einstein manifold.*

Corollary 6.2. *If a Lorentzian para-Kenmotsu manifold admits an η-Ricci soliton and satisfies the curvature condition $Q.P = 0$, then $\mu = -1$, $\lambda = 2 - n$ and the manifold is an Einstein manifold*

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