Geometric characteristics of screen slant lightlike submanifolds of indefinite nearly Kaehler manifolds

Tejinder Kumar, Megha Pruthi, Sangeet Kumar and Pankaj Kumar

Abstract. The aim of present paper is to analyze geometric characteristics of screen slant lightlike submanifolds of indefinite nearly Kaehler manifolds. We establish the existence theorem for screen slant lightlike submanifolds in indefinite nearly Kaehler manifolds. We also derive conditions for the integrability of distributions for such submanifolds. Consequently, we find several characterization results for totally umbilical screen slant lightlike submanifolds in indefinite nearly Kaehler manifolds. Subsequently, minimal screen slant lightlike submanifolds of indefinite nearly Kaehler manifolds are also investigated.

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1 Introduction

Slant immersions and slant submanifolds is one of the most significant contribution in differential geometry. Chen [2] initiated the idea of slant immersions by generalizing invariant and anti-invariant immersions. Further, in [3] the author generalized the concept of slant immersions to define slant submanifolds in complex geometry. Afterwards, Lotta [9, 10] explored slant submanifolds in contact geometry. On the similar note, semi-slant submanifolds, bi-slant submanifolds, hemi-slant submanifolds came into existence and the subject matter was significantly investigated by Papaghiuc [11], Carriazo [1] and Sahin [13].

One may note that most of the work of slant submanifolds has been considered with positive definite metric. But, from last two decades, due to interesting geometric properties of lightlike geometry, the focus of geometers shifted towards lightlike submanifolds. In this context, Sahin [12] played a significant role, when he defined slant lightlike submanifolds in indefinite Hermitian manifolds and further extended this concept in indefinite Sasakian manifolds in [15]. Later on, many authors investigated some other generalized classes of slant lightlike submanifolds viz. pointwise slant

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lightlike submanifolds, semi-slant lightlike submanifolds etc. in indefinite Kaehler manifolds (for detail, see [8], [16], [17]).

In addition, Sahin [14] brought up the concept of screen slant lightlike submanifolds in indefinite Hermitian manifolds. But the concept of screen slant lightlike submanifolds is yet to be explored in indefinite nearly Kaehler manifolds. For this reason, due to broader application area of indefinite nearly Kaehler manifolds, we focus on study of screen slant lightlike submanifolds of indefinite nearly Kaehler manifolds.

To this end, the geometry of screen slant lightlike submanifolds of indefinite nearly Kaehler manifolds is investigated. Consequently, we obtain the existence theorem for such submanifolds. We also establish conditions for the integrability of distributions for such submanifolds. Further, we derive several characterization results for totally umbilical screen slant lightlike submanifolds in indefinite nearly Kaehler manifolds.

2 Preliminaries

In this section, we define indefinite nearly Kaehler manifolds and present the basic notations and formulae for lightlike submanifolds [4].

Consider a submanifold (K_n, g) of semi-Riemannian manifold $(\bar{K}_{m+n}, \bar{g})$ such that $\bar{g}$ is metric with index q satisfying $m, n \geq 1$ and $m + n - 1 \geq q \geq 1$. If metric $\bar{g}$ is degenerate on TK, then T_pK and $T_pK^{\perp}$ both are degenerate and there exists a radical (null) subspace $Rad(T_pK)$ such that $Rad(T_pK) = T_pK \cap T_pK^{\perp}$. If $Rad(TK) : p \in K \to Rad(T_pK)$ is smooth distribution on K with rank $r > 0$, $1 \leq r \leq n$, then K is called r-lightlike submanifold of $\bar{K}$. While the radical distribution $Rad(TK)$ of TK is defined as:

$$Rad(TK) = \cup_{p\in K}\{\xi \in T_pK | g(u, \xi) = 0, \ \forall \ u \in T_pK, \xi \neq 0\}.$$

Further, $S(TK)$ be the screen distribution in TK such that $TK = Rad(TK) \perp S(TK)$ and similarly $S(TK^{\perp})$ is screen transversal vector bundle in $TK^{\perp}$ such that $TK^{\perp} = Rad(TK) \perp S(TK^{\perp})$.
Moreover, there exists a local null frame $\{N_i\}$ of null sections with values in the orthogonal complement of $S(TK^{\perp})$ in $S(TK^{\perp})^{\perp}$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{ for any } i, j \in \{1, 2, .., r\}, \tag{2.1}$$

where $\{\xi_i\}$ is any local basis of $\Gamma(Rad(TK))$. It implies that $tr(TK)$ and $ltr(TK)$, respectively, be vector bundles in $T\bar{K}|_K$ and $S(TK^{\perp})^{\perp}$ with the property

$$tr(TK) = ltr(TK) \perp S(TK^{\perp}),$$

and

$$T\bar{K}|_K = TK \oplus tr(TK) = S(TK) \perp (Rad(TK) \oplus ltr(TK)) \perp S(TK^{\perp}). \tag{2.2}$$

Considering decomposition Eq. (2.2), the Gauss and Weingarten formulae are

$$\bar{\nabla}_P Q = \nabla_P Q + h^l(P, Q) + h^s(P, Q), \tag{2.3}$$

$$\bar{\nabla}_P N = -A_N P + \nabla^l_P N + D^s(P, N), \tag{2.4}$$

$$\bar{\nabla}_P W = -A_W P + D^l(P, W) + \nabla^s_P W, \tag{2.5}$$

where $P, Q \in \Gamma(TK), N \in \Gamma(ltr(TK))$ and $W \in \Gamma(S(TK^{\perp}))$. Further, employing Eqs. (2.3) and (2.5), we have

$$g(A_W P, Q) = \bar{g}(h^s(P, Q), W) + \bar{g}(Q, D^l(P, W)). \tag{2.6}$$

Let us denote the projection morphism of TK on screen distribution $S(TK)$ by S, it follows that

$$\nabla_P SQ = \nabla^*_P SQ + h^*(P, SQ), \quad \nabla_P \xi = -A^*_\xi P + \nabla^{*t}_P \xi, \tag{2.7}$$

where $\{h^*(P, SQ), \nabla^{*t}_P \xi\} \in \Gamma(Rad(TK))$ and $\{\nabla^*_P SQ, A^*_\xi P\} \in \Gamma(S(TK))$. Then considering Eqs. (2.4), (2.5) and (2.7), we attain

$$\bar{g}(h^l(P, SQ), \xi) = g(A^*_\xi P, SQ). \tag{2.8}$$

Let us denote by $\bar{R}$ and R, the curvature tensors of $\bar{\nabla}$ and ∇, respectively, then the Codazzi equation is given by

$$\begin{aligned}(\bar{R}(P, Q)Z)^{\perp} = \; & (\nabla_P h^l)(Q, Z) - (\nabla_Q h^l)(P, Z) + (\nabla_P h^s)(Q, Z) \\ & -(\nabla_Q h^s)(P, Z) + D^l(P, h^s(Q, Z)) - D^l(Q, h^s(P, Z)) \\ & +D^s(P, h^l(Q, Z)) - D^s(Q, h^l(P, Z)),\end{aligned} \tag{2.9}$$

where

$$(\nabla_P h^s)(Q, Z) = \nabla^s_P h^s(Q, Z) - h^s(\nabla_P Q, Z) - h^s(Q, \nabla_P Z), \tag{2.10}$$

$$(\nabla_P h^l)(Q, Z) = \nabla^l_P h^l(Q, Z) - h^l(\nabla_P Q, Z) - h^l(Q, \nabla_P Z), \tag{2.11}$$

for $P, Q, Z \in \Gamma(TK)$.

Definition 2.1. [7] An indefinite almost Hermitian manifold $\bar{K}$ with an almost complex structure $\bar{J}$ and Hermitian metric $\bar{g}$, is said to be an indefinite nearly Kaehler manifold if

$$\bar{J}^2 = -I, \; \bar{g}(\bar{J}P, \bar{J}Q) = \bar{g}(P, Q), \; (\bar{\nabla}_P \bar{J})Q + (\bar{\nabla}_Q \bar{J})P = 0, \tag{2.12}$$

where $P, Q \in \Gamma(T\bar{K})$ and $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{K}$.

An indefinite RK-manifold of constant holomorphic sectional curvature c and of constant type α is called a generalized complex space form and is denoted by $\bar{K}(c, \alpha)$. For a generalized complex space form $\bar{K}(c, \alpha)$, the curvature tensor $\bar{R}$ is given by

$$\begin{aligned}\bar{R}(P, Q)W = \; & \frac{c+3\alpha}{4}\{\bar{g}(Q, W)P - \bar{g}(P, W)Q\} + \frac{c-\alpha}{4}\{\bar{g}(P, \bar{J}W)\bar{J}Q \\ & -\bar{g}(Q, \bar{J}W)\bar{J}P + 2\bar{g}(P, \bar{J}Q)\bar{J}W\},\end{aligned} \tag{2.13}$$

where $P, Q, W \in \Gamma(T\bar{K})$.

3 Screen slant lightlike submanifolds

Following [14], we define screen slant lightlike submanifolds of indefinite nearly Kaehler manifolds as follows.

Definition 3.1. A $2q$ - lightlike submanifold $(K, g, S(TK))$ of an indefinite nearly Kaehler manifold $\bar{K}$ (provided, $2q < dim(K)$) is said to be a screen slant lightlike submanifold of $\bar{K}$ if

(i) $\bar{J}(Rad(TK)) = Rad(TK)$, that is, $Rad(TK)$ is invariant with respect to $\bar{J}$.

(ii) For each non-zero vector field Y tangent to $S(TK)$ at $y \in U \subset K$, the angle $\theta(Y)$ between $\bar{J}Y$ and $S(TK)$ is constant, that is, it is independent of the choice of y and $Y \in (S(TK))$.

In view of Definition (3.1), TK and $S(TK^{\perp})$ of K have following decomposition

$$TK = Rad(TK) \perp S(TK), \quad S(TK^{\perp}) = \bar{J}(S(TK)) \perp \mu. \tag{3.1}$$

Moreover, for $Q \in \Gamma(S(TK))$ and $W \in \Gamma(S(TK^{\perp}))$, we have

$$\bar{J}Q = tQ + nQ, \quad \bar{J}W = bW + cW, \tag{3.2}$$

where tQ and nQ, respectively, represent tangential and transversal component of $\bar{J}Q$ and similarly $bW \in \Gamma(S(TK))$ and $cW \in \Gamma(\mu)$.

Note: In the forthcoming part, we shall denote a screen slant lightlike submanifold by *s.st.l.s.* and an indefinite nearly Kaehler manifold by $\bar{K}$, unless otherwise mentioned. For a *s.st.l.s.* K of $\bar{K}$, let R and S be the projection morphisms of TK on distributions $Rad(TK)$ and $S(TK)$, respectively. Then for any $Q \in \Gamma(TK)$, we have

$$Q = RQ + SQ. \tag{3.3}$$

Further, applying $\bar{J}$, Eq. (3.3) yields to

$$\bar{J}Q = \bar{J}RQ + \bar{J}SQ = tRQ + tSQ + nSQ. \tag{3.4}$$

Theorem 3.1. *A $2q$- lightlike submanifold K of $\bar{K}$ (provided, $2q < dim(K)$), is a* **s.st.l.s***, if and only if*

(i) $ltr(TK)$ is invariant w.r.t. $\bar{J}$.

(ii) $(Sot)^2Q = -cos^2\theta Q$ for $Q \in \Gamma(S(TK))$.

Proof. Let K be a **s.st.l.s** of $\bar{K}$, therefore employing Eq. (3.4), for $Q \in \Gamma(S(TK))$ and $N \in \Gamma(ltr(TK))$, we acquire

$$\bar{g}(\bar{J}N, Q) = -\bar{g}(N, \bar{J}Q) = -\bar{g}(N, tQ) - \bar{g}(N, nQ) = 0.$$

Hence, $\bar{J}N$ does not belong to $S(TK)$. Moreover, from Eq. (3.2), for $W \in \Gamma(S(TK^{\perp}))$, we get

$$\bar{g}(\bar{J}N, W) = -\bar{g}(N, \bar{J}W) = -\bar{g}(N, bW) - \bar{g}(N, cW) = 0.$$

Thus $\bar{J}N$ does not belong to $S(TK^{\perp})$. Next, taking $\bar{J}N \in \Gamma(Rad(TK))$, we get $\bar{J}^2N = -N \in \Gamma(ltr(TK))$. As $Rad(TK)$ is invariant, this leads to a contradiction, which proves (i). In view of statement (ii) and hypothesis, we arrive at

$$cos\theta(Q) = \frac{\bar{g}(\bar{J}Q, tSQ)}{|\bar{J}Q|\,|tSQ|} = -\frac{\bar{g}(Q, \bar{J}tSQ)}{|\bar{J}Q|\,|tSQ|} = -\frac{\bar{g}(Q, (S\circ t)^2Q)}{|Q|\,|tSQ|}. \tag{3.5}$$

On the other hand, we also have

$$cos\theta(Q) = \frac{|tSQ|}{|\bar{J}Q|}. \tag{3.6}$$

Thus from Eqs. (3.5) and (3.6), we obtain

$$cos^2\theta(Q) = -\frac{g(Q, (S\circ t)^2Q)}{|Q^2|}. \tag{3.7}$$

Since angle is constant, we conclude $(Sot)^2Q = -cos^2\theta Q$, which proves (ii).
Conversely, assume that (i) and (ii) hold, then condition (i) implies that $ltr(TK)$ is invariant w.r.t. $\bar{J}$. By virtue of Lemma 3.1 of [14], the vector bundle $S(TK)$ is Riemannian, therefore

$$g(tSQ, tSQ) = -g(t^2SQ, SQ) = cos^2\theta(SQ)g(SQ, SQ), \tag{3.8}$$

for any $Q \in \Gamma(S(TK))$, which gives

$$cos^2\theta(Q) = \frac{g(tSQ, tSQ)}{g(SQ, SQ)}. \tag{3.9}$$

Hence, the result follows. □

Corollary 3.2. *[14] For a* s.st.l.s. *K of $\bar{K}$, one has*

$$g(tSQ_1, tSQ_2) = cos^2\theta g(SQ_1, SQ_2), \tag{3.10}$$

and

$$g(nSQ_1, nSQ_2) = sin^2\theta g(SQ_1, SQ_2), \tag{3.11}$$

for $Q_1, Q_2 \in \Gamma(TK)$.

4 Integrability of the distributions

In this part, we will examine some conditions for integrability of distributions associated with a **s.st.l.s** of $\bar{K}$. Firstly, we prove the following lemma for later use.

Lemma 4.1. *Consider a* s.st.l.s. *K of $\bar{K}$, then*

$$(\nabla_P t)Q + (\nabla_Q t)P = A_{nSQ}P + A_{nSP}Q + 2bh^s(P, Q), \tag{4.1}$$

$$\begin{aligned} 2\bar{J}h^l(P,Q) = \quad & h^l(P,\bar{J}RQ) + h^l(P,tSQ) + D^l(P,nSQ) \\ & + h^l(Q,\bar{J}RP) + h^l(Q,tSP) + D^l(Q,nSP), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} (\nabla_P n)Q + (\nabla_Q n)P = \quad & -h^s(P,\bar{J}RQ) - h^s(P,tSQ) - h^s(Q,\bar{J}RP) \\ & - h^s(Q,tSP) + 2ch^s(P,Q), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} (\nabla_P t)Q &= \nabla_P \bar{J}RQ + \nabla_P tSQ - \bar{J}R\nabla_P Q - tS\nabla_P Q, \\ (\nabla_Q t)P &= \nabla_Q \bar{J}RP + \nabla_Q tSP - \bar{J}R\nabla_Q P - tS\nabla_Q P, \\ (\nabla_P n)Q &= \nabla^s_P nSQ - nS\nabla_P Q, (\nabla_Q n)P = \nabla^s_Q nSP - nS\nabla_Q P, \end{aligned}$$

for any $P, Q \in \Gamma(TK)$.

Proof. Employing Eqs. (2.12), (3.2) and (3.4) and then equating the tangential and transversal components, the result follows. □

Theorem 4.2. *Assume that* K *be a* s.st.l.s. *of* $\bar{K}$. *Then* $Rad(TK)$ *is integrable, if and only if*

$$h^s(P,\bar{J}RQ) + h^s(Q,\bar{J}RP) - 2ch^s(P,Q) = 2nS\nabla_Q P,$$

where $P, Q \in \Gamma(Rad(TK))$.

Proof. Employing Eq. (4.3), for any $P, Q \in \Gamma(Rad(TK))$, we acquire

$$\begin{aligned} nS\nabla_P Q - nS\nabla_Q P + nS\nabla_Q P = \quad & h^s(P,\bar{J}RQ) + h^s(Q,\bar{J}RP) \\ & - 2ch^s(P,Q) - nS\nabla_Q P. \end{aligned} \tag{4.4}$$

Then Eq. (4.4) yields to

$$nS[P,Q] = h^s(P,\bar{J}RQ) + h^s(Q,\bar{J}RP) - 2ch^s(P,Q) - 2nS\nabla_Q P, \tag{4.5}$$

which gives result. □

Theorem 4.3. *For a* s.st.l.s. K *of* $\bar{K}$, $S(TK)$ *is integrable, if and only if*

$$2\bar{J}R\nabla_Q P = R(\nabla_P tSQ + \nabla_Q tSP) - R(A_{nSQ}P + A_{nSP}Q)$$

where $P, Q \in \Gamma(S(TK))$.

Proof. For $P, Q \in \Gamma(S(TK))$, using Eq. (4.1), we obtain

$$\begin{aligned} \bar{J}R\nabla_P Q + tS\nabla_P Q = \quad & \nabla_P tSQ - A_{nSQ}P + \nabla_Q tSP - A_{nSP}Q \\ & - \bar{J}R\nabla_Q P - tS\nabla_Q P - 2bh^s(P,Q), \end{aligned} \tag{4.6}$$

which yields to

$$\begin{aligned} \bar{J}R[P,Q] + tS[P,Q] = \quad & \nabla_P tSQ - A_{nSQ}P + \nabla_Q tSP - A_{nSP}Q \\ & - 2\bar{J}R\nabla_Q P - 2tS\nabla_Q P - 2bh^s(P,Q). \end{aligned} \tag{4.7}$$

Thus

$$\bar{J}R[P,Q] = R(\nabla_P tSQ + \nabla_Q tSP) - R(A_{nSQ}P + A_{nSP}Q) - 2\bar{J}R\nabla_Q P.$$

□

5 Totally umbilical screen slant lightlike submanifolds

Definition 5.1. [5] A lightlike submanifold (K, g) of a semi-Riemannian manifold $(\bar{K}, \bar{g})$ is called totally umbilical, if there is a smooth transversal vector field $H \in \Gamma(tr(TK))$ on K such that

$$h(P,Q) = H\bar{g}(P,Q), \tag{5.1}$$

where $P, Q \in \Gamma(TK)$. Then following Eqs. (2.3) and (2.5), K is totally umbilical, if and only if, there exist smooth vector fields $H^l \in \Gamma(ltr(TK))$ and $H^s \in \Gamma(S(TK^{\perp}))$ such that

$$h^l(P,Q) = H^l g(P,Q); \quad h^s(P,Q) = H^s g(P,Q); \quad D^l(P,W) = 0, \tag{5.2}$$

for $P, Q \in \Gamma(TK)$ and $W \in \Gamma(S(TK^{\perp}))$.

Theorem 5.1. *Assume that K be a totally umbilical* s.st.l.s. *of $\bar{K}$. Then at least one of the following assertions hold:*

(a) K is an anti-invariant submanifold.

(b) $S(TK) = \{0\}$.

(c) If K is a proper s.st.l.s.*, then $H^s \in \Gamma(\mu)$.*

Proof. For any $P \in \Gamma(S(TK))$, from Eqs. (3.11) and (5.1), we have

$$h(tP, tP) = cos^2\theta g(P,P)H. \tag{5.3}$$

Using Gauss formula, we obtain

$$\bar{\nabla}_{tP}tP - \nabla_{tP}tP = cos^2\theta g(P,P)H. \tag{5.4}$$

Then applying $\bar{J}$ on both sides of Eq. (5.4) and in view of nearly Kaehlerian property of $\bar{K}$, we get

$$\bar{\nabla}_{tP}\bar{J}tP - \bar{J}\nabla_{tP}tP = cos^2\theta g(P,P)\bar{J}H,$$

which on using Eq. (3.2) yields

$$\bar{\nabla}_{tP}t^2P + \bar{\nabla}_{tP}ntP - t\nabla_{tP}tP - n\nabla_{tP}tP = cos^2\theta g(P,P)\bar{J}H. \tag{5.5}$$

Taking into account Theorem (3.1), we have $t^2P = -cos^2\theta P$ and hence Eq. (5.5) reduces to

$$\begin{aligned} cos^2\theta g(P,P)\bar{J}H = \; & -cos^2\theta\bar{\nabla}_{tP}P - A_{ntP}tP + D^l(tP, ntP) \\ & +\nabla^s_{tP}ntP - t\nabla_{tP}tP - n\nabla_{tP}tP. \end{aligned} \tag{5.6}$$

Further employing Eq. (5.2) in Eq. (5.6) and then equating transversal components on both sides, one has

$$\begin{aligned} cos^2\theta g(P,P)\bar{J}H = \; & -cos^2\theta g(tP,P)H^l - cos^2\theta g(tP,P)H^s + \nabla^s_{tP}ntP \\ & -n\nabla_{tP}tP. \end{aligned} \tag{5.7}$$

Then, taking the inner product w.r.t. nP in above equation, we obtain

$$-cos^2\theta\bar{g}(\bar{J}H^s,nP)g(P,P)+\bar{g}(\nabla^s_{tP}ntP,nP)-\bar{g}(n\nabla_{tP}tP,nP)=0. \tag{5.8}$$

Employing Eq. (3.11) for $Q_1,Q_2\in\Gamma(S(TK))$ and then considering covariant derivative w.r.t. tP, we obtain

$$\bar{g}(\nabla^s_{tP}nP,nP)=sin^2\theta g(\nabla_{tP}P,P), \tag{5.9}$$

which further gives

$$\bar{g}(\nabla^s_{tP}ntP,nP)=sin^2\theta g(\nabla_{tP}tP,P). \tag{5.10}$$

Next using Eqs. (3.11) and (5.10) in Eq. (5.8), we obtain

$$cos^2\theta g(P,P)\bar{g}(H^s,nP)=0. \tag{5.11}$$

Thus Eq. (5.11) yields that either $P=0$ or $\theta=\pi/2$ or $H^s\in\Gamma(\mu)$, which completes the proof. □

Theorem 5.2. *For a proper totally umbilical* s.st.l.s. *K of $\bar{K}$, we must have $H^l=0$.*

Proof. For $Q\in\Gamma(S(TK))$, from Eq. (2.12), we have $\bar{\nabla}_Q\bar{J}Q=\bar{J}\bar{\nabla}_QQ$, which implies

$$\begin{aligned}&\nabla_QtQ+h^l(Q,tQ)+h^s(Q,tQ)-A_{nQ}Q+\nabla^s_Qn Q+D^l(Q,nQ)\\&=t\nabla_QQ+n\nabla_QQ+\bar{J}h^l(Q,Q)+bh^s(Q,Q)+ch^s(Q,Q).\end{aligned} \tag{5.12}$$

Further, taking inner product w.r.t. $\bar{J}\xi\in\Gamma(Rad(TK))$ on both sides in above equation and using the hypothesis, we obtain $\bar{g}(h^l(Q,tQ),\bar{J}\xi)=\bar{g}(\bar{J}h^l(Q,Q),\bar{J}\xi)$. Again using the hypothesis, we acquire $g(Q,Q)\bar{g}(H^l,\xi)=g(Q,tQ)\bar{g}(H^l,\bar{J}\xi)=0$ and using non-degeneracy of $S(TK)$, we obtain $\bar{g}(H^l,\xi)=0$, this yields

$$H^l=0. \tag{5.13}$$

□

Theorem 5.3. *Consider a proper totally umbilical* s.st.l.s. *of $\bar{K}$ along with $\nabla^s_QZ\in\Gamma(\mu)$, then we have*

$$H^s=0,$$

for $Q\in\Gamma(S(TK))$ and $Z\in\Gamma(S(TK^\perp))$.

Proof. From Theorem (5.1), for a proper totally umbilical *s.st.l.s.* of $\bar{K}$, we have $H^s\in\Gamma(\mu)$. Comparing transversal components on both sides of Eq. (5.12), we obtain

$$\begin{aligned}&h^l(Q,tQ)+h^s(Q,tQ)+\nabla^s_Qn Q+D^l(Q,nQ)\\&=\quad n\nabla_QQ+\bar{J}h^l(Q,Q)+\bar{J}h^s(Q,Q).\end{aligned} \tag{5.14}$$

Then, using Eq. (5.2), we acquire

$$\begin{aligned}&g(Q,tQ)H^l+g(Q,tQ)H^s+\nabla^s_Qn Q\\&=\quad n\nabla_QQ+g(Q,Q)\bar{J}H^l+g(Q,Q)\bar{J}H^s.\end{aligned} \tag{5.15}$$

On taking inner product on both sides w.r.t $\bar{J}H^s$, above equation yields to

$$\bar{g}(\nabla^s_Q nQ, \bar{J}H^s) = g(Q,Q)\bar{g}(H^s, H^s). \tag{5.16}$$

As $\bar{\nabla}$ is a metric connection on $\bar{K}$, thus taking $(\bar{\nabla}_Q g)(nQ, \bar{J}H^s) = 0$, we obtain

$$\bar{g}(\nabla^s_Q nQ, \bar{J}H^s) = -\bar{g}(\nabla^s_Q \bar{J}H^s, nQ) = 0. \tag{5.17}$$

Thus, from Eqs. (5.16) and (5.17), we derive

$$g(Q,Q)\bar{g}(H^s, H^s) = 0. \tag{5.18}$$

As $S(TK)$ is non-degenerate, therefore

$$H^s = 0. \tag{5.19}$$

Hence the proof follows. □

Theorem 5.4. *Every totally umbilical proper* s.st.l.s. *of $\bar{K}$ provided, $\nabla^s_Q Z \in \Gamma(\mu)$, for any $Q \in \Gamma(S(TK))$ and $W \in \Gamma(S(TK^{\perp})$, is totally geodesic.*

Proof. The result follows directly from Theorem (5.2) and Theorem (5.3). □

Theorem 5.5. *There does not exist any proper totally umbilical* s.st.l.s. *in a generalized complex space form $\bar{K}(c,\alpha)$ provided, $c \neq \alpha$.*

Proof. For $P \in \Gamma(S(TK)), Z \in \Gamma(ltr(TK))$ and $\xi \in \Gamma(Rad(TK))$, employing Eq. (2.13), we get

$$\bar{g}(\bar{R}(P,\bar{J}P)Z,\xi) = -\frac{c-\alpha}{2} g(P,P)\bar{g}(\bar{J}Z,\xi). \tag{5.20}$$

On the other hand, from equation of Codazzi (2.9), we derive

$$\bar{g}(\bar{R}(P,\bar{J}P)Z,\xi) = \bar{g}((\nabla_P h^l)(\bar{J}P,Z),\xi) - \bar{g}((\nabla_{\bar{J}P} h^l)(P,Z),\xi). \tag{5.21}$$

Then from Eqs. (5.20) and (5.21), we acquire

$$-\frac{c-\alpha}{2} g(P,P)\bar{g}(\bar{J}Z,\xi) = \bar{g}((\nabla_P h^l)(\bar{J}P,Z),\xi) - \bar{g}((\nabla_{\bar{J}P} h^l)(P,Z),\xi). \tag{5.22}$$

By hypothesis along with Eqs. (2.11) and (5.2), we obtain

$$(\nabla_P h^l)(\bar{J}P,Z) = -\bar{g}(\nabla_P \bar{J}P, Z)H^l - \bar{g}(\bar{J}P, \nabla_P Z)H^l. \tag{5.23}$$

As $\bar{g}(\bar{J}P,Z) = 0$, for $P \in \Gamma(S(TK))$ and $Z \in \Gamma(ltr(TK))$, then taking covariant derivative w.r.t. P, we obtain $\bar{g}(\nabla_P \bar{J}P, Z) = -\bar{g}(\bar{J}P, \nabla_P Z)$. Thus, Eq. (5.23) reduces to

$$(\nabla_P h^l)(\bar{J}P,Z) = 0. \tag{5.24}$$

Similarly, it follows that

$$(\nabla_{\bar{J}P} h^l)(P,Z) = 0. \tag{5.25}$$

Then using Eqs. (5.24) and (5.25) in Eq. (5.22), we acquire $-\frac{c-\alpha}{2} g(P,P)\bar{g}(\bar{J}Z,\xi) = 0$. Hence, in view of non-degeneracy of $S(TK)$, we conclude $c = \alpha$, which completes the proof. □

6 Minimal screen slant lightlike submanifolds

Definition 6.1. A lightlike submanifold $(K, g, S(TK))$ isometrically immersed in a semi-Riemannian manifold $(\bar{K}, \bar{g})$ is called minimal if the following conditions hold:

(i) $h^s(P,Q) = 0$, *for all* $P, Q \in \Gamma(Rad(TK))$.

(ii) *trace* $h|_{S(TK)} = 0$.

Note. In view of Definition (5.1), a *s.st.l.s.* of $\bar{K}$ is minimal, if it is a totally geodesic.

Theorem 6.1. *A necessary and sufficient condition for totally umbilical proper* s.st.l.s. *of* $\bar{K}$ *to be minimal is that, trace* $A_{W_l}|_{S(TK)} = 0$, *for* $W_l \in \Gamma(S(TK^{\perp}))$, *where* $l \in \{1, 2, ..., t\}$.

Proof. According to Definition (3.1), K is minimal *iff*

$$h^s(\xi_i, \xi_j) = 0 \quad and \quad \sum_{j=1}^{q} h(e_j, e_j) = 0,$$

where $\{\xi_i\}_{i=1}^{r}$ and $\{e_j\}_{j=1}^{q}$ are bases of $Rad(TK)$ and $S(TK)$, respectively. From Eq. (5.2), we note that $h(\xi_i, \xi_j) = 0$. Therefore, we get $h^s = 0$ on $Rad(TK)$. Further, employing Eq. (5.13), we conclude $h^l = 0$. As a result, K is minimal *iff* $\sum_{j=1}^{q} h^s(e_j, e_j) = 0$, where

$$\sum_{j=1}^{q} h^s(e_j, e_j) = \sum_{j=1}^{q} \left\{ \frac{1}{t} \sum_{l=1}^{t} g(h^s(e_j, e_j), W_l) W_l \right\}.$$

Then using Eq. (2.6), the above equation reduces to

$$\sum_{j=1}^{q} h^s(e_j, e_j) = \sum_{j=1}^{q} \left\{ \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} e_j, e_j), W_l \right\}.$$

Hence, the result follows. □

Theorem 6.2. *Consider an irrotational* s.st.l.s. K *of* $\bar{K}$. *Then* K *is minimal iff*

$$trace\ A_{W_l}|_{S(TK)} = 0 \quad and \quad trace\ A^*_{\xi_j}|_{S(TK)} = 0,$$

where $W_l \in \Gamma(S(TK)^{\perp})$, $\xi_j \in \Gamma(Rad(TK))$, $l \in \{1, 2, ..., t\}$ *and* $j \in \{1, 2, ..., r\}$.

Proof. Since K is irrotational, therefore $h^s = 0$ on $Rad(TK)$. Moreover, we have

$$trace\ \ h|_{S(TK)} = \sum_{p=1}^{q} h(e_p, e_p).$$

Then, using Eqs. (2.6) and (2.8), we acquire

$$\sum_{p=1}^{q} h(e_p, e_p) = \sum_{p=1}^{q} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} e_p, e_p) N_j + \frac{1}{t} \sum_{k=1}^{t} g(A_{W_l} e_p, e_p) W_l \right\}. \tag{6.1}$$

Thus, we conclude *trace* $h|_{S(TK)} = 0$ *iff trace* $A_{W_l} = 0$ and *trace* $A^*_{\xi_j} = 0$. Hence, the result follows. □

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Authors' addresses:

Tejinder Kumar, Pankaj Kumar
Department of Mathematics,
Maharaja Agrasen University, Baddi (Solan)-174103, H. P., India.
E-mail: tejinder230@gmail.com, pankaj.11dtu@gmail.com

Megha Pruthi and Sangeet Kumar (corresponding author)
Department of Mathematics,
Sri Guru Teg Bahadur Khalsa College,
Sri Anandpur Sahib (Rupnagar)-140118, Punjab, India.
E-mail: meghapruthi4194@gmail.com , sp7maths@gmail.com