

Singularities of null mean curvature flow of null hypersurfaces in Lorentzian manifolds

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Abstract. We classify two main singularities, as type I and type II, associated to null mean curvature flow of screen conformal null hypersurfaces in Lorentzian manifolds. We prove that the flow at a type I singularity is asymptotically self-similar, whereas at a type II singularity there is a blow-up solution which is an eternal solution. For further analysis of the above two singularities, we define null translating solitons and use them to prove some Harnack estimates for null mean curvature flow under certain geometric conditions.

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1 Introduction

The theory of submanifolds of a Riemannian (or semi-Riemannian) manifold is one of the most important topics of differential geometry (see for example, O'Neill [20]). While the geometry of semi-Riemannian submanifolds is fully developed, its counterpart of null (lightlike, degenerate) submanifolds (for which the local geometry is completely different than the non-degenerate case) is relatively new and in a developing stage (see [4, 5, 7, 8, 11, 17, 19] and others referred therein). The theory has numerous applications in mathematical physics, particularly in general relativity and electromagnetism (see [7]). In general relativity, null hypersurfaces can be viewed as models of different black hole horizons (see [3, 2, 7, 8]).

In [19], the authors introduced the concept of evolving null hypersurfaces by their null mean curvature, in which several aspects of such hypersurfaces were investigated. It was shown, under certain geometric conditions such as screen conformality [7], that null mean curvature flow (MCF) also develops singularities. Here we want to study the singularities associated to null mean curvature flow which can occur for non convex initial data. Our aim is to characterize the asymptotic behavior of such surfaces near a singularity using rescaling techniques. These methods have been used in the

study of singularities arising from the classical mean curvature flow of hypersurfaces in Riemannian geometry [13, 14]. For more details on the classical mean curvature flow we refer to [1, 6, 10, 9, 12, 13, 14, 15, 16, 18, 21, 22] and many more references therein.

The paper is arranged as follows. In Section 2, we review the basic concepts on null hypersurfaces. We also recall some known results on null mean curvature flow required in the rest of the paper. In Section 3, we derive a null monotonicity formula which we then use to investigate singularities in null mean curvature flow. Finally, in Section 4, we prove a set of Harnack estimates for null mean curvature flow.

2 Preliminaries

Let $(\overline{M}, \overline{g})$ be a $(n+2)$ -dimensional Lorentzian manifold and let $(M, g = \overline{g}|_{\overline{M}})$ be a hypersurface of \overline{M} . Now, consider the vector bundle TM^\perp whose fibers are defined as

$$T_x M^\perp = \{Y_x \in T_x \overline{M} : \overline{g}_x(X_x, Y_x) = 0, \forall X_x \in T_x M\},$$

for any $x \in M$. Therefore, a hypersurface M of \overline{M} is null if and only if TM^\perp is a distribution of rank 1 on M .

Let M be a null hypersurface of $(\overline{M}, \overline{g})$. We consider a complementary distribution $S(TM)$ to TM^\perp in TM , which is called a *screen distribution*. It is well-known that $S(TM)$ is non-degenerate (see [7]). Thus, we have

$$(2.1) \quad TM = S(TM) \perp TM^\perp.$$

As $S(TM)$ is non-degenerate with respect to \overline{g} , we have $T\overline{M} = S(TM) \perp S(TM)^\perp$, where $S(TM)^\perp$ is the complementary vector bundle to $S(TM)$ in $T\overline{M}|_M$.

From [7, p. 79, Theorem 1.1], we know that for a screen distribution $S(TM)$ on M , there exists a unique vector bundle $\text{tr}(TM)$, called the *null transversal bundle* [7] of M with respect to $S(TM)$, of rank 1 over M such that for any non-zero section E of TM^\perp on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $\text{tr}(TM)$ on \mathcal{U} satisfying $\overline{g}(E, N) = 1$, $\overline{g}(N, N) = \overline{g}(N, Z) = 0$, for any section Z of $S(TM)$. Consequently, we have the following decomposition of $T\overline{M}$.

$$(2.2) \quad T\overline{M}|_M = S(TM) \perp \{TM^\perp \oplus \text{tr}(TM)\} = TM \oplus \text{tr}(TM).$$

Throughout this paper, $\Gamma(\Xi)$ will denote the $\mathcal{F}(M)$ -module of differentiable sections of a vector bundle Ξ .

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, and P be the projection of TM onto $S(TM)$, then the local Gauss-Weingarten equations of M and $S(TM)$ are the following ([7])

$$(2.3) \quad \overline{\nabla}_X Y = \nabla_X Y + h(X, Y) = \nabla_X Y + B(X, Y)N,$$

$$(2.4) \quad \overline{\nabla}_X N = -A_N X + \nabla_X^t N = -A_N X + \tau(X)N,$$

$$(2.5) \quad \nabla_X PY = \nabla_X^* PY + h^*(X, PY) = \nabla_X^* PY + C(X, PY)E,$$

$$(2.6) \quad \nabla_X E = -A_E^* X + \nabla_X^{*t} E = -A_E^* X - \tau(X)E, \quad A_E^* E = 0,$$

for all $X, Y \in \Gamma(TM)$, $E \in \Gamma(TM^\perp)$ and $N \in \Gamma(\text{tr}(TM))$, where $\bar{\nabla}$ is the Levi-Civita connection on \bar{M} . In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A_E^* are the shape operators on TM and $S(TM)$, respectively, while τ is a 1-form on TM . The above shape operators are related to their local fundamental forms by

$$(2.7) \quad g(A_E^*X, Y) = B(X, Y), \quad g(A_NX, PY) = C(X, PY),$$

$$(2.8) \quad \bar{g}(A_E^*X, N) = 0, \quad \bar{g}(A_NX, N) = 0, \quad \forall X, Y \in \Gamma(TM).$$

From (2.8) we notice that A_E^* and A_N are both screen-valued operators. Let $\vartheta = \bar{g}(N, \cdot)$ be a 1-form metrically equivalent to N defined on \bar{M} . Take $\lambda = i^*\vartheta$ to be its restriction on M , where $i : M \rightarrow \bar{M}$ is the inclusion map. Then it is easy to show that

$$(\nabla_X g)(Y, Z) = B(X, Y)\lambda(Z) + B(X, Z)\lambda(Y),$$

for all $X, Y, Z \in \Gamma(TM)$, which indicates that ∇ is generally *not* a metric connection with respect to g . However, the induced connection ∇^* on $S(TM)$ is a metric connection. For more details about null hypersurfaces see the books [7] and [8]. Through the 1-form λ , the authors in [5] considered a non-degenerate metric \hat{g} on M , given by

$$(2.9) \quad \hat{g}(X, Y) = g(X, Y) + \lambda(X)\lambda(Y), \quad \forall X, Y \in \Gamma(TM).$$

The metric \hat{g} is invertible and its inverse, $g^{[\cdot\cdot]}$, was called the *pseudo-inverse* of g (see details in [5]). Also, observe that \hat{g} coincides with g if the latter is non-degenerate. The metric \hat{g} has been used to define (on M) the usual operators such as *gradient*, *divergence*, *D'Alambertian* (see [5]), which one can not afford with the degenerate metric g . In case M is endowed with the metric \hat{g} , we define the gradient $\nabla^s \varrho$, Hessian, $\text{Hess}^s(\varrho)$, and D'Alambertian $\Delta^s \varrho$ of a smooth function ϱ on $\mathcal{U} \subset M$ with respect to the screen distribution $S(TM)$ as

$$(2.10) \quad \nabla^s \varrho = g^{\alpha\beta} X_\alpha(\varrho) X_\beta,$$

$$(2.11) \quad \text{Hess}^s(\varrho) = X_\alpha(X_\beta(\varrho)) - (\nabla_{X_\alpha}^* X_\beta)(\varrho),$$

$$(2.12) \quad \text{and} \quad \Delta^s \varrho = \text{tr}^s(\text{Hess}^s(\varrho)) = g^{\alpha\beta}(X_\alpha(X_\beta(\varrho)) - (\nabla_{X_\alpha}^* X_\beta)(\varrho)),$$

where $\{X_1, \dots, X_n\}$ is a basis of $S(TM)$ and $\text{tr}^s(\cdot)$ denotes the trace with respect to $S(TM)$. Note that if the screen distribution $S(TM)$ is integrable, then (2.10), (2.11) and (2.12) are just, respectively, the gradient, Hessian and Laplacian on a leaf of the screen distribution.

Throughout this paper, we assume that M carries the associated metric \hat{g} and $\text{tr}(\cdot)$ will denote the trace over M with respect to \hat{g} .

In the sequel, we shall make use of the following convention on the range of indices: $1 \leq \alpha, \beta, \gamma, \mu, \sigma \leq n$, $0 \leq a, b, c \leq n$ and $0 \leq i, j, k \leq n+1$.

Let $(M, g, S(TM))$ be a 1-null simply connected $(n+1)$ -dimensional manifold of index $(q-1)$ and with the geometric objects \bar{g} , ∇' , ∇^{tr} , $h^{*'}$ and $A^{*'}$. Then, there exists a null isometric immersion $F : M^{n+1} \rightarrow \bar{M}^{n+2}$ (see [7, Theorem 4.1] for details) satisfying

$$(2.13) \quad g(X, Y) = \bar{g}(F_*X, F_*Y), \quad \forall X, Y \in \Gamma(TM),$$

and a vector bundle isomorphism $\overline{F} : \text{tr}(TM) \longrightarrow \text{tr}(TF(M))$ such that we have $F_*(\nabla'_X Y) = \nabla_{F_*X} F_*Y$, $F_*(A'_U X) = A_{F_*U} F_*X$, $F_*(h^*(X, PY)) = h^*(F_*X, F_*PY)$ and $\overline{F}(\nabla_X^{\text{tr}} V) = \nabla_X^t \overline{F}V$, $\forall X, Y \in \Gamma(TM)$, $U \in \Gamma(\text{Rad } TM)$ and $V \in \Gamma(\text{tr}(TM))$, where $\text{tr}(TF(M))$ is the null transversal vector bundle of $F(M)$ with respect to $F_*S(TM)$, and ∇ , ∇^t , h^* , A^* are the geometric objects induced on $F(M)$ with respect to the immersion F . Certainly, $F(M)$ is nothing but a 1-null submanifold of \overline{M}^{n+2} . Moreover, the immersion F preserves both the radical and screen distribution (see [7] for details).

Next, suppose that $F(M)$ is a null hypersurface as described above. By the method of [7], the null mean curvature vector \mathbf{H} of $F(M)$ at $p \in M$ is a smooth vector field transversal to $F(M)$ and given by

$$(2.14) \quad \mathbf{H} = \text{tr}^s(h) = \text{tr}^s(B)N = \mathcal{S}N,$$

where $N \in \Gamma(\text{tr}(FTM))$ and $\mathcal{S} := \text{tr}^s(B) = \text{tr}^s(A_E^*)$. The function \mathcal{S} is called the *null mean curvature* of $F(M)$.

It is easy to see, from [8, Eq. (2.3.9), p. 59], that \mathcal{S} is independent of a chosen screen distribution $S(TM)$. However, the mean curvature vector $\mathbf{H} = \mathcal{S}N$ is clearly *dependent* on a choice of $S(TM)$, due to its direct dependance on N . In fact, let $S(TM)'$ be another screen distribution on M and $\text{tr}(TM)'$ be its corresponding null transversal bundle. As \mathcal{S} is unique, it follows that

$$\mathbf{H} - \mathbf{H}' = \mathcal{S}(N - N'),$$

from which the mean curvature vector is unique if and only $\text{tr}(TM)$ is unique. Equivalently, when $S(TM)$ is unique. Thus, fixing $S(TM)$ fixes the null transversal vector field N , and so is \mathbf{H} . A null hypersurface M with a fixed screen distribution is sometimes denoted by $(M, g, S(TM))$.

Throughout this paper, we consider a null hypersurface M with a fixed screen distribution, and we may write (M, g) for convenience.

Definition 2.1. [19] Let $(M^{n+1}, g, S(TM))$ be a compact null hypersurface of a semi-Riemannian manifold $(\overline{M}^{n+2}, \overline{g})$. Assume that $F_0 : M^{n+1} \times [0, T) \longrightarrow \overline{M}^{n+2}$ smoothly immerses M as a null hypersurface in \overline{M} . We say that $M_0 := F_0(M)$ evolves by its null mean curvature vector if there is a whole family $F(\cdot, t)$ of smooth immersions with corresponding hypersurfaces $M_t := F(\cdot, t)(M)$ such that

$$(2.15) \quad \frac{\partial F}{\partial t}(p, t) = \mathbf{H}(p, t), \quad F(\cdot, 0) = F_0, \quad p \in M^{n+1},$$

where $\mathbf{H}(p, t) := \mathcal{S}(p, t)N$ is the mean curvature vector of M_t at $F(p, t)$.

3 Singularity formation

In this section, we discuss the singularities which arises during null mean curvature flow. Let us consider an orthonormal basis $\{X_1, \dots, X_n\}$, where $X_\alpha = \frac{\partial}{\partial x_\alpha}$, of $S(TM)$ around a point $p \in M$ such that the induced Levi-Civita connection ∇^* of $S(TM)$ satisfies

$$(3.1) \quad (\nabla_{X_\alpha}^* X_\beta)(p) = 0 \quad \text{and} \quad C(X_0, X_\alpha) = 0 \quad \text{with} \quad X_0 = E.$$

Lemma 3.1. *Under the hypotheses given in (3.1), the screen distribution $S(TM)$ is integrable and its leaves are flat.*

Furthermore, we will assume that the 1-form τ vanishes on $S(TM)$. As an example we have the following.

Example 3.1 (Null cone of \mathbb{R}_1^{n+2}). Let \mathbb{R}_1^{n+2} be the space \mathbb{R}^{n+2} endowed with a semi-Euclidean metric $\bar{g}(x, y) = -x_0y_0 + \sum_{a=1}^{n+1} x_a y_a$, ($x = \sum_{A=0}^{n+1} x^A \partial x_A$) where $\partial x_A := \frac{\partial}{\partial x_A}$. Then, the null cone Λ_0^{n+1} is given by the equation $x_0^2 = \sum_{a=1}^{n+1} x_a^2$, $x_0 \neq 0$. It is well-known (for example see the books [7, 8]) that Λ_0^{n+1} is a null hypersurface of \mathbb{R}_1^{n+2} , in which the radical distribution is spanned by a global vector field $E = \sum_{A=0}^{n+1} x_A \partial x_A$ on Λ_0^{n+1} . The transversal bundle is spanned by a global section N given by $N = \frac{1}{2x_0} \{-x_0 \partial x_0 + \sum_{a=1}^{n+1} x_a \partial x_a\}$. Moreover, E being the position vector field, one gets $\bar{\nabla}_X E = \nabla_X E = X$, for any $X \in \Gamma(TM)$. Consequently, $A_E^* X + \tau(X)E + X = 0$. Noticing that the operator A_E^* is screen-valued, we infer from the last relation that $A_E^* X = -PX$, $\tau(X) = -\bar{g}(X, N) = -\lambda(X)$, for any $X \in \Gamma(TM)$. Next, any $X \in \Gamma(S(T\Lambda_0^{n+1}))$ is expressed as $X = \sum_{a=1}^{n+1} \tilde{X}_a \partial x_a$, where $\{\tilde{X}_1, \dots, \tilde{X}_{n+1}\}$ satisfy $\sum_{a=1}^{n+1} x_a \tilde{X}_a = 0$. From the above calculations, we can clearly see that $\tau(X) = 0$, for any $X \in \Gamma(S(TM))$.

The following result is important to this paper.

Theorem 3.2. [19] *Let (M, g) be a screen integrable null hypersurface of a Lorentzian manifold (\bar{M}, \bar{g}) . Under null MCF, the squared norm $|A_E^*|_s^2$ of the screen shape operator A_E^* evolves according to the following*

$$(3.2) \quad \frac{\partial |A_E^*|_s^2}{\partial t} = \Delta^s |A_E^*|_s^2 + 2\text{tr}^s(A_E^* \circ A_N) |A_E^*|_s^2 - 2|\nabla^s B|_s^2,$$

where $|A_E^*|_s^2 = g^{\alpha\mu} g^{\gamma\beta} B_{\alpha\mu} B_{\gamma\beta} = \text{tr}^s(A_E^{*2})$.

Example 3.2. Under null mean curvature flow, the induced metric g evolves (see [19]) according to

$$(3.3) \quad \frac{\partial g_{\alpha\beta}}{\partial t} = -2\mathcal{S}C_{\alpha\beta},$$

where $\mathcal{S} = \text{tr}^s(A_E^*)$. From [7], the curvature tensors R and R^* of M and $S(TM)$, respectively, are related by

$$(3.4) \quad \begin{aligned} g(R(X, Y)PZ, PW) &= g(R^*(X, Y)PZ, PW) + C(X, PZ)B(Y, PW) \\ &\quad - C(Y, PZ)B(X, PW). \end{aligned}$$

As M is a space of constant curvature c , we have (see [7, p. 41])

$$(3.5) \quad R(X, Y)PZ = c\{g(Y, PZ)X - g(X, PZ)Y\}, \quad \forall X, Y, Z \in \Gamma(TM).$$

Considering (3.4) and (3.5), we derive

$$(3.6) \quad -\mathcal{S}C_{\alpha\beta} = Ric_{\alpha\beta}^*(g) + c(n-1)g_{\alpha\beta} - g(A_E^* X_\alpha, A_N X_\beta),$$

where Ric^* denotes the Ricci tensor of the leaf M' of $S(TM)$. Let k_α^* and k_α be the eigenvalues of A_E^* and A_N , respectively, with respect to $\{X_1, \dots, n\}$. Placing (3.6) in (3.3) we get

$$(3.7) \quad \frac{\partial g_{\alpha\beta}}{\partial t} = 2Ric_{\alpha\beta}^*(g) + 2\sigma g_{\alpha\beta},$$

where $\sigma := c(n-1) - k_\alpha^* k_\beta$. As M' is Einstein, we have $Ric_{\alpha\beta}^*(g(0)) = \varpi g_{\alpha\beta}(0)$, where ϖ is a constant. Set $g_{\alpha\beta} = k g_{\alpha\beta}(0)$, where k is a positive constant. Then $Ric_{\alpha\beta}^*(g) = Ric_{\alpha\beta}^*(g(0)) = \varpi g_{\alpha\beta}(0) = \frac{\varpi}{k} g_{\alpha\beta}$. Let $g_{\alpha\beta}(t) = f(t)g_{\alpha\beta}(0)$ be a solution to (3.7), then

$$(3.8) \quad \begin{aligned} \frac{\partial g_{\alpha\beta}}{\partial t} &= f'(t)g_{\alpha\beta}(0) = 2Ric_{\alpha\beta}^*(f(t)g(0)) + 2\sigma f(t)g_{\alpha\beta}(0) \\ &= 2Ric_{\alpha\beta}^*(g(0)) + 2\sigma f(t)g_{\alpha\beta}(0) \\ &= 2\varpi g_{\alpha\beta}(0) + 2\sigma f(t)g_{\alpha\beta}(0), \end{aligned}$$

in which we have used the fact that M' is an Einstein manifold. From (3.8) we have $f'(t) = 2\sigma f(t) + 2\varpi$. Solving this ODE leads to $f(t) = -\frac{\varpi}{\sigma} + (1 + \frac{\varpi}{\sigma})e^{2\sigma t}$, and hence

$$g_{\alpha\beta}(t) = (-\frac{\varpi}{\sigma} + (1 + \frac{\varpi}{\sigma})e^{2\sigma t})g_{\alpha\beta}(0), \quad \alpha, \beta \in \{1, \dots, n\}.$$

Observe that the flow will become singular at $t = \frac{1}{2\sigma} \ln(\frac{\varpi}{\varpi + \sigma})$.

Now, we discuss some singularities associated to (3.2) in the case M is a screen conformal null hypersurface. A null hypersurface (M, g) of a semi-Riemannian manifold (\bar{M}, \bar{g}) is called *screen conformal* [7, p. 51] if there exist a non-vanishing smooth function ψ on a neighborhood \mathcal{U} in M such that $A_N = \psi A_E^*$, or equivalently,

$$(3.9) \quad C(X, PY) = \psi B(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

We say that M is *screen homothetic* if ψ is a constant function on M .

Example 3.3. Consider the null cone in Example 3.1. By straightforward calculation, one gets $\bar{g}(\nabla_E X, E) = -\sum_{a=1}^{n+1} x_a X_a = 0$, which implies that $\nabla_E X \in \Gamma(S(T\Lambda_0^{n+1}))$. Hence, $A_N E = 0$. Using Gauss-Codazzi equations, we calculate $C(X, Y) = \bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N) = -\frac{1}{2x_0^2} g(X, Y)$, for any $X, Y \in \Gamma(S(T\Lambda_0^{n+1}))$. Consequently, $A_N X = -\frac{1}{2x_0^2} P X$, for any $X, Y \in \Gamma(S(T\Lambda_0^{n+1}))$. Thus, deduce that $A_N X = \frac{1}{2x_0^2} A_E^* X$, for any $X \in \Gamma(S(T\Lambda_0^{n+1}))$. Hence, Λ_0^{n+1} is screen globally conformal null hypersurface of \mathbb{R}_1^{n+2} , with a positive conformal factor $\psi = \frac{1}{2x_0^2}$ globally define on Λ_0^{n+1} .

From the result of Theorem 3.2, we notice that one gets a quadratic term in $|A_E^*|_s^2$ when M is screen conformal. More precisely, when M is screen conformal, we have $\text{tr}^s(A_E^* \circ A_N) = \psi \text{tr}^s(A_E^{*2}) = \psi |A_E^*|_s^2$. Thus from (3.2), we have

$$(3.10) \quad \frac{\partial |A_E^*|_s^2}{\partial t} = \Delta^s |A_E^*|_s^2 + 2\psi |A_E^*|_s^4 - 2|\nabla^s B|_s^2.$$

Relation (3.10) indicates that there will also be *singularities* in null MCF if the initial null hypersurface M_0 is screen conformal, as the quantity $|A_E^*|_s^2$ blows-up in finite time. In fact, the above equation resembles the PDE; $u_t = \Delta u + u^2$, which has solutions which blows up at infinity. When this happens we say that $t := T$ is *singular* time for the null MCF. Moreover, we have the following.

Theorem 3.3. [19] *Let M_0 be a mean convex, screen conformal null hypersurface. If $|A_E^*|_s^2$ is not bounded as $t \rightarrow T < +\infty$ during the null MCF of a compact null hypersurface, then it must satisfy the following lower bound for its blow-up rate:*

$$\max_{p \in M} |A_E^*|_s^2(p, t) \geq \frac{1}{2\psi(T-t)},$$

for every $t \in [0, T)$. Hence,

$$\lim_{t \rightarrow T} \max_{p \in M} |A_E^*|_s^2(p, t) = +\infty,$$

By mean convex null hypersurface, we mean null hypersurface with nonnegative mean curvature everywhere.

In accordance to Theorem 3.3, let T be the maximal time of existence of null mean curvature flow, such that $\psi > 0$. If there exists a constant $\wp > 1$ such that we have the upper bound

$$(3.11) \quad \max_{p \in M} |A_E^*|_s(p, t) \leq \frac{\wp}{\sqrt{2\psi(T-t)}},$$

we say that the flow is developing a *type I singularity* at T . In the event that such a constant does not exist, that is

$$\limsup_{t \rightarrow T} \max_{p \in M} |A_E^*|_s(p, t) \sqrt{T-t} = +\infty,$$

we say that we have a *type II singularity*.

One of the major tools in studying the singularities associated to mean curvature flow of Riemannian surfaces is Huisken's monotonicity formula [13]. It is based on a Gaussian backward heat kernel and the first variation formula for Riemannian surfaces. Due to the inter-relationships between some geometric quantities on null hypersurfaces (and submanifolds in general) and the fact that the normal bundle, TM^\perp , is a subbundle of the tangent bundle, TM , of the null hypersurface M , such monotonicity formula cannot be used to investigate singularities in our case. Thus, we need a null version of it. To state and prove a monotonicity formula for null hypersurfaces moving by their mean curvature, we need a null variation formula. To do this, let U be a vector field on M , whose screen component PU has a compact support in the interior of M_t . Then, according to (2.1), U has a unique decomposition $U = PU + \bar{g}(U, N)E$. Using this decomposition of U , we have the following.

Proposition 3.4. *Let (M, g) be a compact screen conformal null hypersurface of (\bar{M}, \bar{g}) . Let \mathcal{U} be a neighborhood of M and let U be a compactly supported vector field on the screen distribution of M . If \mathcal{U}_t is a 1-parameter family of null hypersurfaces tangent at $t = 0$ to the variation U , then*

$$\int_{\mathcal{U}} \operatorname{div}^{S(TM)}(U) d\mu = - \int_{\mathcal{U}} \bar{g}(H, U) d\mu,$$

where $d\mu$ denotes the volume form with respect to g restricted to $S(TM)$ and $\operatorname{div}^{S(TM)}(\cdot)$ is the divergence operator with respect to $S(TM)$.

Proof. Let (M, g) be a null hypersurface, and let \mathcal{U} be a neighborhood of M . Let $f : \mathcal{U} \times (-\varepsilon, \varepsilon) \rightarrow M$ be a one-parameter variation supported in the interior of \mathcal{U} . Let t denote the coordinate on $(-\varepsilon, \varepsilon)$, and let $U := f_*(\partial_t)$. In this regard, $\mathcal{U} := f(\mathcal{U})$ evolves to $\mathcal{U}_t := f(\mathcal{U}, t)$. Flow U determines an endomorphism field \mathcal{A}_U along f . This endomorphism decomposes into a skew-symmetric part, which rotates $S(TU)$ but preserves volume (here volume is taken with respect to degenerate metric g restricted to $S(TM)$), and a symmetric part. The derivative at $t = 0$ of the area of an infinitesimal plane tangent to $S(T\mathcal{U}_t)$ is then the negative of the trace of \mathcal{A}_U restricted to $S(TU)$. In fact, let $d\mu$ denote the induced volume form on \mathcal{U}_t , then

$$(3.12) \quad \operatorname{vol}(\mathcal{U}_t) = \int_{\mathcal{U}} f_t^* d\mu,$$

where $f_t^* d\mu$ denotes the pullback of the volume form $d\mu$ on \mathcal{U}_t under f_t . But $f_t^* d\mu(X_1, \dots, X_n) = d\mu|_{\mathcal{U}_t}(df_t X_1, \dots, df_t X_n) = \det(df_t) d\mu(X_1, \dots, X_n)|_{\mathcal{U}_t} = \det(df_t)$, where $\{X_1, \dots, X_n\}$ is an orthonormal frame field along \mathcal{U} . Let $g_{\alpha\beta}^t = \bar{g}(df_t X_\alpha, df_t X_\beta)$ and $g(t) = \det(g_{\alpha\beta}^t)$, then $\det(df_t) = \sqrt{g(t)}$. Thus, from (3.12), we have

$$(3.13) \quad \left. \frac{d}{dt} \right|_{t=0} \operatorname{vol}(\mathcal{U}_t) = \int_{\mathcal{U}} \left. \frac{d}{dt} \right|_{t=0} \det(df_t) d\mu_0 = \int_{\mathcal{U}} \frac{1}{2} \left. \frac{dg}{dt} \right|_{t=0} d\mu.$$

By simple calculations, we have

$$(3.14) \quad \frac{1}{2} \left. \frac{\partial}{\partial t} g_{\alpha\alpha}^t \right|_{t=0} = g(\nabla_U X_\alpha^t, X_\alpha^t)|_{t=0} = g(\nabla_{X_\alpha^t} U, X_\alpha^t)|_{t=0},$$

where $X_\alpha^t = df \cdot X_\alpha$. Hence, considering (3.13) and (3.14), we get

$$(3.15) \quad \left. \frac{d}{dt} \right|_{t=0} \operatorname{vol}(\mathcal{U}_t) = \int_{\mathcal{U}} \operatorname{div}^{S(TM)}(U) d\mu.$$

From (3.15) and the decomposition of U , we have

$$(3.16) \quad \left. \frac{d}{dt} \right|_{t=0} \operatorname{vol}(\mathcal{U}_t) = \int_{\mathcal{U}} \operatorname{div}^{S(TM)}(PU) d\mu + \int_{\mathcal{U}} \operatorname{div}^{S(TM)}(\bar{g}(U, N)E) d\mu.$$

As PU is compactly supported on $S(TU)$, Stoke's theorem reduces (3.16) to

$$(3.17) \quad \left. \frac{d}{dt} \right|_{t=0} \operatorname{vol}(\mathcal{U}_t) = \int_{\mathcal{U}} \operatorname{div}^{S(TM)}(\bar{g}(U, N)E) d\mu.$$

By simple calculations, while considering (2.6), we have

$$(3.18) \quad \begin{aligned} g(\nabla_{X_\alpha} \bar{g}(U, N)E, X_\alpha) &= g(X_\alpha, E) X_\alpha \cdot \bar{g}(U, N) + \bar{g}(U, N) g(\nabla_{X_\alpha} E, X_\alpha) \\ &= -B(X_\alpha, X_\alpha) \bar{g}(U, N), \quad \forall \alpha \in \{1, \dots, n\}. \end{aligned}$$

Plugging (3.18) in (3.17), we get

$$(3.19) \quad \frac{d}{dt} \Big|_{t=0} \text{vol}(\mathcal{U}_t) = - \int_{\mathcal{U}} \sum_{\alpha=1}^n B(X_\alpha, X_\alpha) \bar{g}(U, N) d\mu,$$

which concludes the proof. \square

Proposition 3.4 gives the null first variational formula. Now, we are in position to state and prove a general monotonicity formula for null hypersurfaces evolving by their mean curvature. To this end, let us first define a backwards heat kernel (see details in [13]) for such evolution. Let us define a backward heat kernel ρ at $(0, t_0)$ by

$$(3.20) \quad \rho(X, t) := \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \exp\left(-\frac{|X|^2}{4\pi(t_0 - t)}\right), \quad t < t_0,$$

where $|\cdot|^2 := \widehat{g}(\cdot, \cdot)$. Notice that (3.20) is a Gaussian function. This function is frequently used as a backward heat kernel due to its nice properties (see for instance [13]). With such a heat kernel, we have the following null monotocity formula.

Theorem 3.5. *Let M_t be a screen conformal null hypersurface evolving by mean curvature and satisfying Definition 2.1. Then*

$$\frac{d}{dt} \int_{\mathcal{U}_t} \rho(X, t) d\mu = - \int_{\mathcal{U}_t} \rho(X, t) \left\{ \psi \mathcal{S}^2 + \mathcal{S} \frac{\lambda(X)}{(t_0 - t)} + \frac{\lambda(X)^2}{4(t_0 - t)^2} \right\} d\mu,$$

where $\lambda(X) = \bar{g}(X, N)$.

Proof. By a straightforward calculation while considering the evolution equation $\frac{d}{dt} \mu = -\bar{g}(H, \widehat{H}) d\mu$, where $\widehat{H} = \sum_{a=0}^n C(X_a, P X_a) E$ and the fact M is screen conformal we derive

$$(3.21) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathcal{U}_t} \rho(X, t) d\mu \\ &= \int_{\mathcal{U}_t} \left\{ \frac{\partial \rho(X, t)}{\partial t} + \bar{g}(\nabla \rho(X, t), H) \right\} d\mu + \int_{\mathcal{U}_t} \rho(X, t) \frac{\partial d\mu}{\partial t} \\ &= \int_{\mathcal{U}_t} \left\{ \frac{\partial \rho(X, t)}{\partial t} + \bar{g}(\nabla \rho(X, t), H) \right\} d\mu - \int_{\mathcal{U}_t} \rho(X, t) \bar{g}(\widehat{H}, H) d\mu \\ &= \int_{\mathcal{U}_t} \left\{ \frac{\partial \rho(X, t)}{\partial t} + \bar{g}(\nabla \rho(X, t), H) \right\} d\mu - \int_{\mathcal{U}_t} \psi \rho(X, t) \mathcal{S}^2 d\mu, \end{aligned}$$

where ∇f denotes the gradient of a smooth function f with respect to \widehat{g} . From (3.20), we have

$$(3.22) \quad \frac{\partial \rho(X, t)}{\partial t} = \frac{n}{2(t_0 - t)} \rho(X, t) - \frac{|X|^2}{4(t_0 - t)^2} \rho(X, t),$$

$$(3.23) \quad \text{and} \quad \nabla \rho(X, t) = -\frac{1}{2(t_0 - t)} \rho(X, t) X.$$

Putting (3.22) and (3.23) in (3.21), we have

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{U}_t} \rho(X, t) d\mu &= \int_{\mathcal{U}_t} \left\{ \frac{n}{2(t_0 - t)} \rho(X, t) - \frac{|X|^2}{4(t_0 - t)^2} \rho(X, t) \right. \\
&\quad \left. - \frac{\rho(X, t)}{2(t_0 - t)} \bar{g}(X, H) \right\} d\mu - \int_{\mathcal{U}_t} \psi \rho(X, t) \mathcal{S}^2 d\mu \\
&= - \int_{\mathcal{U}_t} \rho(X, t) \left\{ \psi \mathcal{S}^2 + \frac{1}{(t_0 - t)} \bar{g}(X, H) + \frac{|X|^2}{4(t_0 - t)^2} \right\} d\mu \\
(3.24) \quad &+ \int_{\mathcal{U}_t} \frac{\rho(X, t)}{2(t_0 - t)} \bar{g}(X, H) d\mu + \int_{\mathcal{U}_t} \frac{n}{2(t_0 - t)} \rho(X, t) d\mu.
\end{aligned}$$

Now, using the first variation formula in Proposition 3.4 with $U = \frac{\Phi}{2(t_0 - t)} X$, we derive

$$(3.25) \quad \int_{\mathcal{U}_t} \frac{\rho(X, t)}{2(t_0 - t)} \bar{g}(X, H) d\mu = - \int_{\mathcal{U}_t} \rho(X, t) \left\{ \frac{n}{2(t_0 - t)} - \frac{|PX|^2}{4(t_0 - t)^2} \right\} d\mu.$$

Finally, plugging (3.25) in (3.24) and using $X = PX + \bar{g}(X, N)E$, we get

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{U}_t} \rho(X, t) d\mu &= - \int_{\mathcal{U}_t} \rho(X, t) \left\{ \psi \mathcal{S}^2 + \frac{1}{(t_0 - t)} \bar{g}(X, H) + \frac{|X|^2}{4(t_0 - t)^2} \right\} d\mu \\
&\quad + \int_{\mathcal{U}_t} \frac{|PX|^2}{4(t_0 - t)^2} \rho(X, t) d\mu \\
&= - \int_{\mathcal{U}_t} \rho(X, t) \left\{ \psi \mathcal{S}^2 + \frac{\bar{g}(X, H)}{(t_0 - t)} + \frac{\bar{g}(X, N)^2}{4(t_0 - t)^2} \right\} d\mu,
\end{aligned}$$

which proves our result and the proof is complete. \square

Theorem 3.5 gives the null monotonicity formula for a screen conformal lightlike hypersurface evolving by its mean curvature.

Let $(0, T)$ be a Type I singularity and consider the rescaled mean curvature flow

$$(3.26) \quad \tilde{F}(x, \tilde{s}) = \frac{F(x, t)}{\sqrt{2\psi(T - t)}}, \quad \tilde{s}(t) = -\frac{1}{2\psi} \log(T - t).$$

Then

$$(3.27) \quad \frac{d\tilde{F}(x, \tilde{s})}{d\tilde{s}} = \tilde{\mathbf{H}}(x, \tilde{s}) + \tilde{F}(x, \tilde{s}),$$

on $M \times [0, +\infty)$. Using the above rescaling and Theorem 3.5, we have the following important result.

Corollary 3.6. *If the surface $\tilde{M}_{\tilde{s}}$ satisfies the rescaled equation (3.26), then*

$$\frac{d}{d\tilde{s}} \int_{\tilde{F}_{\tilde{s}}} \rho(\tilde{F}, \tilde{s}) d\mu = - \int_{\tilde{F}_{\tilde{s}}} \rho(\tilde{F}, \tilde{s}) \{ \psi \mathcal{S}^2 + 2\mathcal{S}\lambda(\tilde{F}) + \lambda(\tilde{F})^2 \} d\tilde{\mu}_{\tilde{s}},$$

where $\tilde{F}_{\tilde{s}} = \tilde{F}_{\tilde{s}}(M)$, $\rho(\tilde{F}) = \exp(-\frac{1}{2}|\tilde{F}|^2)$ and $\lambda(\tilde{F}) = \bar{g}(\tilde{F}, N)$.

Next, we use Corollary 3.6 to study the behavior of $\widetilde{M}_{\tilde{s}}$ as $\tilde{s} \rightarrow \infty$.

Proposition 3.7. *Let $[0, T)$ be a type I singularity such that (3.11) holds. Then there is $x \in M$ such that $\widetilde{F}(x, \tilde{s})$ as defined in (3.26) remains bounded for $\tilde{s} \rightarrow \infty$.*

Proof. In view of the null mean curvature flow relation of Definition 2.1, it is easy to show that $|F(x, t)| \leq C_0(M)$, where $C_0(M)$ is a non-zero constant on M independent of time, which ends the proof. \square

Notice that $F_t \cap \mathbb{B}_{\mathcal{C}(M)}(0)$ can not be empty for $\mathcal{C}(M)$ large enough, otherwise the time T can not be the first time singularity. Therefore, $\widetilde{F}(\cdot, \tilde{s})$ is indeed locally uniformly bounded.

Theorem 3.8. *Let $[0, T)$ be a type I singularity such that (3.11) holds. Then for each sequence $\tilde{s}_j \rightarrow \infty$ there is a subsequence \tilde{s}_{j_k} such that $\widetilde{M}_{\tilde{s}_{j_k}}$ converges smoothly to an immersed nonempty limiting surface \widetilde{M}_∞ .*

Proof. A proof uses similar arguments as that in [13]. \square

Now, using the rescaled null monotonicity formula in Corollary 3.6 we have the following result about the limiting surface \widetilde{M}_∞ .

Theorem 3.9. *For $\psi \leq 1$, each limiting surface \widetilde{M}_∞ as per Theorem 3.8 satisfies*

$$(3.28) \quad \mathcal{S} = \bar{g}(\mathbf{X}, N), \quad \mathbf{X} := - \left(\frac{\psi}{1 + \sqrt{1 - \psi}} \right)^{-1} X,$$

where X is the position vector and \mathcal{S} is the null mean curvature and N is the lightlike transversal vector, such that $\mathbf{H} = \mathcal{S}N$ is the mean curvature vector.

Proof. From the lightlike monotonicity formula of Corollary 3.6 and the fact that the integral of ρ is positive, we have

$$(3.29) \quad \int_{\tilde{s}_0}^{\infty} \int_{\widetilde{F}_{\tilde{s}}} \rho(\widetilde{F}, \tilde{s}) \{ \psi \mathcal{S}^2 + 2\mathcal{S}\lambda(\widetilde{F}) + \lambda(\widetilde{F})^2 \} d\tilde{\mu}_{\tilde{s}} d\tilde{s} < \infty,$$

and the result follows immediately in view of the uniform estimates in Proposition 2.3 as in [13]. \square

Observe that the relation (3.28) in Theorem 3.9 is a second order elliptic equation. For any solution F_0 to (3.28), there is an associated self-similar solution of the null mean curvature flow, given by $F(t) = (2(T - t))^{1/2} F_0$. Thus, from the above we have the following.

Theorem 3.10. *The null mean curvature flow $F = \{F_t\}$ at a type I singularity (x_0, T) is asymptotically self-similar, in the sense that $[2(T - t)]^{-1/2}(F_t - x_0)$ converges locally uniformly to a solution F of (3.28).*

In Theorem 3.10, we obtained a self-similar blow-up solution at type I singularity, which is an ancient solution, namely it exists from time $t = -\infty$. At type II singularity, there is a blow-up solution which is an eternal solution. To see this, we choose

the blow-up points (x_k, t_k) as follows. For each integer $k \geq 1$, let $t_k \in [0, T - \frac{1}{k}]$ and $x_k \in F_{t_k}$, such that

$$(3.30) \quad |A_E^*|_s^2(x_k, t_k) \left(T - \frac{1}{k} - t_k \right) = \max_{t \leq T - \frac{1}{k}, x \in F_t} |A_E^*|_s^2(x, t) \left(T - \frac{1}{k} - t \right).$$

Let us make the dilation

$$F^k(y, \tilde{s}) = \epsilon_k^{-1} [F(x_k + \epsilon_k y, t_k + \epsilon_k^2 \tilde{s}) - F(x_k, t_k)],$$

where $\epsilon_k = (|A_E^*|_s(x_k, t_k))^{-1}$. Since the singularity is of Type II, the right hand side of (3.30) tends to $+\infty$ as $k \rightarrow +\infty$. Hence, for any fixed $\tilde{s} \in [-\epsilon_k^{-2} t_k, \epsilon_k^{-2} (T - \frac{1}{k} - t)]$, the mean curvature of the rescaled hypersurface F^k satisfies

$$|A_E^*|_s^2 \leq \frac{T - \frac{1}{k} - t_k}{T - \frac{1}{k} - t} = \frac{T - \frac{1}{k} - t_k}{T - \frac{1}{k} - t_k - \epsilon_k^2 \tilde{s}} \rightarrow 1, \text{ as } k \rightarrow +\infty.$$

It follows that for any $\tilde{s}^* > 1$ and any $\epsilon > 0$ there exists k_0 such that

$$\max_{F^k} |A_E^*|_s(\cdot, \tilde{s}) \leq 1 + \epsilon,$$

for any $k \geq k_0$ and $\tilde{s} \in [-\tilde{s}^*, \tilde{s}^*]$. By similar arguments as in [22], the second fundamental form is uniformly bounded and all derivatives of the second fundamental forms of $F^k(\cdot, \tilde{s})$ are uniformly bounded. Hence, by the Arzela-Ascoli theorem, we can extract a subsequence of $F^k(\cdot, \tilde{s})$ which converges uniformly on any compact subsets of $\mathbb{R}^{n+2} \times \mathbb{R}^1$ to a blow-up solution $F^\infty(\cdot, \tilde{s})$ (see [6], [16], [21] for more details).

4 Hanarck estimates

In classical Riemannian (or semi-Riemannian) geometry, a submanifold M in a flat space \bar{M} is called a *translating soliton* [10] if there exists a constant vector \bar{K} in \bar{M} such that $\bar{K} = K + \mathbf{H}$ on \bar{M} , where K is its tangential component and \mathbf{H} is the mean curvature vector of M in \bar{M} . Equivalently, M is a translating soliton if $\mathbf{H} = \bar{K}^\perp$, where \bar{K}^\perp is the projection of \bar{K} in \bar{M} on the normal bundle of M . Consequently, the 1-parameter family of submanifolds M_t defined by $M_t = M + t\bar{K}$ for $t \in \mathbb{R}$ is a solution to the classical MCF (see [9], [14], [15] and many more references therein). Soliton solutions have proved fundamentally important in mean curvature flow studies, for instance singularities are locally modeled on soliton solutions (see [22] and many more references therein). In [12], Hamilton used translating solitons to prove some Harnack estimates for MCF of Riemannian hypersurfaces.

We have seen, in the previous section, that the evolution equations for null MCF possesses some analogies with those in classical MCF of Riemannian hypersurfaces (see some of such evolution equations in [13], [14], [15] and many more references therein). Thus, singularities are bound to occur as in the usual MCF. Furthermore, translators or translating solitons can equally be defined in null MCF to facilitate the study of such singularities as well as in the establishment of *null Harnack inequalities* which is the main aim of this section. Note, however, that in null MCF we can not use the above definition of translating solitons due to the fact that the normal bundle of a

null hypersurface is a subbundle of its tangent bundle. Furthermore, in null MCF the null hypersurface under consideration is evolving in the transversal direction. From the decomposition of $T\bar{M}$ in (2.2), any vector \bar{K} of \bar{M} can be uniquely written as $\bar{K} = K + K^{tr}$, where K and K^{tr} are its tangential and transversal components, respectively. Thus, for null MCF, we have the following definition.

Definition 4.1. A solution M_t of null MCF will be called a *null translating soliton* if its null mean curvature vector \mathbf{H} satisfy $\mathbf{H} = K^{tr}$.

As $\mathbf{H} = \mathcal{S}N$, we immediately get $\mathcal{S} = \bar{g}(K^{tr}, E)$, where E spans the radical distribution of M_t . Suppose a solution M_t to the null MCF in Definition 2.1 translates in the direction of the constant vector $\bar{K} = (\bar{K}^0, \dots, \bar{K}^{n+1})$, for $i \in \{0, \dots, n+1\}$, and let $K = (K^0, \dots, K^{n+1})$ be its tangential component. Then the transversal component of \bar{K} must be $\mathcal{S}N^i$ to solve the null MCF. Locally, the tangential component K can be written as $K^i = K^a \frac{\partial y^i}{\partial x_a} := K^a \nabla_{X_a} y^i$, where $y^i = F^i(x_a, t)$. Also, with respect to the non-degenerate metric \hat{g} its covector is $K_a = \hat{g}_{ab} K^b$, for any $a, b \in \{0, \dots, n\}$. Then differentiating $\bar{K}^i = K^i + \mathcal{S}N^i$ and using the facts $\nabla_{X_a} \bar{K}^i = 0$ (\bar{K} is a constant vector), (2.4) and $\tau = 0$ we get

$$(4.1) \quad g^{[bc]}(\nabla_{X_a} K_b - \mathcal{S}C_{ab})\nabla_{X_c} y^i + (\nabla_{X_a} \mathcal{S} + g^{[bc]}B_{ab}K_c)N^i = 0.$$

Taking the \bar{g} -product of (4.1) with N^i and observing that $\bar{g}(\nabla_{X_c} y^i, N^i) \neq 0$, we get the two relations

$$(4.2) \quad \nabla_{X_a} K_b = \mathcal{S}C_{ab} \quad \text{and} \quad \nabla_{X_a} \mathcal{S} + B_{ab}K_b = 0.$$

By simple calculations while considering the first relation of (4.2), we have

$$(4.3) \quad (\mathcal{L}_K g)(X_\alpha, X_\beta) = \nabla_{X_\alpha} K_\beta + \nabla_{X_\beta} K_\alpha = \mathcal{S}(C_{\alpha\beta} + C_{\beta\alpha}).$$

Hence, from the evolution equation of g (see [19]), if the local second fundamental form C is symmetric, the degenerate metric g evolves by the Lie derivative with respect to K , that is;

$$(4.4) \quad \frac{\partial}{\partial t} g_{\alpha\beta} = -(\mathcal{L}_K g)_{\alpha\beta}, \quad \alpha, \beta \in \{1, \dots, n\}.$$

The vector field \bar{K} is said to be conformal [7, p. 259] if

$$(\mathcal{L}_{\bar{K}} \bar{g})(\bar{X}, \bar{Y}) = 2\Psi \bar{g}(\bar{X}, \bar{Y}), \quad \forall \bar{X}, \bar{Y} \in \Gamma(T\bar{M}),$$

where Ψ is a scalar function on \bar{M} . In particular, \bar{K} is *homothetic* or *Killing* vector field according as Ψ is non-zero constant or zero respectively. \bar{K} is called a proper conformal vector field if Ψ is a non-constant function. Accordingly, the following holds.

Proposition 4.1. *If M_t is null translating soliton in which \bar{K} is conformal, then $\Psi = 0$, that is; \bar{K} is a killing vector field.*

Proof. As M_t is a null translating soliton, we have $\bar{K} = K + \mathbf{H}$, where \mathbf{H} is the null mean curvature and given by $\mathbf{H} = \mathcal{S}N$. Then by simple calculations we have

$$\begin{aligned}
(\mathcal{L}_{\bar{K}}\bar{g})(X, Y) &= (\mathcal{L}_{K+\mathbf{H}}\bar{g})(X, Y) \\
&= (\mathcal{L}_K g)(X, Y) + \bar{g}(\bar{\nabla}_X \mathbf{H}, Y) + \bar{g}(X, \bar{\nabla}_Y \mathbf{H}) \\
&= (\mathcal{L}_K g)(X, Y) + \mathcal{S}\bar{g}(\bar{\nabla}_X N, Y) + \mathcal{S}\bar{g}(X, \bar{\nabla}_Y N) \\
&\quad + X(\mathcal{S})\lambda(Y) + Y(\mathcal{S})\lambda(X),
\end{aligned}
\tag{4.5}$$

for any $X, Y \in \Gamma(TM)$. Setting $X = X_\alpha$, $Y = X_\beta$ and the fact \bar{K} is conformal, we get

$$\begin{aligned}
(\mathcal{L}_K g)(X_\alpha, X_\beta) &= 2\Psi g(X_\alpha, X_\beta) - \mathcal{S}\bar{g}(\bar{\nabla}_{X_\alpha} N, X_\beta) - \mathcal{S}\bar{g}(X_\alpha, \bar{\nabla}_{X_\beta} N) \\
&= 2\Psi g(X_\alpha, X_\beta) + \mathcal{S}g(A_N X_\alpha, X_\beta) + \mathcal{S}g(X_\alpha, A_N X_\beta) \\
&= 2\Psi g(X_\alpha, X_\beta) + 2\mathcal{S}C(X_\alpha, X_\beta),
\end{aligned}
\tag{4.6}$$

in which we have used (2.4), (2.7) and symmetry of C . Then, using (4.4) and (4.6) we get

$$\frac{\partial g_{\alpha\beta}}{\partial t} = -2\Psi g_{\alpha\beta} - 2\mathcal{S}C_{\alpha\beta}.
\tag{4.7}$$

Comparing (4.7) and the evolution of g in [19], we get $-2\Psi g_{\alpha\beta} = 0$, which implies that $\Psi = 0$. Hence, \bar{K} is killing and the proof is complete. \square

Observe that on a null translating soliton, (4.2) will imply that $\nabla_{X_\alpha} K_\beta = \nabla_{X_\beta} K_\alpha$. Consequently, there is a function ϕ with $\nabla_{X_\alpha} \phi = K_\alpha$. Since the vector field is a gradient, we observe that null translating solitons are also *gradient solitons* [12], satisfying $\nabla_{X_\alpha} \nabla_{X_\beta} \phi = \mathcal{S}C_{\alpha\beta}$. Next, let us set

$$\mathcal{B}_{\alpha\beta\gamma\mu} := B_{\alpha\gamma} C_{\beta\mu} K_\mu + C_{\alpha\gamma} \nabla_{X_\beta} \mathcal{S}, \quad R_{\alpha\beta\gamma\mu} := g(R(X_\alpha, X_\beta)X_\gamma, X_\mu),$$

for $\alpha, \beta, \gamma, \mu \in \{1, \dots, n\}$. Then, the following follows from the above discussion.

Proposition 4.2. *On a null translating soliton, the following holds*

$$\nabla_{X_\alpha} \nabla_{X_\beta} \phi = \mathcal{S}C_{\alpha\beta} \quad \text{and} \quad \mathcal{B}_{\alpha\beta\gamma\mu} = \mathcal{B}_{\beta\alpha\gamma\mu}.
\tag{4.8}$$

Moreover, if M is screen conformal and the local second fundamental form B satisfy $B_{\alpha\beta} < \mathcal{S}g_{\alpha\beta}$, we have

$$\nabla_{X_\alpha} \mathcal{S} + B_{\alpha\beta} K_\beta = 0.
\tag{4.9}$$

Proof. Differentiating the first relation of (4.8) we get

$$\nabla_{X_\alpha} \nabla_{X_\beta} \nabla_{X_\gamma} \phi = C_{\beta\gamma} \nabla_{X_\alpha} \mathcal{S} + \mathcal{S} \nabla_{X_\beta} C_{\alpha\gamma}.
\tag{4.10}$$

Inter-changing X_α and X_β and subtracting the two equations, we have

$$\nabla_{X_\alpha} \nabla_{X_\beta} \nabla_{X_\gamma} \phi - \nabla_{X_\beta} \nabla_{X_\alpha} \nabla_{X_\gamma} \phi = C_{\beta\gamma} \nabla_{X_\alpha} \mathcal{S} - C_{\alpha\gamma} \nabla_{X_\beta} \mathcal{S}.
\tag{4.11}$$

Then, from Gauss-Codazzi equation (3.7) of [7, p. 94], we have

$$(4.12) \quad R_{\alpha\beta\gamma\mu} = B_{\alpha\gamma}C_{\beta\mu} - B_{\beta\gamma}C_{\alpha\mu}.$$

From (4.11), (4.12) and the fact $\nabla_{X_\mu}\phi = K_\mu$, we get

$$(4.13) \quad B_{\alpha\gamma}C_{\beta\mu}K_\mu + C_{\alpha\gamma}\nabla_{X_\beta}\mathcal{S} = B_{\beta\gamma}C_{\alpha\mu}K_\mu + C_{\beta\gamma}\nabla_{X_\alpha}\mathcal{S},$$

which proves the second relation of (4.8). Then when M is screen conformal [7], we get

$$(4.14) \quad B_{\alpha\gamma}(B_{\beta\mu}K_\mu + \nabla_{X_\beta}\mathcal{S}) = B_{\beta\gamma}(B_{\alpha\mu}K_\mu + \nabla_{X_\alpha}\mathcal{S}).$$

Setting $\mathcal{W}_\alpha := B_{\alpha\mu}K_\mu + \nabla_{X_\alpha}\mathcal{S}$, such that (4.14) reduces to

$$(4.15) \quad B_{\alpha\gamma}\mathcal{W}_\beta = B_{\beta\gamma}\mathcal{W}_\alpha.$$

Taking the trace on β and γ in (4.15), we obtain

$$(4.16) \quad (\mathcal{S}g_{\alpha\beta} - B_{\alpha\beta})\mathcal{W}_\beta = 0.$$

Since $B_{\alpha\beta} < \mathcal{S}g_{\alpha\beta}$, then the matrix $\mathcal{S}g_{\alpha\beta} - B_{\alpha\beta}$ is strictly positive and hence, $\mathcal{W}_\beta = 0$ and the proof is complete. \square

By the method of Hamilton [12], we have the following differential Harnack-type inequality for null MCF.

Theorem 4.3. *Let (M, g) be a null hypersurface of the Lorentzian manifold $(\overline{M}, \overline{g})$. If M moves by null MCF, then*

$$(4.17) \quad \frac{\partial \mathcal{S}}{\partial t} + \frac{\mathcal{S}}{2t} + 2\nabla \mathcal{S}(K) + B(K, PK) \geq 0,$$

for any tangent vector K , where $\mathcal{S} = \text{tr}^s(A_E^*)$.

Proof. From [19] we have

$$(4.18) \quad \frac{\partial \mathcal{S}}{\partial t} = \Delta^s \mathcal{S} + \mathcal{S} \text{tr}^s(A_E^* \circ A_N).$$

Now, differentiating the second relation of (4.2) and then applying the first relation of (4.2) and the fact $X_\alpha \cdot B_{\beta\gamma} = X_\beta \cdot B_{\alpha\gamma}$ around $p \in M$, we have

$$(4.19) \quad \nabla_{E_\alpha} \nabla_{E_\beta} \mathcal{S} + K_\gamma \nabla_{E_\gamma} B_{\alpha\beta} + \mathcal{S} B_{\beta\gamma} C_{\gamma\alpha} = 0.$$

Taking trace of (4.19) we get

$$(4.20) \quad \Delta \mathcal{S} + K_\gamma \nabla_{E_\gamma} \mathcal{S} + \mathcal{S} \text{tr}^s(A_E^* \circ A_N) = 0.$$

Then substituting equation (4.20) in (4.18), we deduce that

$$(4.21) \quad \frac{\partial \mathcal{S}}{\partial t} + K_\alpha \nabla_{E_\alpha} \mathcal{S} = 0.$$

But from the second relation of (4.2), we have

$$(4.22) \quad K_\alpha \nabla_{E_\alpha} \mathcal{S} + B_{\alpha\beta} K_\alpha K_\beta = 0.$$

Adding (4.21) to (4.22), we deduce

$$(4.23) \quad \frac{\partial \mathcal{S}}{\partial t} + 2\nabla \mathcal{S}(K) + B(PK, PK) = 0.$$

Equation (4.23) is the basic differential null Harnack equation. As mentioned in [12], for solutions on $t > 0$, we should really consider homothetically expanding solitons, which differ from the translating ones by terms that are basically lower order with a $\frac{1}{t}$ in them. This accounts for the existence of the term $\frac{1}{2t}\mathcal{S}$ in our inequality, hence the proof. \square

Using Hamilton's method [12] and Theorem 4.3, we state the following integrated version of null Hanarck inequality above, along paths in spacetime \overline{M}^{n+2} .

Theorem 4.4. *For any weakly convex solution to the null mean curvature flow for $t > 0$ we have*

$$\mathcal{S}(y_2, t_2) \geq \sqrt{\frac{t_1}{t_2}} e^{-\frac{1}{4}\Omega} \mathcal{S}(y_1, t_1),$$

for any two points y_1 and y_2 on the evolving null hypersurface at times t_1 and t_2 , with $T \geq t_2 > t_1 > 0$, where

$$\Omega := \inf_y \int_{t_1}^{t_2} g\left(P \frac{dy}{dt}, P \frac{dy}{dt}\right) dt,$$

is the infimum over all paths $y(t)$ remaining on the null hypersurface at time t with $\frac{dy}{dt}$ as the velocity vector of the path.

Proof. Along any path $y(t) = F(x(t), t)$ we have

$$(4.24) \quad \frac{d\mathcal{S}}{dt} = \frac{\partial \mathcal{S}}{\partial t} + \nabla \mathcal{S}\left(\frac{dx}{dt}\right).$$

Using (4.24) and Theorem 4.3, with $K = \frac{1}{2} \frac{dx}{dt}$, we get

$$(4.25) \quad \frac{d\mathcal{S}}{dt} \geq -\frac{1}{4} B\left(P \frac{dx}{dt}, P \frac{dx}{dt}\right) - \frac{1}{2t} \mathcal{S}.$$

But for a convex hypersurface, $B\left(P \frac{dx}{dt}, P \frac{dx}{dt}\right) \leq Sg\left(P \frac{dx}{dt}, P \frac{dx}{dt}\right)$ and hence (4.25) gives

$$(4.26) \quad \frac{d \ln(\mathcal{S})}{dt} \geq -\frac{1}{4} g\left(P \frac{dx}{dt}, P \frac{dx}{dt}\right) - \frac{1}{2t},$$

where $P \frac{dx}{dt}$ is the screen component of $\frac{dy}{dt}$, such that (4.26) leads to

$$(4.27) \quad \log\left(\frac{\mathcal{S}(y_2, t_2)}{\mathcal{S}(y_1, t_1)}\right) \geq -\frac{1}{2} \log\left(\frac{t_2}{t_1}\right) - \frac{1}{4}\Omega,$$

where $\Omega := \inf_y \int_{t_1}^{t_2} g\left(P \frac{dy}{dt}, P \frac{dy}{dt}\right) dt$, and the result follows by exponentiating, which completes the proof. \square

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